Proceedings of the International MultiConference of Engineers and Computer Scientists 2014 Vol II,
IMECS 2014, March 12 - 14, 2014, Hong Kong

Ulam-Hyers Stability Results for Fixed Point Problems via Generalized Multivalued Almost Contraction

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***Abstract*—In this paper, we introduced the notion of a generalized multivalued (α, φ)-almost contractions and establish the existence of fixed point theorems for this class of mapping. The results presented in this paper generalize and extend some recent results in multivalued almost contraction. Also, we show its applications in the Ulam-Hyers stability of fixed point problems for multivalued operators.**

***Index Terms*—Almost contraction, Fixed point theorems, Generalized multivalued almost contraction, Ulam-Hyers stability.**

I. Introduction

A Study of fixed point for a multivalued (set-valued) mappings was originally initiated by von Neumann [23] in the study of game theory. The development of geometric fixed point theory for multivalued mapping was initiated with the work of Nadler [16] in 1969. He combined the ideas of multivalued mapping and Lipschitz mapping and used the concept of Hausdorff metric to establish the multivalued contraction principle, usually referred as Nadler's contraction mapping principle.

Definition 1. [16] Let (X, d) be a complete metric space and $S : X \to \mathcal{CB}(X)$ be multivalued mapping such that for all $x, y \in X$, we have

$$H(Sx, Sy) \leq kd(x, y), \quad \text{where} \quad k \in (0, 1). \tag{1}$$

Then there exists $z \in X$ such that $z \in Sz$.

In 2003, Berinde [3] introduced almost contractions that satisfy a simple but general contraction condition that includes most of the conditions in Rhoadesclassification [18]. He obtained a fixed point theorem for such mappings which generalized the results of Kannan [13]. The weakly contractive metric-type fixed point result in [4] is almost covered by the related altering metric one due to Khan et al.[13]. A number of papers appeared in which fixed points of almost contractions for single valued mapping have been discussed (see [1, 4–7, 20] and references therein).

In 2007, M. Berinde and V. Berinde [2] extended almost contractions of self-mappings to the case of multivalued almost contractions. Afterward, several researches have extended and proved the fixed point theorems of multivalued almost contractions (see [5–7] and references therein).

On the other hand, stability problem of functional analysis is the another one which play the most important in mathematics analysis. It was introduced by Ulam [22], he was concern the stability of group homomorphisms. Afterward, Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces, this type of stability is called Ulam-Hyers stability. Several authors consider Ulam-Hyers stability results in fixed point theory and remarkable result on the stability of certain classes of functional equations via fixed point approach (see [8–11, 15, 21] and references therein).

In this work, we give fixed point results for some new class of multivalued almost contractions. Our results generalize and extend several multivalued almost contraction results in the existing literature. Moreover, we show its applications in the Ulam-Hyers stability of fixed point problems for multivalued operators.

II. Preliminaries

Throughout this paper, let (X, d) be a metric space and $\mathcal{CB}(X)$ be the family of nonempty closed bounded subsets of X. For a point x in X and a nonempty subset A of X, we define the distance $d(x, A)$ from x to A by

$$d(x, A) = \inf\{d(x, a) : a \in A\},$$

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

For $A, B \in \mathcal{CB}(X)$, we define the *Hausdroff distance*, between A and B by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, \tag{2}$$

which is symmetric in A and B. It well known that $(\mathcal{CB}(X), H)$ is a complete metric space.

Definition 2. Let $S : X \to (\mathcal{CB}(X))$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of S if $x \in Sx$.

Lemma 3 ([16]). *Let (X, d) be a metric space and $A, B \in \mathcal{CB}(X)$, then for each $a \in A$,*

$$d(a, B) \leq H(A, B).$$

Lemma 4 ([16]). *Let (X, d) be a metric space and $A, B \in \mathcal{CB}(X)$, then for each $a \in A$, $\varepsilon > 0$, there exists an element $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.*

Lemma 5. *Let (X, d) be a metric space. Let $A, B \in X$ and $q > 1$. Then, for every $a \in A$, there exists $b \in B$ such that*

$$d(a, b) \leq qH(A, B). \tag{3}$$

Proof. If $H(A, B) = 0$, then $a \in B$ and (3) holds for $b = a$. If $H(A, B) > 0$, then let us denote

$$\varepsilon = (q - 1)H(A, B) > 0. \tag{4}$$

This work was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission.

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ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Using the definition of $d(a, B)$ and $H(A, B)$, it follows that, for any $\varepsilon > 0$, there exists $b \in B$ such that

$$d(a, b) \leq d(a, B) + \varepsilon \leq H(A, B) + \varepsilon. \quad (5)$$

Now, by inserting (4) in (5), we get (3).

III. Fixed Point Theorems

In this section, we introduce and prove the concept of generalized multivalued (α, φ)-almost contraction mappings.

Theorem 6. *Let (X, d) be a complete metric space and $S : X \to \mathcal{CB}(X)$ be a generalized multivalued (α, φ)-almost contraction, i.e., a mapping satisfying for which there exists a function $\alpha : [0, \infty) \to [0, 1)$ and $\varphi : X \to [0, \infty)$, satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$, such that*

$$\begin{aligned} &H(Sx, Sy) \\ &\quad \leq \alpha(d(x, y))d(x, y) + \varphi(d(x, y)) \\ &\qquad \min\{d(x, Sx), d(y, Sy), d(x, Sy), d(y, Sx)\} \end{aligned} \quad (6)$$

for all $x, y \in X$. Then S has a fixed point in X.

Proof. Let $q > 1$. Let $x_0 \in X$ and $x_1 \in Sx_0$. If $H(Sx_0, Sx_1) = 0$ then $Sx_0 = Sx_1$, i.e., $x_1 \in Sx_1$, then x_1 is fixed point of S. Let $H(Sx_0, Sx_1) \neq 0$. By Lemma 5, there exists $x_2 \in Sx_1$ such that

$$d(x_1, x_2) \leq qH(Sx_0, Sx_1).$$

By (6), we obtain

$$\begin{aligned} &d(x_1, x_2) \\ &\quad \leq q[\alpha(d(x_0, x_1))d(x_0, x_1) + \varphi(d(x_0, x_1)) \\ &\qquad \min\{d(x_0, Sx_0), d(x_1, Sx_1), d(x_0, Sx_1), d(x_1, Sx_0)\}] \\ &\quad \leq q[\alpha(d(x_0, x_1))d(x_0, x_1) + \varphi(d(x_0, x_1)) \\ &\qquad \min\{d(x_0, x_1), d(x_1, x_2), d(x_0, x_2), d(x_1, x_1)\}] \\ &\quad \leq q[\alpha(d(x_0, x_1))d(x_0, x_1)]. \end{aligned}$$

Because

$$\min\{d(x_0, Sx_0), d(x_1, Sx_1), d(x_0, Sx_1), d(x_1, Sx_0)\} = 0$$

and we take $\theta = q\alpha(d(x_0, x_1))$, hence

$$d(x_1, x_2) \leq \theta d(x_0, x_1).$$

If $H(Sx_1, Sx_2) = 0$ then $Sx_1 = Sx_2$, i.e., $x_2 \in Sx_2$, then x_2 is fixed point of S. Let $H(Sx_1, Sx_2) \neq 0$. By Lemma 5, there exists $x_3 \in Sx_2$ such that

$$d(x_2, x_3) \leq qH(Sx_1, Sx_2).$$

Again by (6), we have

$$\begin{aligned} &d(x_2, x_3) \\ &\quad \leq q[\alpha(d(x_1, x_2))d(x_1, x_2) + \varphi(d(x_1, x_2)) \\ &\qquad \min\{d(x_1, Sx_1), d(x_2, Sx_2), d(x_1, Sx_2), d(x_2, Sx_1)\}] \\ &\quad \leq q[\alpha(d(x_1, x_2))d(x_1, x_2) + \varphi(d(x_1, x_2)) \\ &\qquad \min\{d(x_1, x_2), d(x_2, x_3), d(x_1, x_3), d(x_2, x_2)\}] \\ &\quad \leq q[\alpha(d(x_1, x_2))d(x_1, x_2)]. \end{aligned}$$

Because

$$\min\{d(x_1, Sx_1), d(x_2, Sx_2), d(x_1, Sx_2), d(x_2, Sx_1)\} = 0$$

and $\theta = q\alpha(d(x_1, x_2))$. So, we have

$$d(x_2, x_3) \leq \theta d(x_1, x_2) \leq \theta^2 d(x_0, x_1).$$

By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$ and

$$d(x_n, x_{n+1}) \leq \theta d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. By inductive, we obtain

$$d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1).$$

Hence,

$$d(x_{n+k}, x_{n+k+1}) \leq \theta^{k+1} d(x_{n-1}, x_n)$$

for all $n, k \in \mathbb{N}$. Now, for positive integers m and n with $m > n$, we have

$$\begin{aligned} &d(x_n, x_m) \\ &\quad \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\quad \leq \theta^n d(x_0, x_1) + \theta^{n+1} d(x_0, x_1) + \ldots + \theta^{m-1} d(x_0, x_1) \\ &\quad \leq (\theta^n + \theta^{n+1} + \ldots + \theta^{m-1}) d(x_0, x_1) \\ &\quad \leq \frac{\theta^n}{1 - \theta} d(x_0, x_1). \end{aligned}$$

Since, $\theta < 1$, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. Now, from the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Then

$$\begin{aligned} &d(x^*, Sx^*) \\ &\quad \leq d(x^*, x_{n+1}) + d(x_{n+1}, Sx^*) \\ &\quad \leq d(x^*, x_{n+1}) + H(Sx_n, Sx^*) \\ &\quad \leq d(x^*, x_{n+1}) + \alpha(d(x_n, x^*))d(x_n, x^*) + \varphi(d(x_n, x^*)) \\ &\qquad \min\{d(x_n, Sx_n), d(x^*, Sx^*), d(x_n, Sx^*), d(x^*, Sx_n)\} \\ &\quad \leq d(x^*, x_{n+1}) + \alpha(d(x_n, x^*))d(x_n, x^*) + \varphi(d(x_n, x^*)) \\ &\qquad \min\{d(x_n, x_{n+1}), d(x^*, Sx^*), d(x_n, Sx^*), d(x^*, x_{n+1})\}. \end{aligned}$$

Letting $n \to \infty$ implies that $d(x^*, Sx^*) = 0$. Since Sx^* is closed, this implies $x^* \in Sx^*$. This completes the proof.

Let $\delta \in (0, 1)$ and $L \geq 0$. If $\alpha(r) := \delta$ and $\varphi(r) := L$ for each $r \in [0, \infty)$ in Theorem 6 then it reduce to next Corollary which defined by Berinde and Pacurar.

Corollary 7. *[5] Let (X, d) be a complete metric space and $S : X \to \mathcal{CB}(X)$ be a generalized multivalued (δ, L)-almost contraction, i.e., a mapping satisfying for which there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$, such that*

$$\begin{aligned} H(Sx, Sy) \leq\ & \delta d(x, y) + L \min\{d(x, Sx), d(y, Sy), \\ & d(x, Sy), d(y, Sx)\} \end{aligned} \quad (7)$$

for all $x, y \in X$, Then S has a fixed point in X.

Proof. The proof could be easily adapted after that of Theorem 6 which replace $\alpha(d(x, y)) := \delta$ and $\varphi(d(x, y)) := L$

If $\min\{d(x, Sx, d(y, Sy), d(x, Sy), d(y, Sx)\} = d(y, Sx)$ in Corollary 7 then it reduce to Corollary 8 which defined by M. Berinde and V. Berinde [2]

Corollary 8. *[2] Let (X, d) be a complete metric space and $S : X \to \mathcal{CB}(X)$ be a multivalued almost contraction which there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that*

$$H(Sx, Sy) \leq \delta d(x, y) + Ld(y, Sx) \quad (8)$$

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

for all $x, y \in A$, Then S has a fixed point in X.

IV. The Ulam-Hyers Stability

We start this section by presenting the Ulam-Hyers stability concepts for the fixed point problem associated to a multivalued operator.

Definition 9. Let (X, d) be complete metric space and $S : X \to \mathcal{CB}(X)$ be an operator. By definition, the fixed point inclusion

$$x \in Sx \tag{9}$$

for all $x \in X$ and for each $\varepsilon > 0$ real number the following inequality

$$d(y, Sy) \leq \varepsilon, \tag{10}$$

is said to be generalized Ulam-Hyers stable if there exists an increasing operator $\psi : [0, \infty) \to [0, \infty)$, continuous at 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ real number and each solution $y^* \in X$ an solution of the inequality (10) there exists a solution $x^* \in X$ of the fixed point inclusion (9) such that

$$d(y^*, x^*) \leq \psi(\varepsilon). \tag{11}$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in [0, \infty)$, then the fixed point inclusion (9) is said to be Ulam-Hyers stable.

Now, we prove a generalized Ulam-Hyers stability for fixed point problems which Theorem 6 hold.

Theorem 10. *Let (X, d) be a complete metric space. Suppose that all the hypotheses of Theorem 6 hold and also that the function $\psi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) := t - t\alpha(t)$ is strictly increasing and onto. Then, the fixed point inclusion (9) is generalized Ulam-Hyers stable.*

Proof. By Theorem 6, we have $x^* \in Sx^*$, that is, $x^* \in X$ is a solution of the fixed point inclusion (9). Let $\varepsilon > 0$ and $y^* \in Sy^*$ is a solution of the inequality (10), that is

$$d(y^*, Sy^*) \leq \varepsilon.$$

Now, we obtain

$$\begin{aligned} d(x^*, y^*) &= d(Sx^*, y^*) \\ &\leq d(Sx^*, Sy^*) + d(Sy^*, y^*) \\ &\leq d(x^*, Sy^*) + d(Sy^*, y^*) \\ &\leq H(Sx^*, Sy^*) + d(Sy^*, y^*) \\ &\leq [\alpha(d(x^*, y^*))d(x^*, y^*) + \varphi(d(x^*, y^*)) \\ &\quad \min\{d(x^*, Sx^*), d(y^*, Sy^*), d(x^*, Sy^*), \\ &\quad d(y^*, Sx^*)\}] + \varepsilon \\ &\leq \alpha(d(x^*, y^*))d(x^*, y^*) + \varepsilon. \end{aligned}$$

It follows that

$$d(x^*, y^*) - \alpha(d(x^*, y^*))d(x^*, y^*) \leq \varepsilon.$$

Since $\psi(t) := t - t\alpha(t)$, we have

$$\psi(d(x^*, y^*)) := (d(x^*, y^*) - \alpha(d(x^*, y^*)))d(x^*, y^*).$$

It implies that

$$d(x^*, y^*) \leq \psi^{-1}(\varepsilon).$$

Notice that $\psi^{-1} : [0, \infty) \to [0, \infty)$ exists, is increasing, continuous at 0 and $\psi^{-1}(0) = 0$. Therefore, the fixed point inclusion (9) is generalized Ulam-Hyers stable. This completes the proof.

Corollary 11. *Let (X, d) be a complete metric space. Suppose that all the hypotheses of Corollary 7 hold. Then the fixed point inclusion (9) is Ulam-Hyers stable.*

Proof. By Corollary 7, we have $x^* \in Sx^*$, that is, $x^* \in X$ is a solution of the fixed point inclusion (9). Let $\varepsilon > 0$ and $y^* \in Sy^*$ is a solution of the inequality (10), that is

$$d(y^*, Sy^*) \leq \varepsilon.$$

Now, we obtain

$$\begin{aligned} d(x^*, y^*) &= d(Sx^*, y^*) \\ &\leq d(Sx^*, Sy^*) + d(Sy^*, y^*) \\ &\leq d(x^*, Sy^*) + d(Sy^*, y^*) \\ &\leq H(Sx^*, Sy^*) + d(Sy^*, y^*) \\ &\leq [\delta d(x^*, y^*) + L \min\{d(x^*, Sx^*), d(y^*, Sy^*), \\ &\quad d(x^*, Sy^*), d(y^*, Sx^*)\}] + \varepsilon \\ &\leq \delta d(x^*, y^*) + \varepsilon. \end{aligned}$$

It follows that

$$d(x^*, y^*) - \delta d(x^*, y^*) \leq \varepsilon.$$

Since $\delta \in (0, 1)$, we have

$$d(x^*, y^*) \leq \frac{1}{1 - \delta}\varepsilon.$$

Because $\frac{1}{1-\delta} > 0$. Therefore, the fixed point inclusion (9) is generalized Ulam-Hyers stable. This completes the proof.

Remark 12. If suppose that all the hypotheses of Corollary 8 holds. Also, the fixed point inclusion (9) is Ulam-Hyers stable.

Corollary 13. *Let (X, d) be a complete metric space. Suppose that all the hypotheses of Corollary 8 hold. Then, the fixed point inclusion (9) is generalized Ulam-Hyers stable.*

Proof. By Corollary 8, we have $x^* \in Sx^*$, that is, $x^* \in X$ is a solution of the fixed point inclusion (9). Let $\varepsilon > 0$ and $y^* \in Sy^*$ is a solution of the inequality (10), that is

$$d(y^*, Sy^*) \leq \varepsilon.$$

Now, we obtain

$$\begin{aligned} d(x^*, y^*) &= d(Sx^*, y^*) \\ &\leq d(Sx^*, Sy^*) + d(Sy^*, y^*) \\ &\leq d(x^*, Sy^*) + d(Sy^*, y^*) \\ &\leq H(Sx^*, Sy^*) + d(Sy^*, y^*) \\ &\leq [\delta d(x^*, y^*) + L d(y^*, Sx^*)] + \varepsilon \\ &\leq (\delta + L)d(x^*, y^*) + \varepsilon. \end{aligned}$$

It follows that

$$d(x^*, y^*) - (\delta + L)d(x^*, y^*) \leq \varepsilon.$$

Since $\delta \in (0, 1)$ and for some $L \geq 0$, we have

$$d(x^*, y^*) \leq \frac{1}{1 - \delta - L}\varepsilon.$$

Therefore, the fixed point inclusion (9) is generalized Ulam-Hyers stable. This completes the proof.

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Acknowledgments

The first author would like to thank the Faculty of Science and Department of Mathematics, KMUTT for financial support. Also, the authors were supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission.

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