Proceedings of the International MultiConference of Engineers and Computer Scientists 2014 Vol I,
IMECS 2014, March 12 - 14, 2014, Hong Kong

Multiprogram Digital Control

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Abstract—**In this paper, we analyze potentials and describe methods of synthesis of multiprogram controls in different classes of dynamic systems. The main idea is to construct the control that provides predesigned finite set of asymptotically stable program motions for closed-loop system. Special attention is paid to modification of algorithms to extend possibilities of its application. For this aim, special class of multiprogram controls is involved. So-called multiprogram digital control we use for realization some program motions in a difference system, and for further discrete stabilization of these motions. Practically, the system closed with a multiprogram control is a nonlinear program automate that can realize any program motion of predesigned class depending on initial values. Developed theoretical base allows constructing of multiprogram control systems in cases of both complete and incomplete feedback when information in feedback channels is discrete.**

Index Terms—**Discrete dynamic systems, difference system, multiprogram digital control, stabilization.**

I. Introduction to MPS Problem

In this part we give a brief overview of the synthesis of the multiprogram stable (MPS) controls. For the first time the method of stabilization control synthesis for a set of program motions in linear systems was proposed by Zubov [1]. According to that work let us consider the linear control system

$$\dot{x} = Ax + Bu + f(t), \tag{1}$$

where $x \in R^n$ is a phase state vector, $u \in R^r$ is a control (input) vector, A, B are real constant matrices of appropriate dimensions, and $f(t)$ is a real continuous vector-function for $t \in [0,+\infty)$.

Assume that for system (1) the bounded program controls $u_1(t),\dots,u_N(t)$ and the corresponding program motions $x_1(t),\dots,x_N(t)$ are obtained. Here N is an arbitrary natural number.

MPS problem (*Multiprogram Stabilization problem*) is to construct the control

$$u(x,t) = u(x,u_1(t),\dots,u_N(t),x_1(t),\dots,x_N(t)) \tag{2}$$

under which system (1) has the given program and asymptotically stable motions $x_1(t),\dots,x_N(t)$.

The following theorem, formulated in [1], solves the MPS problem for linear system (1).

Theorem 1. Let us assume that the following conditions are fulfilled:

1) the system $\dot{x} = Ax + Bu$ at control of the form $u = Cx$ is stabilized, i.e. this system has an arbitrary stability store at the control $u = Cx$;

2) the program motions $x_1(t),\dots,x_N(t)$ of system (1) are distinguishable at $t \ge t_0 \ge 0$, i.e. $\inf_{t\ge 0}\| x_i - x_j \| > 0$, $i \ne j$.

Then there exists the control (2) under which system (1) has the given set of asymptotically stable program motions $x_1(t),\dots,x_N(t)$.

The detailed proof of the Theorem 1 has been done in [2]. Control (2) has been constructed in the form of the Hermit's interpolational polynomial

$$\begin{aligned} u(x,t) = \sum_{j=1}^{N} \Bigg(& u_j(t) + C(x - x_j(t)) - \\ & - 2u_j(t) \sum_{i=1, i\ne j}^{N} \frac{(x_j(t)-x_i(t))(x-x_j(t))}{(x_j(t)-x_i(t))^2} \Bigg) p_j(x) \end{aligned} \tag{3}$$

where

$$p_j(x) = \prod_{i=1, i\ne j}^{N} \frac{(x - x_i(t))^2}{(x_j(t)-x_i(t))^2}, \quad j = \overline{1,N}. \tag{4}$$

In formulas (3), (4), and further, the expressions $(x_j(t)-x_i(t))(x-x_j(t))$ and $(x_j(t)-x_i(t))^2$ are the scalar products of the corresponding vectors.

The control (3), (4) is a kind of nonlinear feedback and closed-loop control system (1), (3), (4) is a nonlinear multiprogram automaton. This automaton can realize every program motion from family $x_1(t),\dots,x_N(t)$ depending on their initial value.

As an example, in [3] the research results are illustrated on real technical application. The multiprogram control problem for providing of several modes of flywheel work in a car engine is considered. In [4–6], the second (direct) Lyapunov's method is used to propose approaches to the

Manuscript received December 23, 2013; revised January 08, 2014.

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design of bounded feedback controls ensuring the asymptotic stability of the zero equilibrium state including the global stability. In the paper [7] the problem of multiprogram stabilization of the given set of equilibrium states in the class of nonlinear time invariant systems is considered. In articles [8–10] the problem of multiprogram control is solved in the case of incomplete feedback.

II. MPDS Problem Statement

Let us consider a system of difference equations in general form

$$x(k+1) = F(k, x(k), u(k)), \tag{5}$$

where $x(k)$ – the n-dimensional phase state vector with the components $x_1(k),\dots,x_n(k)$; $u(k)$ – r- dimensional vector of controls with the components $u_1(k),\dots,u_r(k)$; an integer argument k accepts values $k = 0,1,\dots$; n-dimensional vector-function $F(k,x,u)$ is defined at $k = 0,1,\dots$, $x \in G$, $u \in U$, where G and U – some areas of Euclidean spaces E^n and E^r respectively.

Assume that for any fixed k the function $F(k,x,u)$ is continuous in x, and u.

Definition 1. Admissible control is a sequence of vectors

$$u_d(k) = \{u(0), u(1), \dots, u(m-1)\}, \quad 0 \le k \le m-1.$$

The integer parameter m specifies the length of solution of control problem.

Definition 2. The function $x(k)$, defined for $k' \le k < k''$, where $k' \ge 0$, $k'' \le \infty$, is called the solution of the system (5), corresponding to the control $u_d(k)$, if it turns this system into identity when $u(k) = u_d(k)$.

The Cauchy problem for a difference system (5) is to find using the specified values $k_0 \ge 0$, $x_0 \in G$, $u_d(k) \in U$ the solution $x(k)$, satisfying the initial condition $x(k_0) = x_0$. Obviously, if the initial data k_0, x_0 and the admissible control $u_d(k)$ are known, we can clearly define the sequence $x(k_0+1)$, $x(k_0+2)$, … by the formula (5) until the resulting solution is not leave the area G. In particular, when $G = E^n$, the solution of the Cauchy problem will be defined for all $k \ge k_0$.

Definition 3. Admissible control $u_p(k)$ is called program, if the system (5), closed with this control has a solution satisfying the boundary conditions $x(0) = x_0$, $x(m) = x_1$.

Here x_0, x_1 are two preassigned constant vectors in state space. The pair x_0, x_1 is called controllable at the interval $[0,m]$.

Definition 4. System (5) is called fully controllable at the interval $[0,m]$, if any pair of states x_0, x_1 is controllable at $[0,m]$.

Definition 5. The solution of the system (5), corresponding to the program control $u_p(k)$, is called the program motion of these system. We will denote this function $x_p(k) = x_p(k, k_0, m, x_0, x_1)$.

MPDS problem (*Multiprogram Digital Stabilization problem*). Assume that the difference system (5) describes the movement of some digital controlled object. And for this system at the interval $k \in [0,m]$ the bounded program controls

$$u_j(k) = u_{pj}(k), \quad j = \overline{1,N}, \tag{6}$$

and the corresponding bounded program motions

$$x_j(k) = x_{pj}(k), \quad j = \overline{1,N}, \tag{7}$$

are constructed.

It is required to construct the control

$$u = u(x, u_1(k), \dots, u_N(k), x_1(k), \dots, x_N(k)) \tag{8}$$

that realize prescribed program motions (7) using program controls (6). Moreover, the asymptotic stability of program motions (7) with the control (8) is required [3, 7].

Note that the number N of program motions (7) is not depending on the dimension of the system (5) and the dimension of the space of controls.

III. MPDS Problem for linear difference system

Let us consider a linear difference system with constant coefficients

$$x(k+1) = Ax(k) + Bu(k) + F(k), \tag{9}$$

where $x(k)$ is the n-dimensional phase state vector; $u(k)$ is r- dimensional vector of controls; A, B are constant real matrices of appropriate dimensions; $F(k)$ is defined n-dimensional vector-function; integer argument k takes values $k = 0,1,\dots$

Notice, that by analogy with the linear continuous systems, system (9) is fully controllable at $[0,m]$ if and only if $rang\,(B, AB, \dots A^{m-1}B) = n$.

The solution of MPDS problem for system (9) gives the Theorem:

Theorem 2. Let us assume that the following conditions are fulfilled:

1) the linear homogeneous system $x(k+1) = Ax(k) + Bu(k)$ is fully controllable;

2) the program motions (7) are distinguishable at $k \ge 0$, in other words, $\inf_{k \ge 0} \| x_i(k) - x_j(k) \| > 0$, $i \ne j$.

Then there exists the control (8), under which system (9) has the given set of asymptotically stable program motions (7).

Proof. Let us consider the control (8) in the form

$$u(x,k) = \sum_{j=1}^{N} \Bigg(u_j(k) + C(x(k) - x_j(k)) - \\ - 2u_j(k) \sum_{i=1, i\ne j}^{N} \frac{(x_j(k) - x_i(k))(x(k) - x_j(k))}{(x_j(k) - x_i(k))^2} \Bigg) p_j(x) \tag{10}$$

ISBN: 978-988-19252-5-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where

$$p_j(x) = \prod_{i=1, i\neq j}^{N} \frac{(x(k) - x_i(k))^2}{(x_j(k) - x_i(k))^2}, \quad j = \overline{1, N}. \tag{11}$$

Here and further, the expressions

$$(x_j(k) - x_i(k))(x(k) - x_j(k)), \quad (x_j(k) - x_i(k))^2$$

are the scalar products of the corresponding vectors.

In subsequent formulae and where this does not prevent understanding of the essence of transformations, we will omit the argument k of vector functions x_j, x_i, …

For scalar functions (11) and for control (10) at $k = 0,1,\ldots$ the following obvious properties are fulfilled:

$$\begin{gathered} p_j(x_i(k)) = 0, \ i \neq j; \quad p_j(x_j(k)) = 1; \\ u(x_j(k), k) = u_j(k) \end{gathered} \tag{12}$$

In virtue of (12), it is clear that the system (9) with the control (10), (11) has prescribed program motions $x_1(k), \ldots, x_N(k)$, i.e., the system (9) will follow one of them in case of precision installation the appropriate initial data.

Thus, the control (10), (11) is a nonlinear feedback, and system (9), closed by this control, is a non-linear multiprogram automate, which is capable to implement any program motion from the family (7) depending on the choice of the initial data.

Let us proof the asymptotic stability of these motions. Consider an arbitrary $x_s(k)$ and construct for this motion the system in deviations. Index s in this case denotes some program motion from a given set of (7), i.e. the motion, that it is required to realize and stabilize in the specific situation.

However, the control system must be universal in relation to the given set and must depend only on the initial values of the selected motion.

For deviation $y_s(k) = x(k) - x_s(k)$ we obtain the system

$$\begin{aligned} y_s(k+1) &= Ay_s(k) + \\ &+ B\sum_{j=1}^{N}\Bigg(u_j + C(y_s(k) + x_s - x_j) - \\ &- 2u_j \sum_{i=1, i\neq j}^{N} \frac{(x_j - x_i)(y_s(k) + x_s - x_j)}{(x_j - x_i)^2}\Bigg)\times \\ &\times p_j(y_s(k) + x_s) - Bu_s. \end{aligned} \tag{13}$$

Let allocate in (13) linear approximation. For this purpose let us take into account properties of functions $p_j(y_s + x_s)$, $j = \overline{1, N}$:

1) for $j \neq s$ in the structure of product

$$p_j(y_s + x_s) = \prod_{i=1, i\neq j}^{N} \frac{(y_s + x_s - x_i)^2}{(x_j - x_i)^2}, \quad j = \overline{1, N},$$

at $i = s$ we have a factor

$$\frac{y_s^{\,2}}{(x_j - x_s)^2}.$$

This means that all terms of a sum in the right-hand side of (13) at $j \neq s$ are of the order not less than the second on the components vector y_s and, consequently, not included in the system of linear approximation;

2) for $j = s$

$$p_s(y_s + x_s) = \prod_{i=1, i\neq s}^{N} \frac{(y_s + x_s - x_i)^2}{(x_s - x_i)^2}. \tag{14}$$

Obviously that every factor in (14) has the form

$$\frac{(y_s + x_s - x_i)^2}{(x_s - x_i)^2} = \frac{y_s^2}{(x_s - x_i)^2} + 2\frac{(x_s - x_i)y_s}{(x_s - x_i)^2} + 1,$$
$$i = \overline{1, N}, \quad i \neq s.$$

Then (14) takes the form

$$p_s(y_s + x_s) = 1 + 2\sum_{i=1, i\neq s}^{N} \frac{(x_s - x_i)y_s}{(x_s - x_i)^2} + h_s(y_s),$$

where $h_s(y_s)$ is scalar function – the sum of terms, the order of which for the components of the vector y_s is not less than two.

Return to the right side of the system (13). Taking into consideration the properties 1), 2) of functions $p_j(y_s + x_s)$, $j = \overline{1, N}$, it is clear that additional linear terms on y_s may appear only when $j = s$. Consider separately the term, corresponding to $j = s$:

$$\begin{aligned} &B\left(u_s + Cy_s - 2u_s \sum_{i=1, i\neq s}^{N} \frac{(x_s - x_i)y_s}{(x_s - x_i)^2} \right)\times \\ &\quad\times\left(1 + 2\sum_{i=1, i\neq s}^{N} \frac{(x_s - x_i)y_s}{(x_s - x_i)^2} + h_s(y_s)\right) = \\ &\quad = Bu_s + BCy_s + \widetilde{H}_s(y_s), \end{aligned}$$

where

$$\begin{aligned} \widetilde{H}_s(y_s) = &B\left(Cy_s - 2u_s \sum_{i=1, i\neq s}^{N} \frac{(x_s - x_i)y_s}{(x_s - x_i)^2} \right)\times \\ &\times\left(2\sum_{i=1, i\neq s}^{N} \frac{(x_s - x_i)y_s}{(x_s - x_i)^2} + h_s(y_s)\right) + \\ &+ Bu_s h_s(y_s). \end{aligned}$$

With regard to this form the system (13) can be written as

$$y_s(k+1) = (A + BC)y_s(k) + H_s(y_s(k)), \tag{15}$$

where

$$\begin{aligned} H_s(y_s) = \widetilde{H}_s(y_s) + B &\sum_{j=1, j\neq s}^{N}\Bigg(u_j + C(y_s + x_s - x_j) - \\ &- 2u_j \sum_{i=1, i\neq j}^{N} \frac{(x_j - x_i)(y_s + x_s - x_j)}{(x_j - x_i)^2}\Bigg)\times \\ &\times p_j(y_s + x_s). \end{aligned}$$

ISBN: 978-988-19252-5-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

The first condition of the theorem means that the eigenvalues of the matrix of linear approximation $A + BC$ can be made any preassigned, including module less than unity [11]. In this case, the zero solution of each system (15) at $s = \overline{1, N}$ is asymptotically stable in the linear approximation [11]. Therefore, all program motions of system (9) with the control (10), (11) will be asymptotically stable.

The theorem is proved.

IV. Conclusion

In summary, we emphasize the key points of the multiprogram stabilization system once more. The chief advantage of the multiprogram controller is that the choice and implementation of the stable operation of a specific preprogrammed mode from the predefined family is ensured solely by setting the initial values in its neighborhood. The only inconvenience is the relatively lengthy form of control (10), (11) which later manifests itself in the additional nonlinear terms in the deviation system (13). However, the nonlinear (in the phase variables) terms in representation (15) are polynomial, which does not complicate the analysis of stability of the zero solution to system (13). The theorem 2 was formulated having the solution of the MPDS problem for linear system (9) because it is the most useful class of models under consideration. If needed, this result can be adapted for other dynamic systems.

The construction of estimates of the asymptotic stability domains [11, 12] for the equilibrium states of system (13) implemented by the multiprogram control system remained beyond the scope of this paper. On the one hand, this is a problem in its own right. On the other hand, the preliminary work for its solution has been done. In system (15), an explicit representation of the nonlinear terms was given, and the stabilizing feedbacks for the linear approximation systems can be constructed using well known methods [11].

In our opinion the solution of MPDS problem is of particular importance for the creation of control systems for marine vessels [13–15].

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