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Restricted Poly-Time SAT-Solvable CNF's

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Abstract—**This article, from an algorithmic point of view, studies the satisfiability problem SAT restricted to a CNF fraction MHF whose members consist of a Horn formula and a 2-CNF part. In general SAT is NP-complete for such formulas. We consider various structurally defined subclasses of MHF, for which SAT can be solved in polynomial time. The instances defined here are required to have a linear, or even exact-linear Horn part. Moreover, the 2-CNF part is required to be of specific graph structure, e.g., disjoint edges or disjoint triangles. Further poly-time classes are presented, for which the Horn part is not forced to be linear.**

Index Terms—**propositional satisfiability, Mixed Horn formula, linear formula, polynomial-time algorithm**

I. Introduction

The propositional satisfiability problem (SAT) of conjunctive normal form (CNF) formulas is an essential combinatorial problem, namely one of the first problems that have been proven to be NP-complete [4]. More precisely, it is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover SAT plays a fundamental role in the theory of designing exact algorithms, and it has a wide range of applications because many problems can be encoded as a SAT problem via reduction [9], [7] due to the rich expressiveness of the CNF language. The applicational area is pushed by the fact that meanwhile several powerful solvers for SAT have been developed (cf. e.g. [11], [19] and references therein). Also from a theoretical point of view one is interested in classes for which SAT can be solved in polynomial time. There are known several subclasses of CNF retricted to which SAT behaves polynomial-time solvable, so for instance 2-CNF-SAT, where clauses have length at most two [1], and Horn-SAT [12], confer also [17]. Furthermore SAT has been investigated for several variants of Horn formulas [2], [3]. Combining a Horn and a 2-CNF formula yields a mixed Horn formula (MHF) according to [15]. For the unrestricted class MHF and also for several of its subclasses SAT is NP-complete [15], [14], [13]. In this context it turns out that reducing numerous combinatorial NP-complete problems to SAT, formulas in MHF are generated quite immediately. This holds true for many prominent NP-complete problems [5] like Feedback Vertex Set, Vertex Cover, Dominating Set, Hitting Set etc. [14], [18], [20]. Therefore it is worthwhile to design good exact algorithms for solving unrestricted MHF formulas. On the other hand, one is interested to isolate subclasses of MHF which are polynomial-time solvable w.r.t. SAT. In this paper we define several subclasses of MHF for which SAT is efficiently decidable, or is trivial in the sense that its members all are either satisfiable or unsatisfiable. The Horn part is required to consist of negative clauses only. Furthermore clauses are allowed to overlap only sparsely. More precisely, we require linear, exact-linear or even pairwise disjoint Horn clauses. Also the positive monotone 2-CNF part of MHF's is restricted in such a way that its formula graph consists of disjoint edges, respectively, disjoint triangles, only. However, in case that there are no further restrictions on the linear or even exact-linear Horn part, then SAT remains NP-complete as we shall prove also. Further poly-time classes are presented, for which the Horn part is not forced to be linear, but is closely related to the number of triangles in the 2-CNF part of the instances.

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II. Notation and Preliminaries

Let CNF denote the set of formulas (free of duplicate clauses) in conjunctive normal form over propositional variables $x \in \{0, 1\}$. A *positive (negative)* literal is a (negated) variable. A clause c is a disjunction of different literals, and is represented as a set $c = \{l_1, \dots, l_{|c|}\}$. Each formula $C \in$ CNF is considered as a set of its clauses $C = \{c_1, \dots, c_{|C|}\}$ having in mind that it is a conjunction of these clauses. The *negation (or complement)* of a literal l is $\bar{l}$. Throughout we assume that clauses are free of pairs of complemented literals, like $x, \overline{x}$. For formula C, clause c, by $V(C), V(c)$ we denote the variables contained (neglecting negations), correspondingly. Given $x \in V(c)$, by $l(x)$ we denote the literal over x that is contained in c. Let $\|C\|$ denote the number of literals in C, i.e., its length, whereas $|C|$ is the size of formula C. A positive (negative) clause consists of positive (negative) clauses only. Furthermore, CNF_+ denotes the set of *monotone* formulas, i.e., every clause is positive. As usual k-CNF denotes the subclass of CNF, where each member clause has length at most k, for fixed integer $k \geq 2$. Moreover a formula C is called *k-uniform* if $|c| = k$, for all clauses $c \in C$. A clause of size k is sometimes called a *k-clause*.

The satisfiability problem (SAT) asks, whether input $C \in$ CNF has a *model*, which is a truth value assignment $t : V(C) \to \{0, 1\}$ assigning at least one literal in each clause of C to 1. Let UNSAT denote the set of all unsatisfiable members of CNF, and let SAT denote the set of all satisfiable members of CNF. By convention, we have $\emptyset \in$ SAT, i.e., the empty clause set is satisfiable.

Recall that clauses of a Horn formula are allowed to contain at most one positive literal. Because all 2-clauses which are not positive monotone are Horn, every formula $M \in$ MHF has the unique clause-set representation $M = H \cup P$, where P is the collection of all positive monotone 2-clauses in M and H is the remaining Horn subformula. Throughout the paper, we restrict all Horn parts H of a mixed Horn formula to contain exclusively negative clauses, unless stated otherwise. Whenever k-uniformity of the Horn part is required, we indicate that by adding a $(= k)$-prefix to the notation.

A CNF formula C is called *linear* if for all $c_1, c_2 \in C$: $c_1 \neq c_2$ we have $|V(c_1) \cap V(c_2)| \leq 1$. C is called *exact-*

linear if for all $c_1, c_2 \in C : c_1 \neq c_2$ we have $|V(c_1) \cap V(c_2)| = 1$. So, linear MHF's are denoted as LMHF whereas exact-linear ones are collected in XLMHF, similarly we write LCNF for the set of unrestricted linear formulas.

For a monotone formula P we can construct its *formula graph* G_P with vertex set $V(P)$ in linear time. Two vertices are joined by an edge iff there is a clause in P containing the corresponding variables. Recall that a *vertex cover* U of a graph is a subset of its vertex set such that each edge containa at least one vertex of U. As usual, a vertex cover is called *minimal* if it contains no proper vertex cover, i.e., if it is minimal w.r.t. inclusion [6]. We shall use further notation for restricting the structure of G_P, for the component P in $M = H \cup P$. If G_P consists of disjoint edges only, we shall add an upper index d to the formula class notation, and an upper index $\triangle$, if G_P consists of disjoint triangles only. So $(= k)$-LMHF^d, for example, denotes the class of all MHF's with k-uniform linear Horn part and whose P component has pairwise disjoint clauses.

III. Mixed Horn Formulas With Linear Horn Part

At first we consider the class $(= k)$-LMHF$^d \subset$ MHF which turns out to be NP-complete for SAT. Here, $P \in$ CNF$_+$ also is a specific linear formula in which each variable occurs exactly once. Recall that P always is 2-uniform by construction.

Theorem 1: SAT is NP-complete restricted to $(= k)$-LMHF^d.

PROOF. SAT restricted to the set $(= k)$-LCNF of k-uniform linear formulas is NP-complete according to [16]. Providing a polynomial-time reduction from this class to $(= k)$-LMHF^d enables us to establish the assertion. Therefore, let $C \in (= k)$-LCNF be chosen arbitrarily. Every variable $x \in V(C)$ occurring positively in C is replaced by $\overline{y_x}$ where y_x is a new variable. Moreover the clause $\{x, y_x\}$ is added. Let $M = H \cup P \in$ MHF be the resulting formula, where H consists of k-uniform, negative monotone, linear clauses and P consists of pairwise disjoint positive 2-clauses. Now suppose C is satisfiable with model t. Then t' defined as follows is a model for M: $t'(x) := t(x)$, $t'(y_x) := 1 - t(x)$. Indeed, P is obviously satisfied because either x or y_x is set to 1 for every clause in P. Moreover, H is satisfied also, since each of its clauses c' is obtained from a clause c of C which is satisfied let say through setting literal $l(x) := 1$, $x \in V(C)$. If $l(x) = \overline{x}$, i.e., $t(x) = 0 = t'(x)$, then c' is satisfied likewise. Else, c' contains $\overline{y_x}$ and c' is satisfied because $t'(y_x) = 1 - t(x) = 0$.

Conversely, assume that M is satisfiable by t'. Then its restriction $t := t'|_{V(C)}$ to the variables in C is a model for C. Indeed, let $c' \in M$ be a k-clause with counterpart $c \in C$ then it is either satisfied by an original variable that occurs negatively in both c, c'. Or c' is satisfied by a new variable, i.e., $t'(y_x) = 0$. But then we have $t'(x) = 1$, for the original variable $x \in V(C)$ because M contains the 2-clause $\{x, y_x\}$. Since by construction c contains the literal x it is satisfied also.□

NP-completeness still holds if G_P consists of disjoint triangles only, as is proven in [18]. Next we require the Horn part to be exact-linear, but have no restrictions on P yielding also a fragment of MHF for which SAT is NP-complete. As we shall see later, allowing only disjoint edges for G_P, makes the class polynomial-time SAT-solvable.

Theorem 2: SAT is NP-complete restricted to XLMHF.

PROOF. Using the preceding theorem enables us to provide a reduction to the asserted class XLMHF w.r.t. SAT. Let $M = H \cup P \in (= k)$-LMHF^d be chosen arbitrarily with variable set $V(M) = \{x_1, \ldots, x_n\}$. While there exist two clauses c_i, c_j in the Horn part of the current formula that have empty intersection, create a variable y_{ij} not occurring in the current formula and enlarge both c_i and c_j by $\overline{y_{ij}}$. This procedure yields a MHF with exact-linear Horn part H'. Let V be the set of all new variables. To guarantee SAT-equivalence we add to P the 2-clauses: $\{x_1, y_{ij}\}, \{x_2, y_{ij}\}, \ldots, \{x_n, y_{ij}\}$, for every $y_{ij} \in V$. Moreover we add a 2-clause for each pair of new variables. Let $M' := H' \cup P' \in$ XLMHF be the resulting formula. Assume M is satisfiable. Then also M' is satisfiable by setting all variables in V to 1, and the remaining variables according to a model of M. If $M' \in$ SAT with model t', then the clauses in H' that are satisfied by an original variable, i.e., $t'(x_i) = 0$ have satisfied counterparts in H. If there is a clause c for which all original variables are set to 1, then it contains a new variable set to 0. By construction of P' this means that all other variables in M' are set to 1. A contradiction occurs because c then is the only clause that can be contained in $H' \in$ SAT and therefore in H. But then there would not have been generated a new variable for it, because H is assumed to be exact-linear. So, we have only clauses of the first kind in H' and thus $H \in$ SAT. □

Restricting P in the preceding situation such that G_P consists of disjoint edges leads to a tractable class in the sense that SAT becomes even trivial.

Theorem 3: Formulas of the class XLMHF^d are always satisfiable.

PROOF. Let $M = H \cup P \in$ XLMHF^d be arbitrarily chosen. If for each clause $c \in H$ there exists a positive monotone 2-clause $p \in P$ such that $V(p) \subset V(c)$, then we set all variables of an arbitrary minimal vertex cover of G_P to 1 and all remaining variables to 0 thus obtaining a model for M in this situation. Indeed, then in each clause $p \in P$ exactly one variable is set to 0 and the other is set to 1 and as each clause of H contains all variables of at least one clause of P, both P and H are satisfied. Now suppose H contains a clause c for which there does not exist a clause of P whose variables are contained in c, then we set all variables of c to 0 and the remaining variables of M to 1. As H is exact-linear and consists of negative clauses only, each remaining clause of H shares exactly one variable with c and thus is satisfied also. Moreover P is satisfied because c contains no two variables of any clause in P. Therefore in every clause of P at most one variable is now set to 0 and at least one variable is set to 1. □

The last result can be transfered to the case where G_P consists of disjoint triangles only as is shown in [18].

Next we consider linear MHF's $M = H \cup P$ whose Horn part H even consists of disjoint clauses only and G_P consists of disjoint triangles only. Further we require $V(P) = V(H)$ which is no loss of generality, because variables either occurring only in P or only in H can be set independently for satisfying the corresponding clauses immediately. For instructive reasons we provide an argumentation in three steps:

(1) *Theorem 4:* Formulas as above with Horn clauses of length at least three are always satisfiable.
PROOF. Let $M = H \cup P$ with the properties as stated above be chosen arbitrarily, and let I_M be the bipartite incidence graph associated to M as follows: the vertex set partition $V_1 \cup V_2$ is composed of the clauses of H yielding V_1 and the triangles of G_P yielding V_2. A vertex in V_1 and a vertex in V_2 are joined by an edge whenever the corresponding clause in H and the triangle in G_P have a variable in common. Let n be the number of variables in M then $|V_2| = n/3$ and $|V_1| \leq n/3$. Further, because the clauses of H are pairwise disjoint we obtain the following: Every i vertices of V_1 have at least i neighbours in V_2, for all $i \in \{1, \ldots, n/3\}$, because every i clauses of H contain at least $3 \cdot i$ different variables and every i triangles in P contain exactly $3 \cdot i$ different variables. Now one can apply the Theorem of König-Hall for bipartite graphs [8], [10] stating that there exists a matching in I_M covering the component V_1 of the vertex set. In terms of the formula this means that to each clause c of H can be assigned exactly one triangle which is denoted as $\triangle_c$. Setting any of the variables to 0 that have c and $\triangle_c$ in common, and the remaining two variables of $\triangle_c$ to 1 provides a model for M. □

(2) *Theorem 5:* Formulas as above with Horn clauses of length at most two are unsatisfiable.
PROOF. Let $M = H \cup P$ be such a formula arbitrarily chosen, and let r be the number of disjoint triangles in G_P. Let n be the number of variables in M. Then $3r = n \leq 2|H|$ hence $r \leq \frac{2}{3}|H|$. To satisfy H in each clause of H set at least one variable to 0, that means one has to assign 0 to $|H|$ variables. Since there are more clauses in H than triangles in P, by the pigeonhole principle it follows that in order to satisfy H one has to set two variables to 0 in at least one triangle. However, this violates the satisfiability of subformula P, hence M is unsatisfiable. □

(3) *Theorem 6:* SAT can be solved in polynomial time for formulas as above if its H component contains at least one clause of length at most two and at least one clause of length at least three.
PROOF. The following algorithm provides a proof of the assertion. Let $\ell(x)$ denote a function, which assigns to all three variables of a triangle of G_P the same label. Variables of different triangles get different labels.
INPUT: M as above.
OUTPUT: SATISFIABLE, if $M \in$ SAT, else UNSATISFIABLE.
BEGIN

a) To every variable $x \in V(M)$ assign a label $\ell(x)$, such that variables of the same triangle get the same label and variables of different triangles get different labels.
b) Sort the clauses of H in ascending order with respect to their sizes and variable-labels: $H = \{c_1, c_2, \ldots, c_{|H|}\}$, where $|c_1| \leq \ldots \leq |c_{|H|}|$, s.t. clauses of the same size and with the same labels are neighbours. Let $c_i = \{\overline{x_{i_1}}, \ldots, \overline{x_{i_k}}\}$ and let r be the number of triangles in G_P.
c) Initialize $Z := \emptyset$.
d) If $r < |H|$ then output UNSATISFIABLE.
e) **For** $i := 1$ **to** $|H|$ **do** (*$c_i = \{\overline{x_{i_1}}, \ldots, \overline{x_{i_k}}\}$*)
   i) **For** $j := i_1$ **to** i_k **do** $Z := Z \cup \{\ell(x_j)\}$;
   ii) **If** $|Z| < i$ **then** UNSATISFIABLE; **else** $i := i + 1$;
f) M is SATISFIABLE

END

$O(||M||^2)$ time is consumed to label all variables of M, and sorting the clauses in H can be performed in time $O(\log |H| \cdot |H|)$. Moreover, the running time of both for-loops can be upper bounded by $O(|H| \cdot |c_H|)$, where c_H denotes the longest clause of H and $|c_H| \leq |V(M)|$. So $O(||M||^2)$ is an upper bound for the running time of the algorithm above.
To prove the correctness of this algorithm, it is clear by the pigeonhole principle that, if M has more clauses in H than triangles in G_P, we cannot satisfy M. Next if the first i clauses, for each $i \in \{1, \ldots, |H|\}$, contain less than i different labels then we cannot satisfy M, because otherwise at least two variables of the same triangle in G_P have to be set to 0. Else we can satisfy M using the König-Hall Theorem.□

Remark 1: Note that the algorithm also works when G_P consists of disjoint edges only, or if G_P is composed of both disjoint edges and disjoint triangles.
In summary, we have

Corollary 1: SAT can be solved in polynomial time for the class of MHF's, whose H component consists of pairwise disjoint clauses and for whose positive 2-CNF part P it holds that the corresponding graph G_P consists of disjoint triangles only.

For the next result the usual requirement that the Horn clauses have to be negative is dropped but still the Horn clauses are required to be pairwise disjoint.

Theorem 7: SAT can be solved in polynomial time for LMHF$^{\triangle}$, where the Horn component consists of pairwise disjoint clauses.

PROOF. Let $M = H \cup P$ be an arbitrary formula with properties as stated above. If H contains negative clauses only then we can apply to M the algorithm above. Else we set each variable x occurring positive in H to 1 and evaluate the formula. So, a formula $M' = H' \cup P'$ is obtained where H' consists of pairwise disjoint negative clauses and $G_{P'}$ consists of disjoint triangles and edges. Note that by setting the variables occurring positive in H to 1 we do not make any restrictions concerning the satisfiability of M. Then we apply the algorithm above to M' solving SAT in polynomial time according to Remark 1. □

Returning to MHF's with strict negative Horn component, we consider the subclass of LMHF^d such that the Horn clauses are pairwise disjoint again. Further assume $V(P) = V(H)$.

Theorem 8: SAT can be solved in polynomial time for the formulas as above.

PROOF. Let $M = H \cup P$ be an arbitrary such formula. By unit-propagation all 1-clauses can be eliminated either

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yielding (1) a partial model, or (2) a contradiction with output $M \in$ UNSAT. In case (1), the remaining formula M' still has the properties as above. Moreover every variable x occurs exactly once in P and exactly once in H, because of $V(P) = V(H)$. Thus, we have that every $x \in V(M')$ occurs exactly twice in M', and moreover $|c| \geq 2$, for all $c \in M'$. Due to a result in [21] we can conclude that M' is satisfiable in this situation. □

IV. Further Subclasses Of MHF

In this section we are able to provide several polynomial-time subclasses of MHF w.r.t. SAT, for which the negative Horn part is not required to be linear at all. However, then we have to relate the number of triangles in G_P to the size of the Horn part, a first result of this kind is:

Theorem 9: Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$ with $V(P) = V(H)$ and let r be the number of triangles in G_P. If each clause in H has length exactly k, $k \geq 1$, and H contains more than $\binom{3r}{k} - \binom{2r}{k}$ clauses, then M is unsatisfiable.

Proof. Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$, where H is k-uniform and let r be the number of triangles in G_P. Since there are r triangles in G_P, M has $3r$ variables. The Horn part H has only clauses of length k, thus H can contain at most $\binom{3r}{k}$ different k-clauses.

In each triangle of G_P at most one variable can be set to 0 and the other (at least two) variables of this triangle are forced to 1. Further to satisfy M in each clause of H at least one variable must be set to 0. Setting in each of the r triangles at most one variable to 0 yields at least $2r$ variables which have to be set to 1. Thus for every truth assignment which sets at most one variable of each triangle to 0 and at least two variables to 1 at least $\binom{2r}{k}$ unsatisfiable k-clauses are obtained among all $\binom{3r}{k}$ possible k-clauses. It follows that every k-uniform formula $M = H \cup P \in \mathrm{MHF}^{\triangle}$ with $|H| \geq \binom{3r}{k} - \binom{2r}{k}$ is unsatisfiable. □

Lemma 1: Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$, r the number of triangles in G_P and let $|c| \geq 2r + 1$, for all $c \in H$. Then M is satisfiable.

Proof. Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$ with $|c| \geq 2r + 1$, for all $c \in H$. Then for each $c \in H$ there exists at least one triangle $\triangle_c$ in G_P such that all variables of $\triangle_c$ are contained in c as can be easily shown. Setting exactly one variable of each triangle to 0 and the remaining two variables to 1 yields a model for M. □

Lemma 2: Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$ and let r be the number of triangles in G_P. Further let $|c| = 2r$ for all $c \in H$. If $|H| < 3^r$ then M is satisfiable.

Proof. For arbitrary $M = H \cup P \in \mathrm{MHF}^{\triangle}$, let r be the number of triangles in G_P. Since $|c| = 2r$ for every clause c in H there exists at least one triangle $\triangle_c$ in G_P such that at least two variables of $\triangle_c$ occur in c. W.l.o.g. we do not consider clauses c for which there exists a triangle in G_P whose variables all occur in c, because these clauses are always satisfiable as explained in the proof of Lemma 1. Thus we assume that each clause of H has exactly two variables in common with each triangle of G_P. No further case can occur because of the condition $|c| = 2r$, for all $c \in H$. There are exactly 3^r different clauses with this property. Suppose H contains all these 3^r clauses, then H is unsatisfiable. This is due to the fact that then for each truth assignment which sets in each triangle exactly one variable to 0 and the remaining two variables to 1 exactly one clause in H remains unsatisfied, namely that one consisting of all variables set to 1. So, if $|H| < 3^r$ then there exists a model for H. □

Theorem 10: Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$ with $V(P) = V(H)$ and let $r \geq 3$ be the number of triangles in G_P. Further let $|c| = 2r - 1$ for all clauses $c \in H$. If $|H| \leq 2r$ then M is satisfiable.

Proof. We treat the worst-case $|H| = 2r$. W.l.o.g. we do not consider clauses $c \in H$, for which there exists at least one triangle $\triangle$ in G_P whose variables all occur in c, because these clauses are always satisfiable by setting an arbitrary variable of $\triangle$ to 0.

Since $|c| = 2r - 1$ for all $c \in H$ it follows that each clause of H contains at least one variable from every triangle. There are $3r$ different variables in M. The $2r$ clauses of H contain $2r(2r - 1)$ literals where $2r(2r - 1) \geq 10r$, for $r \geq 3$. That means, that some of the $3r$ different variables occur more than once in H. We have

$$\frac{2r(2r-1)}{3r} = \frac{2}{3}(2r-1)$$

motivating the following case distinction:

- In case $2r \equiv 1 \pmod 3$ each of the $3r$ different variables occurs either exactly $\frac{2}{3}(2r-1)$ times in H or there is a variable x occurring more than $\frac{2}{3}(2r-1)$ times in H. It is sufficient to have a variable x occurring at least $\frac{2}{3}(2r-1)$ times in H, because setting x to 0, and the remaining two variables in the triangle containing x to 1, satisfies already $\frac{2}{3}(2r-1)$ clauses of H. Thus $2r - \frac{2}{3}(2r-1) = \frac{2}{3}r + \frac{2}{3}$ clauses remain. We have $(\frac{2}{3}r + \frac{2}{3}) - (r - 1) = -\frac{r}{3} + \frac{5}{3}$, and for $r \geq 5$ it holds that $-\frac{r}{3} + \frac{5}{3} \leq 0$ (observe that for $r = 3, 4$ we have $2r \not\equiv 1 \pmod 3$) which means that there are less than $r - 1$ remaining clauses. Therefore we can satisfy H using appropriate variables from the remaining $r - 1$ triangles. That works because each clause of H contains a variable from every of the $r - 1$ triangles in G_P. Choosing a unique triangle for every clause and setting the common variable to 0 and the remaining variables to 1 satisfies all clauses left.
- In case $2r \not\equiv 1 \pmod 3$ there exists a variable occurring at least $\lceil \frac{2}{3}(2r-1) \rceil$ times in M enabling us to proceed as argumented above.

□

Finally, we have two assertions stating that SAT resp. UNSAT are trivial for specific formulas $M = H \cup P$ where the length of clauses in H, or the size of H itself is related to the number of triangles in G_P. Both can easily be proven using the preceding results.

Corollary 2: Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$ s.t. G_P contains $r \geq 1$ triangles. If $|c| \geq 2r - 1$, for all $c \in H$, and $|H| \leq 2r$ then M is satisfiable.

Proof. The assertion follows immediately from Lemma 2 and Theorem 10. □

Lemma 3: Let $M = H \cup P \in \mathrm{MHF}^{\triangle}$ s.t. G_P contains $r \geq 1$ triangles. If $|c| = 2$ for all $c \in H$, $3r \equiv 0 \pmod 2$ and $|H| = \frac{3r}{2}$ then $M \in$ UNSAT.

Proof. For $3r \equiv 0 \pmod 2$ assume that $|H| = \frac{3r}{2}$. In consequence H consists of disjoint negative 2-clauses and thus is unsatisfiable according to Theorem 5. □

V. Concluding Remarks

In the present paper we studied various subclasses of linear mixed Horn formulas with restricted 2-CNF part and negative Horn part. While for several of them SAT is shown to be NP-complete, we could detect numerous classes for which SAT can be decided in polynomial time. Moreover we provided polynomial-time algorithms for other subclasses of mixed Horn formulas with a Horn part that is not required to be linear or exact-linear. For the discussion of further polynomial-time subclasses of MHF we refer to [18].

References

[1] Aspvall, B., Plass, M.R., Tarjan, R.E., A linear-time algorithm for testing the truth of certain quantified Boolean formulas, Inform. Process. Lett. 8 (1979) 121-123.

[2] Boros, E., Crama, Y., Hammer, P.L., Polynomial time inference of all valid implications for Horn and related formulae, Annals of Math. Artif. Intellig. 1 (1990) 21-32.

[3] Boros, E., Hammer, P.L., Sun, X., Recognition of q-Horn formulae in linear time, Discrete Appl. Math. 55 (1994) 1-13.

[4] Cook, S.A., The Complexity of Theorem Proving Procedures. In: Proceedings of the 3rd ACM Symposium on Theory of Computing, pp. 151-158, 1971.

[5] Garey M.R., Johnson, D.S., Computers and Intractability: A Guide to the Theory of NP-Completeness, W. H. Freeman and Company, San Francisco, 1979.

[6] Golumbic, M.C., Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.

[7] Gu, J., Purdom, P.W., Franco, J., Wah, B.W., Algorithms for the Satisfiability (SAT) Problem: A Survey, in: D. Du, J. Gu, P. M. Pardalos (Eds.), Satisfiability Problem: Theory and Applications, DIMACS Workshop, March 11-13, 1996, DIMACS Series, vol. 35, pp. 19-151, American Mathematical Society, Providence, Rhode Island, 1997.

[8] Hall, P., On representatives of subsets, J. London Math. Soc. 10 (1935) 26-30.

[9] Karp, R.M., Reducibility Among Combinatorial Problems. In Complexity of Computer Computations, Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y.. New York: Plenum, pp. 85-103. 1972.

[10] König, D., Graphen und Matrizen, Math. Fiz. Lapok 38 (1931) 116-119.

[11] Le Berre, D., Simon, L., The Essentials of the SAT 2003 Competition, in: Proceedings of the 6th International Conference on Theory and Applications of Satisfiability Testing (SAT'03), Springer-Verlag LNCS 2919 (2004) 172-187.

[12] Minoux, M., LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation, Inform. Process. Lett. 29 (1988) 1-12.

[13] Namasivayam, N., Truszczynski, M., Simple but Hard Mixed Horn Formulas. In: Proc. SAT 10, LNCS vol. 6175, pp. 382-387, 2010.

[14] Porschen, S., Schmidt, T., Speckenmeyer, E.: Some Aspects of Mixed Horn Formulas. In: Proc. SAT 09, LNCS vol. 5584, pp. 86-100, 2009.

[15] Porschen, S., Speckenmeyer, E., Satisfiability of Mixed Horn Formulas, Discrete Appl. Math. 155 (2007) 1408-1419.

[16] Porschen, S., Speckenmeyer, E., Zhao, X., Linear CNF formulas and satisfiability, Discrete Appl. Math. 157 (2009) 1046-1068.

[17] Schaefer, T.J., The complexity of satisfiability problems, in: Proc. STOC 1978. ACM (1978) 216-226.

[18] Schmidt, T., Computational complexity of SAT, XSAT and NAE-SAT for linear and mixed Horn CNF formulas, dissertation, Univ. Köln, 2010.

[19] Speckenmeyer, E., Min Li, C., Manquinho V., Tacchella, A., (Eds.), Special Issue on the 2007 Competitions, Journal on Satisfiability, Boolean Modeling and Computation, 4 (2008).

[20] Stamm-Wilbrandt, H., Programming in propositional Logic or Reductions: Back to the Roots (Satisfiability), Technical Report, Universität Bonn, 1991.

[21] Tovey, C.A., A Simplified NP-Complete Satisfiability Problem, Discrete Appl. Math. 8 (1984) 85-89.

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