Viscosity Approximation Methods for Split Variational Inclusion and Fixed Point Problems in Hilbert Spaces

Nimit Nimana and Narin Petrot

Abstract—**The main objective of this paper is to find a common solution of split variational inclusion problem and fixed point problem of infinite family of nonexpansive operators in a setting of real Hilbert spaces. To reach this goal, the iterative algorithms which combine Moudafi's viscosity approximation method with some fixed point technically proving methods are utilized for solving the problem. We prove that the iterative schemes with some suitable control conditions converge strongly to a common solution of the considered problem. We also show that many interesting problems can be solved by using our presented results.**

Index Terms—**Split variational inclusion problem, fixed point problem, nonexpansive operators, resolvent operators, strong convergence.**

I. Introduction

CENSOR and Elfving [11] initially introduced the celebrated split feasibility problem (SFP), which can be mathematically formulated as the problem of finding a point $x^* \in C$ such that $Ax^* \in Q$, where C and Q are nonempty closed convex subsets of R^n and R^m, respectively, and A is an $m \times n$ matrix. Also, they proposed an algorithm for solving such introduced problem. Nevertheless, the algorithm involves the complicated computations of matrix inverses. Hereupon, a new iterative algorithm for solving the (SEP) problem was presented by Byrne [6], namely CQ-algorithm, which is defined and considered by the following iterative step:

$$x_{n+1} = P_C(x_n + \gamma A^T(P_Q - I)Ax_n), \qquad \forall n \geq 0,$$

where an initial $x_0 \in R^n$, $\gamma \in (0, 2/\|A\|^2)$ and P_C and P_Q denote the metric projections onto C and Q, respectively. Thenceforward, the split feasibility problem has been considered by many authors in many aspect, for more information, readers may consult [5], [6], [7], [9], [11], [12], [20], [27], [28], [29] and reference therein. It is worth to mentioning that the split feasibility problem in finite-dimensional Hilbert spaces had already been used in practice as a model in the intensity-modulation radiation therapy (IMRT) treatment planning, see [11], [12], [13]. Moreover, this formalism is in itself at the core of the modeling of many inverse problems in various area of mathematics, physical, medical, technical, and information sciences, see [10] for more details.

Appropriately, in 2010, Xu [28] extended the split feasibility problem to the case of infinite-dimensional Hilbert spaces and proposed a modified CQ-algorithm: Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be bounded linear operator. For given $x_0 \in H_1$, and consider an iterative scheme via the procedure

$$x_{n+1} = P_C(x_n + \gamma A^*(P_Q - I)Ax_n), \qquad \forall n \geq 0,$$

where $\gamma \in (0, 2/\|A\|^2)$. He proved that his iterative sequence converges weakly to the solution of the split feasibility problem, provided that they exist.

Recently, Censor - Gibali and Reich [14] introduced a concept of Split Variational Inequality Problem (SVIP) which is formulated as follows: find a point $x^* \in H_1$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C, \tag{1}$$

and such that the point $y^* = Ax^* \in H_2$ solves

$$\langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q, \tag{2}$$

where C and Q are closed convex subset of Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given operators. In order to solve (SVIP) problem, they proposed the following algorithm: let λ be a positive real number and select an arbitrary starting point $x_0 \in H_1$. Given the current iterate x_n, compute

$$x_{n+1} = P_C^{f,\lambda}(x_n + \gamma A^*(P_Q^{g,\lambda} - I)Ax_n)), \qquad \forall n \geq 0, \tag{3}$$

where $\gamma \in (0, 1/\|A\|^2)$, and $P_C^{f,\lambda}$ and $P_Q^{g,\lambda}$ are abbreviated stand for $P_C(I - \lambda f)$ and $P_Q(I - \lambda g)$, respectively. Under some suitable conditions imposed upon on the operators f and g, they proved the weakly convergent result of the generated sequence $\{x_n\}$ to a solution point of split variational inequality problem.

To generalize the gorgeously written paper [14], Moudafi [21] introduced the following Split Monotone Variational Inclusion (SMVI):

$$\text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*), \tag{4}$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \tag{5}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given single-valued operators. Moudafi proposed an iterative method for solving (4) - (5), he showed that the sequence generated by the proposed algorithm weakly converges to a solution of split monotone variational inclusion problem. Note that if C and Q are nonempty closed convex subset

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of H_1 and H_2, (resp.), and set $B_1 = N_C$ and $B_2 = N_Q$; where N_C and N_Q are normal cone to C and Q, (resp.); then the split monotone variational inclusion problem (4) - (5) reduces to split variational inequality problem (1) - (2).

On the other hand, let us recall some iterative methods for solving the fixed point problems of nonexpansive mappings. We know that most of the methods can be acquired from Mann' s iterative procedure [18], namely, for given element x_0 in a nonempty closed convex subset of H, compute

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \qquad \forall n \geq 0, \qquad (6)$$

where T is a nonexpansive mapping from such nonempty closed convex subset of H into itself and $\{\alpha_n\}$ is a control sequence, which we must impose some control conditions to force the (weak) convergent result of the sequence $\{x_n\}$ to a fixed point of T. Meanwhile, to obtain the strong convergence results, it is necessary to apply some regularizing procedures. In 2000, Moudafi [19] proposed the viscosity approximation method which is done by considering the approximate well-posed problem and combining the nonexpansive mapping of T with a contraction of a given mapping f over the nonempty closed convex subset. He proposed an iterative scheme: given an arbitrary x_0 in a nonempty closed convex subset, compute iterative sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \qquad (7)$$

where $\{\alpha_n\} \subset (0,1)$ goes slowly to zero. Under this iterative procedure, the strong convergent result was successfully obtained.

Motivated by the methods of finding solutions of split variational inclusion problem and Moudafi' s viscosity approximation, Kazmi and Rizvi [22] presented an explicit viscosity approximation method for approximate a common solution of fixed point problem for a nonexpansive mapping and the following type of split variational inclusion problem and in real Hilbert space: given two set-valued maximal monotone operators $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$, a bounded linear operator $A : H_1 \to H_2$, and two single-valued operators $f : H_1 \to H_1$, $g : H_2 \to H_2$, the split variational inclusion problem can be stated that:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in B_1(x^*), \qquad (8)$$

and such that

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in B_2(y^*). \qquad (9)$$

Further, from now on, we will denote the solution set of the problem (8) - (9) by

$$\Gamma = \{x \text{ which solves } (8) : Ax \text{ solves } (9)\}.$$

In [22], Kazmi and Rizvi proposed iterative scheme and proved that such sequences converges strongly to a common solution of split variational inclusion problem and fixed point problem. Note that, in fact, the problem that considered by Kazmi and Rizvi [22] is nothing but the problems (4) - (5), when the operators f and g in are zero operators.

In the present paper, inspired by the above cited works, we suggest and analyze the iterative methods for approximating a common solution of split variational inclusion problem (8) - (9) and the fixed point problem of infinitely family of nonexpansive mappings by using the viscosity approximation method and some fixed point technically proving methods. Using our results, we can obtain some interesting applications.

II. Preliminaries

Throughout this paper unless otherwise stated, we let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H.

We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let T be a mapping of H into H. Then, T is said to be

(i) ***k-contraction*** if for all $x, y \in H$, there exists $k \in [0,1)$ such that
$$\|Tx - Ty\| \leq k\|x - y\|;$$

(ii) ***nonexpansive*** if for all $x, y \in H$,
$$\|Tx - Ty\| \leq \|x - y\|;$$

(iii) ***firmly nonexpansive*** if $2T - I$ is nonexpansive, or equivalently for all $x, y \in H$,
$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle,$$
where I is denoted for the identity operator on H.

We denoted by $\mathrm{Fix}(T)$ the set of all fixed point of a mapping $T : H \to H$, that is $\mathrm{Fix}(T) = \{x \in H : x = Tx\}$.

Next, let us consider a set-valued operator $B : H \to 2^H$, we define a graph of B by $\{(x,y) \in H \times H : y \in B(x)\}$, and denote it by $\mathrm{Graph}(B)$. Moreover, an inverse operator of B, denoted B^{-1}, is defined through its graph, i.e., for every $(x,y) \in H \times H, x \in B^{-1}y \Leftrightarrow y \in Bx$.

A set-valued operator $B : H \to 2^H$ is called ***monotone*** if for all $x, y \in H$, $u \in B(x)$ and $v \in B(y)$ satisfy

$$\langle x - y, u - v \rangle \geq 0.$$

Such monotone operator is said to be ***maximal monotone*** if there exists no any other monotone operator such that its graph properly contains the graph of B. Furthermore, consider a maximal monotone operator B, we note that for each element $x \in H$ and a positive real number λ, there is a unique element $z \in H$ such that $x \in (I + \lambda B)z$. The operator $J_\lambda^B := (I + \lambda B)^{-1}$ is called the ***resolvent*** of B with parameter λ, which we know that it is a single-valued and firmly nonexpansive mapping.

III. Convergence Results

To deal with an infinitely family of nonexpansive mappings, Aoyama *et al.* [1] gave the following condition and lemma: let C be a nonempty subset of real Hilbert space H and let $\{T_n\}$ be a family of mappings of C into itselt such that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$, and we denote the common fixed point set of infinitely family of mapping $\{T_n\}$ by Ω, that is $\Omega = \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_n)$. We say that $\{T_n\}$ satisfies the *AKTT-condition* if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty.$$

Lemma-AKTT [[1], Lemma 3.2] Let C be a nonempty closed subset of a real Hilbert space H, and let $\{T_n\}$ be a sequence of mappings from C into itself. Suppose that

$\{T_n\}$ satisfies *AKTT-condition*. Then, for each $x \in C$, $\{T_n x\}$ converges strongly to a point in C. Furthermore, let $T : C \to C$ be defined by

$$Tx := \lim_{n\to\infty} T_n x \qquad \forall x \in C.$$

Then, for each bounded subset B of C,

$$\lim_{n\to\infty} \sup\{\|Tz - T_n z\| : z \in B\} = 0.$$

It should be noted that, if the sequence $\{T_n\}$ satisfies the *AKTT-condition* and $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, then it is not necessary that $\mathrm{Fix}(T) = \Omega$, for a counterexample, see [25]. In the sequel, we shall say that $\{T_n, T\}$ satisfies *AKTT-condition* if $\{T_n\}$ satisfies *AKTT-condition* and $\mathrm{Fix}(T) = \Omega$.

Based on Lemma-AKTT and the concept of *AKTT-condition*, we prove a convergence theorem for an iterative method for approximating a common solution of the problems (8) - (9) and fixed point problems as follows.

Theorem 1 Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone operators, $\{T_n\}$ be a family of nonexpansive mappings of H_1 into itself such that $\{T_n, T\}$ satisfying AKTT-condition and $f : H_1 \to H_1$ be a k-contraction mapping. For a given $x_1 \in H_1$ be arbitrary and let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} u_n &= J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)T_n u_n, \quad \forall n \ge 1, (10) \end{aligned}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \textit{and} \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$\{\gamma_n\} \subset (0, \frac{1}{\|A\|^2})$, and A^* is the adjoint operator of A. If $\Omega \cap \Gamma \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to $z \in \Omega \cap \Gamma$, where $z = P_{\Omega\cap\Gamma} f(z)$.

Next, in another approaching, to avoid the summable assumption of $\sup\{\|T_{n+1}z - T_n z\| : z \in B\}$ over a natural number n, by *AKTT-condition*, let us consider the following Bruck [4]' s lemma.

Lemma-Bruck [[4], Lemma 3] Let C be a nonempty closed convex subset of a strictly convex Banach space E and $\{T_n\}$ be a sequence of nonexpansive mappings from C into E. Then there exists a nonexpansive mapping $L : C \to E$ such that $\mathrm{Fix}(L) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

Motivated by Lemma-Bruck, He and Guo [17] showed the following useful fact.

Lemma-HG [[17], Lemma 2.7] Let E be a Banach space, $\{T_k\}$ a sequence of nonexpansive mappings on E with $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \neq \emptyset$, and $\{\omega_k\}$ a sequence of positive real numbers with $\sum_{k=1}^{\infty} \omega_k = 1$. Let $L = \sum_{k=1}^{\infty} \omega_k T_k, L_n = \sum_{k=1}^{n} \frac{\omega_k}{S_n} T_k$, and $S_n = \sum_{k=1}^{n} \omega_k$. Then L_n uniformly converges to L on each bounded subset S of E.

It should be note by Bruck' s lemma and He-Guo' s lemma that each L_n is also a nonexpansive mapping and $\mathrm{Fix}(L) = \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$.

By using the above fact we obtain the following result.

Theorem 2 Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone operators, $f : H_1 \to H_1$ be a k-contraction mapping, $\{T_k\}$ be a family of nonexpansive mappings of H_1 into itself and $\{\omega_k\}$ be a sequence of positive real numbers with $\sum_{k=1}^{\infty} \omega_k = 1$. Let $L = \sum_{k=1}^{\infty} \omega_k T_k, L_n = \sum_{k=1}^{n} \frac{\omega_k}{S_n} T_k$, and $S_n = \sum_{k=1}^{n} \omega_k$. For a given $x_1 \in H_1$ be arbitrary and let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} u_n &= J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)L_n u_n, \quad \forall n \ge 1, (11) \end{aligned}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \textit{and} \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$\{\gamma_n\} \subset (0, \frac{1}{\|A\|^2})$, and A^* is the adjoint operator of A. If $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \cap \Gamma \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to $z \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \cap \Gamma$, where $z = P_{\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)\cap\Gamma} f(z)$.

IV. Some Applications

Here, we present two interesting problems which can be solved by using our presented results.

A. Split Minimization Problem

Let us consider the split minimization problem (SNP), namely

find $x^* \in H_1$ such that $x^* = \mathrm{argmin}_{x\in H_1} \phi(x)$,

and such that

find $y^* = Ax^* \in H_2$ such that $y^* = \mathrm{argmin}_{y\in H_2} \varphi(y)$,

where $\phi : H_1 \to R$ and $\varphi : H_2 \to R$ be convex lower semicontinuous functions. Recall that the subdifferentials of a function $h : H \to R$ at $\bar{x}$ is the set-valued operator on H defined by

$$\partial h(\bar{x}) := \{z \in H : h(\hat{x}) \ge h(\bar{x}) + \langle z, \hat{x} - \bar{x}\rangle \text{ for all } \hat{x} \in H\}.$$

Since $\partial\phi$ and $\partial\varphi$ are maximal monotone operators and we know that

$$J_\lambda^{\partial\phi} = \mathrm{prox}_{\lambda\phi} \quad \text{and} \quad J_\lambda^{\partial\varphi} = \mathrm{prox}_{\lambda\varphi},$$

where proximal operators $\mathrm{prox}_{\lambda\phi}$ and $\mathrm{prox}_{\lambda\varphi}$ of ϕ and φ with parameter $\lambda > 0$ defined by

$$\mathrm{prox}_{\lambda\phi}(x) = \mathrm{argmin}_{u\in H_1}\{\phi(u) + \frac{1}{2\lambda}\|x - u\|\},$$

for each $x \in H_1$

$$\mathrm{prox}_{\lambda\varphi}(y) = \mathrm{argmin}_{v\in H_2}\{\varphi(v) + \frac{1}{2\lambda}\|y - v\|\},$$

for each $y \in H_2$.

By taking $B_1 = \partial\phi$ and $B_2 = \partial\varphi$, the iterative scheme (11) becomes

$$\begin{aligned} u_n &= \mathrm{prox}_{\lambda\phi}(x_n + \gamma_n A^*(\mathrm{prox}_{\lambda\varphi} - I)Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)L_n u_n, \quad \forall n \ge 1, \end{aligned}$$

and we can assert a strong convergence of this proposed iteration, which solves a common solution of split minimization problem (SMP) and fixed point problem for a family of nonexpansive mappings.

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B. Equilibrium problem

For a nonempty closed convex subset C of a real Hilbert space H, let us assume that $f : C \times C \to R$ is a bifunction satisfying the following condition:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is a convex and lower semicontinuous function.

The equilibrium problem is a problem of finding $z \in C$ such that

$$f(z, x) \geq 0,$$

for all $x \in C$. We denote its solution set by $EP(f)$. In 1994, Blum and Oettli [3] asserted that, if the bifunction f satisfying (A1)-(A4) and let $r > 0$ and $x \in H$, then there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,$$

for all $y \in C$. Furthermore, if we set

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\},$$

we have from Combettes and Hirstoaga [15] that

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive;

(iii) $F(T_r) = EP(f)$

(iv) $EP(f)$ is a closed convex subset of C.

Recently, Takahashi *et al.* [24] defined a multivalued mapping A_f of H into itself by

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \geq \frac{1}{r}\langle y - x, z\rangle, \forall y \in C\}; & x \in C, \\ \emptyset & ; \ x \notin C. \end{cases}$$

They proved that $EP(f) = A_f^{-1}(0)$, A_f is a maximal monotone operator with $\mathrm{dom}(A_f) \subset C$ and

$$T_r(x) = (I + rA_f)^{-1}(x).$$

If we put $B_1 = A_f$ and $B_2 = A_g$, where f, g are bifunctions satisfy (A1)-(A4) that, then ones can see that the problem of finding $x^* \in C$ such that $f(x^*, y) \geq 0$ for all $y \in C$ is equivalent to the problem of finding $x^* \in C$ such that $0 \in A_f(x^*)$. Hence, we can obtain a strong convergence result for a common solution of split equilibrium problem (SEP), namely

find $x^* \in C$ such that $f(x^*, x) \geq 0$ for all $x \in C$,

and such that

find $y^* = Ax^* \in Q$ such that $g(y^*, y) \geq 0$ for all $y \in Q$,

and fixed point problem for a family of nonexpansive mappings.

V. Conclusion

In this work, motivated by Moudafi's viscosity approximation method, two iterative algorithms are constructed for finding a common solution of split variational inclusion problem and fixed point problem of infinite family of nonexpansive operators in a setting of real Hilbert spaces. The first one (10), requires some additional conditions on the considered operators, such as the *AKTT-condition*. Meanwhile, the second one (11), we do not need to assume any additional conditions on the considered operators. However, in the practical point of view, ones may point out that the first algorithm (10) is more effective. By the way, for further works, considering the methods for finding a solution of the more general classes of operators and problems are required.

Appendix A
Proof of Theorem 1

Proof: We proceed along several steps.

Step 1. We show that $\{x_n\}$ is a bounded sequence.

Let $\bar{x} \in \Omega \cap \Gamma$. We know that $\bar{x} = J_\lambda^{B_1}(\bar{x})$, $A\bar{x} = J_\lambda^{B_2}(A\bar{x})$ and $\bar{x} = T_n(\bar{x})$ for each n. Also, we note that

$$\|x_{n+1} - \bar{x}\| \leq \alpha_n k\|x_n - \bar{x}\| + \alpha_n\|f(\bar{x}) - \bar{x}\| + (1 - \alpha_n)\|u_n - \bar{x}\|. \quad (12)$$

On the other hand, we nconsider

$$\begin{aligned} 2\gamma_n\langle A^*(J_\lambda^{B_2} - I)Ax_n, x_n - \bar{x}\rangle & \\ &= 2\gamma_n\langle (J_\lambda^{B_2} - I)Ax_n, Ax_n - A\bar{x}\rangle \\ &= 2\gamma_n[\langle (J_\lambda^{B_2} - I)Ax_n, Ax_n - A\bar{x} + (J_\lambda^{B_2} - I)Ax_n\rangle \\ &\quad -\|(J_\lambda^{B_2} - I)Ax_n\|^2] \\ &= 2\gamma_n[\langle J_\lambda^{B_2}Ax_n - Ax_n, Ax_n - A\bar{x}\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2] \\ &\leq 2\gamma_n\left[\frac{1}{2}\|J_\lambda^{B_2}Ax_n - Ax_n\|^2 - \|(J_\lambda^{B_2} - I)Ax_n\|^2\right] \\ &= -\gamma_n\|(J_\lambda^{B_2} - I)Ax_n\|^2, \end{aligned}$$

since $J_\lambda^{B_2}$ is a nonexpansive mapping. Using this one, we obtain

$$\begin{aligned} \|u_n - \bar{x}\|^2 &= \|J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n) - \bar{x}\|^2 \\ &\leq \|x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n - \bar{x}\|^2 \\ &= \|x_n - \bar{x}\|^2 + \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\quad +2\gamma_n\langle A^*(J_\lambda^{B_2} - I)Ax_n, x_n - \bar{x}\rangle \\ &= \|x_n - \bar{x}\|^2 + \gamma_n^2\langle AA^*(J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &\quad +2\gamma_n\langle A^*(J_\lambda^{B_2} - I)Ax_n, x_n - \bar{x}\rangle \\ &\leq \|x_n - \bar{x}\|^2 + \gamma_n(\gamma_n\|A\|^2 - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2, \quad (13) \end{aligned}$$

and in view of $\{\gamma_n\} \in (0, \frac{1}{\|A\|^2})$, we get

$$\|u_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2. \quad (14)$$

Now, by using (14), the inequality (12) becomes

$$\begin{aligned} \|x_{n+1} - \bar{x}\| &\leq (1 - \alpha_n(1 - k))\|x_n - \bar{x}\| + \alpha_n\|f(\bar{x}) - \bar{x}\| \\ &\leq \max\left\{\|x_n - \bar{x}\|, \frac{\|f(\bar{x}) - \bar{x}\|}{1 - k}\right\}, \quad (15) \end{aligned}$$

for each $n \geq 1$. Accordingly, by using an inductive argument, we can conclude that

$$\|x_{n+1} - \bar{x}\| \leq \max\left\{\|x_1 - \bar{x}\|, \frac{\|f(\bar{x}) - \bar{x}\|}{1 - k}\right\}, \quad \forall n \geq 1. \quad (16)$$

This means that $\{x_n\}$ is a bounded sequence, as required. Subsequently, we have $\{f(x_n)\}$ is a bounded sequence and also $\{T_k u_n\}$ are bounded sequences, for each k.

Step 2. We show that the sequence $\{x_n\}$ is asymptotically regular, i.e., $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$.

Let M be a constant such that

$$M = \max\left\{\sup_{n\geq 1}\|f(x_n)\|, \sup_{k,n\geq 1}\|T_k u_n\|\right\}.$$

Since $J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is a nonexpansive mapping, we have

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\|.$$

Now, by the definition of $\{x_n\}$, we notice that

$$\|x_{n+1}-x_n\| \leq (1-\alpha_n(1-k))\|x_n-x_{n-1}\|+2M|\alpha_n-\alpha_{n-1}|+L_n, \tag{17}$$

where $L_n = \sup\left\{\|T_n z - T_{n-1}z\| : z \in \{u_n\}\right\}$ Subsequently, we get

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \tag{18}$$

as required.

Step 3. We show that $\lim_{n\to\infty}\|Tu_n - u_n\| = 0$.

Note that

$$\begin{aligned}\|T_n u_n - x_n\| &\leq \|T_n u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \alpha_n\|T_n u_n - f(x_n)\| + \|x_{n+1} - x_n\|.\end{aligned}$$

Taking n approaches to infinity, we have

$$\lim_{n\to\infty}\|T_n u_n - x_n\| = 0. \tag{19}$$

Next, we claim that $\|x_n - u_n\| \to 0$ as $n \to \infty$. From (13), we note that

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)T_n u_n - \bar{x}\|^2 \\ &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + (1-\alpha_n)\|T_n u_n - \bar{x}\|^2 \\ &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + \|u_n - \bar{x}\|^2 \\ &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 \\ &\quad +\gamma_n(\gamma_n\|A\|^2 - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2,\end{aligned}$$

this implies

$$\begin{aligned}&\gamma_n(1-\gamma_n\|A\|^2)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &\qquad \leq \alpha_n\|f(x_n)-\bar{x}\|^2+\|x_{n+1}-x_n\|(\|x_n-\bar{x}\|+\|x_{n+1}-\bar{x}\|).\end{aligned} \tag{20}$$

Since $\gamma_n(1-\gamma_n\|A\|^2) > 0$, $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we can conclude that

$$\lim_{n\to\infty}\|(J_\lambda^{B_2} - I)Ax_n\| = 0. \tag{21}$$

Now, the firmly nonexpansiveness of $J_\lambda^{B_1}$ implies that

$$\begin{aligned}2\|u_n - \bar{x}\|^2 &= 2\|J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n) - \bar{x}\|^2 \\ &\leq 2\langle u_n - \bar{x}, x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n - \bar{x}\rangle \\ &= \|u_n - \bar{x}\|^2 + \|x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n - \bar{x}\|^2 \\ &\quad -\|u_n - x_n - \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \\ &= \|u_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\quad +2\langle u_n - \bar{x}, \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &\leq \|u_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\quad +2\frac{1}{\|A\|^2}\|u_n - \bar{x}\|\|A^*(J_\lambda^{B_2} - I)Ax_n\|.\end{aligned}$$

This gives

$$\begin{aligned}\|u_n - \bar{x}\|^2 &\leq \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\quad +2\frac{1}{\|A\|^2}\|u_n - \bar{x}\|\|A^*(J_\lambda^{B_2} - I)Ax_n\|.\end{aligned}$$

Subsequently,

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + (1-\alpha_n)\|u_n - \bar{x}\|^2 \\ &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + (1-\alpha_n)[\|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\quad +2\frac{1}{\|A\|^2}\|u_n - \bar{x}\|\|A^*(J_\lambda^{B_2} - I)Ax_n\|] \\ &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\quad +2\frac{1}{\|A\|^2}\|u_n - \bar{x}\|\|A^*(J_\lambda^{B_2} - I)Ax_n\|,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\|u_n - x_n\|^2 &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|x_{n+1} - \bar{x}\|^2 \\ &\quad +2\frac{1}{\|A\|^2}\|u_n - \bar{x}\|\|A^*(J_\lambda^{B_2} - I)Ax_n\|, \\ &\leq \alpha_n\|f(x_n) - \bar{x}\|^2 \\ &\quad +\|x_{n+1} - x_n\|(\|x_n - \bar{x}\| + \|x_{n+1} - \bar{x}\|) \\ &\quad +2\frac{1}{\|A\|^2}\|u_n - \bar{x}\|\|A^*(J_\lambda^{B_2} - I)Ax_n\|.\end{aligned}$$

Using this one, from (18), (21) and $\alpha_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{22}$$

This proves the claim. Next, we obtain

$$\lim_{n\to\infty}\|T_n u_n-u_n\| \leq \lim_{n\to\infty}\|T_n u_n-x_n\|+\lim_{n\to\infty}\|x_n-u_n\| = 0. \tag{23}$$

Note that

$$\begin{aligned}\|Tu_n - u_n\| &\leq \|Tu_n - T_n u_n\| + \|T_n u_n - u_n\| \\ &\leq \sup\{\|T_n z - Tz\| : z \in \{x_n\}\} \\ &\quad +\|T_n u_n - u_n\|.\end{aligned} \tag{24}$$

Using this one, in view of (23), we have

$$\lim_{n\to\infty}\|Tu_n - u_n\| = 0, \tag{25}$$

as required.

Step 4. We show that $x_n \to z \in \Omega \cap \Gamma$, where $z = P_{\Omega\cap\Gamma}f(z)$.

Note that, by (22) and the boundedness of $\{x_n\}$, by passing to a subsequence (if necessary), we can assume without loss of generality that $u_n \rightharpoonup u$ for some $u \in H_1$. Since T is a nonexpansive mapping, then by Step 3 and in view of the demiclosedness of T, we know that $u = Tu$. This means $u \in \Omega$. Further, since $u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n)$, we know that

$$\frac{x_n - u_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n}{\lambda} \in B_1 u_n,$$

by passing n to infinity, and using (21) and (22), we can show that $0 \in B_1 u$. This means that u solves (8). Further, by (22), we have Ax_n weakly converges to Au. Thus, by using (21) and applying the demiclosed principle to a nonexpansive mapping $J_\lambda^{B_2}$, we obtain that $0 \in B_2(Au)$. Therefore, $u \in \Omega \cap \Gamma$.

Since $z = P_{\Omega\cap\Gamma}f(z)$, we have

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \langle f(z) - z, u - z\rangle \leq 0. \tag{26}$$

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Finally, we show that $x_n \to z$ as $n \to \infty$. We observe that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \langle \alpha_n f(x_n) + (1-\alpha_n)T_n u_n - z, x_{n+1}-z\rangle \\
&= \alpha_n\langle f(x_n)-z, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n)\langle T_n u_n - z, x_{n+1}-z\rangle \\
&= \alpha_n\langle f(x_n)-f(z), x_{n+1}-z\rangle \\
&\quad +\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +(1-\alpha_n)\langle T_n u_n - z, x_{n+1}-z\rangle \\
&\le \frac{\alpha_n}{2}\left[\|f(x_n)-f(z)\|^2 + \|x_{n+1}-z\|^2\right] \\
&\quad +\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +\frac{1-\alpha_n}{2}\left[\|T_n u_n - z\|^2 + \|x_{n+1}-z\|^2\right] \\
&\le \frac{\alpha_n k^2}{2}\|x_n - z\|^2 + \frac{\alpha_n}{2}\|x_{n+1}-z\|^2 \\
&\quad +\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\
&\quad +\frac{1-\alpha_n}{2}\left[\|u_n - z\|^2 + \|x_{n+1}-z\|^2\right] \\
&\le \frac{\alpha_n k^2}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1}-z\|^2 \\
&\quad +\alpha_n\langle f(z)-z, x_{n+1}-z\rangle + \frac{1-\alpha_n}{2}\|x_n - z\|^2
\end{aligned}$$

and so

$$\|x_{n+1}-z\|^2 \le (1-\alpha_n(1-k^2))\|x_n-z\|^2 + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle, \tag{27}$$

Since $\sum_{n=1}^{\infty}\alpha_n(1-k^2)=\infty$, we have

$$\lim_{n\to\infty}\|x_n - z\| = 0. \tag{28}$$

This completes the proof. ■

Appendix B
Proof of Theorem 2

Proof: By using the similar arguments and techniques as those showed in proving Theorem 1, we can show that $\{x_n\}$ is bounded sequence and

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le (1-\alpha_n(1-k))\|x_n - x_{n-1}\| \\
&\quad +|\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad +\|L_n u_{n-1} - L_{n-1}u_{n-1}\| \\
&\quad +|\alpha_n - \alpha_{n-1}|\|L_{n-1}u_{n-1}\|, \qquad (29)
\end{aligned}$$

for each $n \ge 1$. Further, we can compute that

$$\sum_{n=1}^{\infty}\|L_n u_{n-1} - L_{n-1}u_{n-1}\| \le 2\sum_{n=1}^{\infty}\frac{\omega_n}{S_n}M \le 2\sum_{n=1}^{\infty}\frac{\omega_n}{\omega_1}M, \tag{30}$$

where $M := \sup\{\|T_m u_n\| : m,n \ge 1\}$. Since $\{\omega_n\}$ is summable, we obtain that

$$\sum_{n=1}^{\infty}\|L_n u_{n-1} - L_{n-1}u_{n-1}\| < \infty.$$

By using this one, in view of the control conditions on $\{\alpha_n\}$, we see that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

By following step 3. of the proof in Theorem 1 we see that

$$\lim_{n\to\infty}\|L_n u_n - u_n\| = 0.$$

Note that

$$\|Lu_n - u_n\| \le \|Lu_n - L_n u_n\| + \|L_n u_n - u_n\|$$

By Lemma III, we have

$$\lim_{n\to\infty}\|Lu_n - u_n\| = 0. \tag{31}$$

From this point, by replacing T by L and following the lines proof of Theorem 1, we can obtain the desired result. ■

References

[1] Aoyama, K., Kimura, Y., Takahashi, W., Toyoda, M.: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350–2360 (2007)
[2] Bauschke, H. H., Combettes, P. L.: Convex analysis and monotone operator theory in Hilbert spaces. Springer, New York (2011)
[3] Blum, E., Oettli, W.: From optimization and variational inequalities. Math. Stud. **63**, 123-146 (1994)
[4] Bruck, R. E.: Properties of fixed-point sets of nonexpansive mappings in Banach soaces. Trans. Amer. Math. Soc. **179**, 251-262 (1973)
[5] Byrne, C.:Iterative projection onto convex sets using multiple Bregman distances. Inverse Probl. **15**, 1295-1313 (1999)
[6] Byrne, C.: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441-453 (2002)
[7] Byrne, C.: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103-120 (2004)
[8] Byrne, C., Censor, Y., Gibali, A., Reich, S.: The split common null point problem. J. Nonlinear Convex Anal. **13**, 759-775(2012)
[9] Cegielski, A.: Generalized relaxations of nonexpansive operators and convex feasibility problems. Contemp. Math. **513**, 111-123 (2010)
[10] Cegielski, A.: Iterative methods for fixed point problems in Hilbert spaces. Springer, New York (2010)
[11] Censor, Y., Elfving, T.: A multiprojection algorithm using Bregman projections in product space. Numer. Algor. **8**, 221–239 (1994)
[12] Censor, Y., Elfving, T., Kopf, N., Bortfeld, T.: The multiple-sets split feasibility problem and its applications for inverse problems. Inverse Probl. **21**, 2071-2084 (2005)
[13] Censor, Y., Bortfeld, T., Martin, B., Trofimov, A.: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. **51**, 2353-2365 (2006)
[14] Censor, Y., Gibali, A., Reich, S.: Algorithms for the split variational inequality problem. umer Algor. **59**, 301-323 (2012)
[15] Combettes, P. L., Hirstoaga, S. A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
[16] Crombez, G.:A geometrical look at iterative methods for operators with fixed points. Numer. Funct. Anal. Optim. **26**, 157-175 (2005)
[17] He, S., Guo, J.: Iterative algorithm for common fixed points of infinite family of nonexpansive mappings in Banach spaces. J. Appl. Math. (2012). DOI 10.1155/2012/787419
[18] Mann, W. R.: Mean value methods in iteration. Proc. Amer. Math. Soci. **4**, 506-510 (1953)
[19] Moudafi, A. : Viscosity approximation methods for fixed points problems. J. Math. Anal. Appl. **241**, 46-55 (2000)
[20] Moudafi, A.: The split common fixed-point problem for demicontractive mappings. Inverse Probl. **26**, 1-6 (2010)
[21] Moudafi, A.: Split monotone variational inclusions. J. Optim. Theory Appl. **150**, 275-283 (2011)
[22] Kazmi, K. R., Rizvi, S. H.: An iterative method for split variational inclusion problem and fixed point problem for nonexpansive mapping. Optimm. Lett. (2012). DOI 10.1007/s11590-013-0629-2.
[23] Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Amer. Math. Soc. **73**, 591-597 (1976)
[24] Takahashi, S., Takahashi, W., Toyoda, M.: Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. J. Optim. Theory Appl. **147**, 27-41 (2010)
[25] S. Wang, Hybrid iterative algorithms for nonexpansive and nonspreading mappings in Hilbert spaces, 314 (2013), DOI 10.1186/1687-1812-2013-314.
[26] Xu, H. K. : Viscosity approximation methods for nonexpansive mapping. J. Math. Anal. Appl. **298**, 279-291 (2004)
[27] Xu, H. K.: A variable KrasnoselskiiMann algorithm and the multiple-set split feasibility problem. Inverse Probl. **22**, 2021-2034 (2006)
[28] Xu, H. K.: Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. Comput. Math. Appl. **26**, (2010) 105018 (17pp)
[29] Zhao, J., Yang, Q.: Several solution methods for the split feasibility problem. Inverse Probl. **21**, 1791–1800 (2005)

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