Numerical Method for the Heat Equation with Dirichlet and Neumann Conditions

A. Cheniguel

Abstract— **In this paper, one-dimensional heat equation subject to both Neumann and Dirichlet initial boundary conditions is presented and a Homotopy Perturbation Method (HPM) is utilized for solving the problem. The obtained results as compared with previous works are highly accurate. Also HPM provides continuous solution in contrast to finite difference method, which only provides discrete approximations. It is found that this method is a powerful mathematical tool and can be applied to a large class of linear and nonlinear problem in different fields of science and technology**

Index Terms— **Homotopy perturbation method (HPM), Partial differential equations, Heat conduction, Dirichlet and Neumann boundary Conditions**

I. Introduction

Recently, new analytical methods have gained the interest of researchers for finding approximate solutions to partial differential equations. This interest was driven by the needs from applications both in industry and sciences. Theory and numerical methods for solving initial boundary value problems were investigated by many researchers see for instance [4-13, 15-17,19-21,25-30] and the reference therein. In the last decade, there has been a growing interest in the new analytical techniques for linear and nonlinear initial boundary value problems. The widely applied techniques are perturbation methods. He [23] has proposed a new perturbation technique coupled with the homotopy technique, which is called the homotopy perturbation method (HPM). In contrast to the traditional perturbation methods. a homotopy is constructed with an embedding parameter π 2 [0; 1], which is considered as a small parameter. HPM has gained reputation as being a powerful tool for solving linear or nonlinear partial differential equations. This method has been the subject of intense investigation during recent years and many researchers have used it in their works involving differential equations see in [14,18]. He [22], applied HPM to solve initial boundary value problems which is governed by the nonlinear ordinary (Partial) differential equations, the results show that this method is efficient and simple. Thus, the main goal of this work is to apply the homotopy perturbation method (HPM) for solving one-dimensional heat conduction problem with Dirichlet and Neumann boundary conditions. The obtained results are more accurate than those obtained by Damrongsak et al [3]. The general form of equation is given as:

$$u_t = \alpha u_{xx} + f(x,t), 0 < x < l, t > 0 \quad (1)$$

Subject to the initial condition:

$$u(x,0) = u_0(x), 0 < x < a \quad (2)$$

And the boundary conditions

$$u(0,t) = g_0(t), u(l,t) = g_1(t), t > 0 \quad (3)$$

$$u_x(0,t) = g_2(t), u_x(l,t) = g_3(t), t > 0 \quad (4)$$

Where the diffusion coefficient α is positive, $u(x,t)$ represents the temperature at point (x,t) and $f(x,t), g_0(t), g_1(t), g_2(t), g_3(t)$ are sufficiently smooth known functions.

A. Cheniguel is with Department of Mathematics and Computer Science, Faculty of Sciences, Kasdi Merbah University Ouargla, Algeria (e-mail: cheniguelahmedl@yahoo.fr)

II. Analysis of Homotopy Perturbation Method

To illustrate the basic ideas, let X, and Y be two topological spaces. If f and $g$are continuous maps of the spaces X into Y , it is said that f is homotopic to g, if there is continuous map $F: X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x \in X$, then the map is called homotopy between f and g.

We consider the following nonlinear partial differential equation:

$$A(u) - f(r) = 0, r \ \Omega \quad (5)$$

Subject to the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, r \in \Gamma \quad (6)$$

Where A is a general differential operator. f is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial \eta}$ denotes directional derivative in outward normal direction to Ω. The operator A, generally divided into two parts, L and N, where L is linear, while N is nonlinear. Using A =L +N, eq. (5) can be rewritten as follows:

$$L(v) + N(v) - f(r) = 0 \quad (7)$$

By the homotopy technique, we construct a homotopy defined as

$$H(v,p): \Omega \times [0,1] \longrightarrow R \quad (8)$$

Which satisfies:

$$H(v,p) = (1-p)\big(L(v) - L(u_0)\big) + p\big(A(v) - f(r)\big), p \in [0,1], r \in \Omega \quad (9)$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\big(N(v) - f(r)\big) = 0, p \in [0,1], r \in \Omega \quad (10)$$

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Where $p \in [0,1]$ is an embedding parameter, v_0 is an initial approximation of equation (5), which satisfies the boundary conditions. It follows from equation (10) that:

$$H(v,0) = L(v) - L(u_0) = 0 \qquad (11)$$

$$H(v,1) = A(v) - f(r) = 0 \qquad (12)$$

The changing process of p from 0 to 1 monotonically is a trivial problem. $H(v,0) = L(v) - L(u_0) = 0$ is continuously transformed to the original problem

$$H(v,1) = A(v) - f(r) = 0. \qquad (13)$$

In topology, this process is known as continuous deformation. $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. We use the embedding parameter p as a small parameter, and assume that the solution of equation (10) can be written as power series of p :

$$v = p^0 v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots + p^n v_N + \cdots \qquad (14)$$

Setting $p = 1$ we obtain the approximate solution of equation (5) as:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots \qquad (15)$$

The series of equation (15) is convergent for most of the cases, but the rate of the convergence depends on the nonlinear operator $N(v)$. He (1999) has suggested that:

- The second derivative of $N(v)$ with respect to v should be small because the parameter may be relatively large i.e $p = 1$ and the norm of $L^{-1}(\frac{\partial N}{\partial u})$ must be smaller than one in order for the series to converge.

III. EXAMPLES

A. Example 1

We consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 \le x \le 1, t > 0 \qquad (16)$$

With the initial condition:

$$u(x,0) = \sin(\pi x),$$

And the boundary conditions:

$$u(0,t) = 0, u(1,t) = 0 \qquad (17)$$

For solving this problem, we construct the HPM as follows:

$$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0 \qquad (20)$$

The component v_i of (15) are obtained as follows:

$$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u(x,0) = \sin(\pi x) \qquad (21)$$

$$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} = 0, v_1(x,0) = 0 \qquad (22)$$

$$\frac{\partial^2 v_0}{\partial x^2} = -\pi^2 \sin(\pi x)$$

$$\frac{\partial v_1}{\partial t} = -\pi^2 \sin(\pi x)$$

Hence

$$v_1 = -\pi^2 \sin(\pi x) t \qquad (23)$$

$$\frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = 0, v_2(x,0) = 0 \qquad (24)$$

$$\frac{\partial^2 v_1}{\partial t^2} = 12t^2 + 24t, \frac{\partial v_1}{\partial x} = \frac{\partial^2 v_0}{\partial x^2} = 0$$

$$\frac{\partial v_2}{\partial t} = \pi^4 \sin(\pi x) t$$

Then, we have

$$v_2 = -\pi^4 \sin(\pi x)\frac{t^2}{2!} \qquad (25)$$

For the next component:

$$\frac{\partial v_3}{\partial t} - \frac{\partial^2 v_2}{\partial x^2} = 0, v_3(x,0) = 0$$

$$v_3 = -\pi^6 \sin(\pi x)\frac{t^3}{3!} \qquad (26)$$

And so on, we obtain the approximate solution as follows:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$$

And this leads to the following solution

$$u(x,t) = \sin(\pi x) e^{-\pi^2 t} \qquad (27)$$

We can, immediately observe that this solution is exact.

B. Example 2

Consider the following nonlinear reaction-diffusion equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \; 0 \le x \le 1, t > 0 \qquad (28)$$

Subject to the initial condition

$$u(x,0) = \cos(\pi x), \qquad (29)$$

And the boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(1,t)}{\partial x} = 0 \qquad (30)$$

Solving the equation (28) with the initial condition (29), yields:

$$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u_0 = \cos(\pi x)$$

$$\frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} = 0, v_1 = -\pi^2 \cos(\pi x)\, t, v_1(x,0) = 0$$

$$\frac{\partial v_2}{\partial t} - \frac{\partial^2 v_1}{\partial x^2} = 0, v_2 = \pi^4 \cos(\pi x)\frac{t^2}{2!},$$

And we can deduce the remaining components as:

$$v_3 = -\pi^6 \cos(\pi x)\frac{t^3}{3!}, \ldots, v_n = (-1)^n \pi^{2n} \cos(\pi x)\frac{t^n}{n!}, \qquad (32)$$

Using equations in the above, we get:

$$u(x,t) = \cos(\pi x)\left(1 - \frac{\pi^2 t}{1!} + \frac{(\pi^2 t)^2}{2!} - \frac{(\pi^2 t)^3}{3!} + \cdots\right)$$

And finally the approximate solution is obtained as :

$$u(x,t) = e^{-\pi^2 t}\cos(\pi x) \qquad (33)$$

C. Example 3

As a last example, consider the following problem:

$$u_t = u_{xx} + (\pi^2 - 1)e^{-t}\cos(\pi x) + 4x - 2, \qquad (37)$$

$$0 \le x \le 1, t > 0$$

With the initial condition

$$u(x,0) = \cos(\pi x) + x^2 \qquad (38)$$

And the boundary conditions:

$$u(0,t) = e^{-t}, u(1,t) = -e^{-t} + 4t + 1$$

According to the HPM, we have:

$$H(v,p) = (1-p)\left(\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - f\right) = 0 \qquad (39)$$

Where $f = (\pi^2 - 1)e^{-t} + 4x - 2$

By equating the terms with the identical powers of π , yields

$$p^0: \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \frac{\partial v_0}{\partial t} = 0, v_0 = \cos(\pi x) + x^2 \qquad (40)$$

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$$p^1: \ \frac{\partial v_1}{\partial t} - \frac{\partial^2 v_0}{\partial x^2} = 0, v_1(x,0) \ = 0,$$

$$\frac{\partial v_1}{\partial t} = 4x + \cos(\pi x)\,(-\pi^2 + (\pi^2 - 1)e^{-t})$$

$$v_1 = 4xt + \cos(\pi x)\,(-\pi^2 t + (\pi^2 - 1)(1 - e^{-t}))$$

$$p^2: \ \frac{\partial v_2}{\partial t} - \frac{\partial {v_1}^2}{\partial x^2} = 0, v_2(x,0) \ = 0,$$

$$\frac{\partial v_2}{\partial t} = \cos(\pi x)\,(\pi^4 t - \pi^2(\pi^2\text{-}1)(1\text{-}e^{-t}))$$

Continuing like-wise we get:

$$v_2 = \cos(\pi x)\,((\pi^4 - \pi^2)(1 - \tfrac{t}{1!} - e^{-t}) + \tfrac{(\pi^2 t)^2}{2!})$$

$$v_3 = \cos(\pi x)\,((\pi^6 - \pi^4)\left(1 - \frac{t}{1!} + \frac{t^2}{2!} - e^{-t}\right) - \frac{(\pi^2 t)^3}{3!})$$

$$v_4 = \cos(\pi x)\,((\pi^8 - \pi^6)(1 - \tfrac{t}{1!} + \tfrac{t^2}{2!} - \tfrac{t^3}{3!}\text{-}e^{-t}) + \tfrac{(\pi^2 t)^4}{4!})$$

And so on we then have:

$$u_{5hpm} = x^2 + 4xt + \cos(\pi x)\,[\pi^8(1 - \tfrac{t}{1!} + \tfrac{t^2}{2!} - \tfrac{t^3}{3!}\text{-} e^{-t}) + e^{-t}] \qquad (41)$$

From this result we deduce that the series solution converges to the exact one :

$$u(x,t) = x^2 + 4xt + \cos\,(\pi x) e^{-t}$$

Example 4

Once again, consider the non-homogeneous heat equation with non-homogeneous Neumann boundary conditions:

$$u_t = u_{xx} + (\tfrac{\pi^2}{2})e^{\frac{-\pi^2}{2}t}\cos(\pi x) + x - 2$$

$$0 \le x \le 1, t > 0 \qquad (42)$$

$$u(x,0) = \cos(\pi x) + x^2, u_x(0,t) = t$$

$$u_x(1,t) = 2 + t$$

The theoretical solution is:

$$u(x,t) = x^2 + xt + e^{\frac{-\pi^2}{2}t}\cos\,(\pi x).$$

According to HPM, we get the components of (15):

$$v_{0t} - u_{0t} = 0, v_0 = \cos(\pi x) + x^2 \qquad (43)$$

$$v_{1t} = v_{0xx} + \tfrac{\pi^2}{2} e^{\frac{-\pi^2}{2}t}\cos(\pi x) + x - 2, v_1(x,0) = 0$$

$$v_{1t} = x + \cos(\pi x)\,(-\pi^2 + \tfrac{\pi^2}{2} e^{\frac{-\pi^2}{2}t})$$

$$v_1 = xt + \cos(\pi x)\,(1 - \pi^2 t - e^{\frac{-\pi^2}{2}t}) \qquad (44)$$

$$v_{2t} = v_{1xx} = \cos(\pi x)\,(-\pi^2 + \pi^4 t + \pi^2 e^{\frac{-\pi^2}{2}t})$$

$$v_2 = \cos(\pi x)\,(2 - \pi^2 t + \tfrac{(\pi^2 t)^2}{2!} - 2e^{\frac{\pi^2}{2}t})$$

$$v_{3t} = v_{2xx} = \cos(\pi x)\,(-2\pi^2 + \pi^4 t - \frac{\pi^6 t^2}{2!} + 2\pi^2 e^{\frac{-\pi^2}{2}t})$$

$$v_3 = \cos(\pi x)\,(4 - 2\pi^2 t + \tfrac{(\pi^2 t)^2}{2!} - \tfrac{(\pi^2 t)^3}{3!} - 4e^{\frac{-\pi^2}{2}t}) \qquad (45)$$

$$v_{4t} = v_{3xx} = \cos(\pi x)\,(-4\pi^2 + 2\pi^4 t - \pi^2\frac{(\pi^2 t)^2}{2!} + \pi^2\frac{(\pi^2 t)^3}{3!} + 4\pi^2 e^{\frac{-\pi^2}{2}t})$$

$$v_4 = \cos(\pi x)\,(8 - 4(\pi^2 t) + (\pi^2 t)^2 - \tfrac{(\pi^2 t)^3}{3!} + \tfrac{(\pi^2 t)^4}{4!} - 8e^{\frac{-\pi^2}{2}t}) \qquad (45)$$

And so on, we obtain the approximate solution as follows:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + v_4 + \cdots$$

Or

$$u(x,t) = x^2 + xt + \cos(\pi x)\, e^{\frac{-\pi^2}{2}t} + 15\left\{(1 - \frac{\left(\frac{\pi^2}{2}t\right)}{1!} + \frac{(\frac{\pi^2}{2}t)^2}{2!} - \frac{(\frac{\pi^2}{2}t)^3}{3!} + \frac{(\frac{\pi^2}{2}t)^4}{4!} -.+\cdots.) - e^{\frac{-\pi^2}{2}t}\right\}$$

And this leads to the following solution

$$u(x,t) = x^2 + xt + \cos\,(\pi x) e^{\frac{-\pi^2}{2}t}$$

This solution coincides with the exact one.

Table 1 Example 1

$h_x = 0.1\,, h_t = 0.004\,,$ 3-Iterates

x_i	u_{ex}	u_{hpm}	$\lvert u_{ex} - u_{hpm}\rvert$
0.0	0. 0	0.0	0.0
0.1	0.2971	0.2971	0.0
0.2	0.05650	0.5650	0.0
0.3	0.7777	0.7777	0.0
0.4	0.9142	0.9142	0.0
0.5	0.9613	0.9613	0.0
0.6	0.9142	0.9142	0.0
0.7	0.7777	0.7777	0.0
0.8	0.5650	0.5650	0.0
0.9	0.2971	0.2971	0.0
1.0	0. 0	0.0	0.0

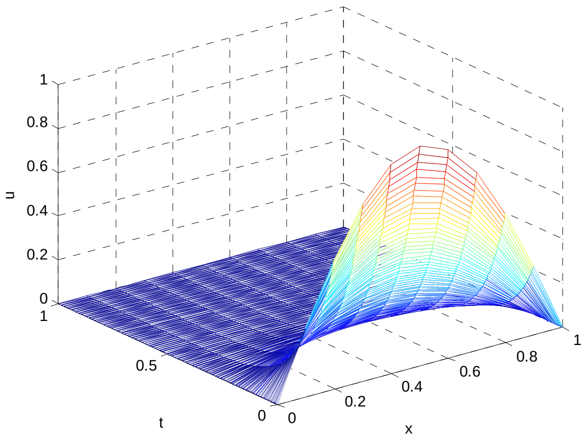

Fig.1.Variation of the approximate solution for different values of x and t

Table 2 Example 2

$h_x = 0.1\,, h_t = 0.004,$ 3-Iterates

x_i	u_{ex}	u_{hpm}	$\lvert u_{ex} - u_{hpm}\rvert$
0.0	0.9613	0.9613	0.0
0.1	0.9142	0.9142	0.0
0.2	0.7777	0.7777	0.0
0.3	0.5650	0.5650	0.0
0.4	0.2971	0.2971	0.0
0.5	0.0	0.0	0.0
0.6	-0.2971	-0.2971	0.0
0.7	-0.5650	-0.5650	0.0
0.8	-0.7777	-0.7777	0.0
0.9	-0.9142	-0.9142	0.0
1.0	-0.9613	-0.9613	0.0

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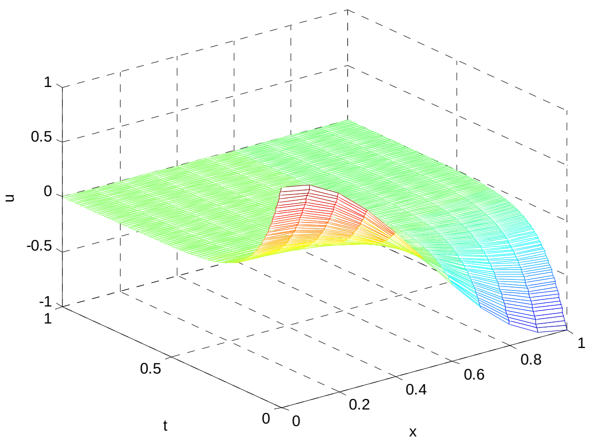
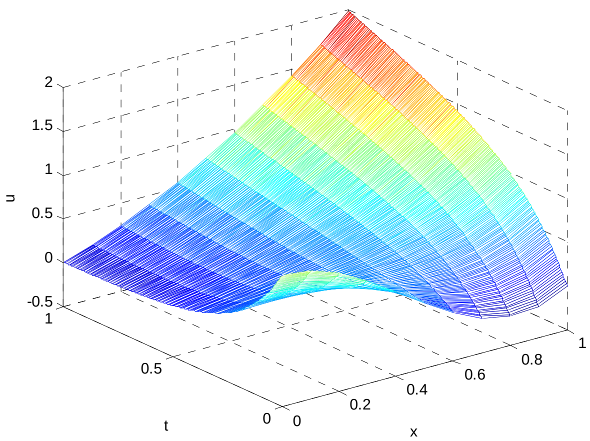


Fig.2. Variation of the approximate solution for different values of x and t



Fig;4;Variation of the approximate solution for different values of x and t

Table 3 Example 3

$h_x = 0.1, h_t = 0.004, \quad 2 - Iterates$

x_i	u_{ex}	u_{2hpm}	$\lvert u_{ex} - u_{2hpm}\rvert$
0.0	0.9960	0.9960	0.0
0.1	0.9589	0.9589	0.0
0.2	0.8490	0.8490	0.0
0.3	0.6802	0.6802	0.0
0.4	0.4742	0.4742	0.0
0.5	0.2580	0.2580	0.0
0.6	0.0618	0.0618	0.0
0.7	-0.0842	-0.0842	0.0
0.8	-0.1530	-0.1530	0.0
0.9	-0.1229	-0.1229	0.0
1.0	0.0200	0.0200	0.0

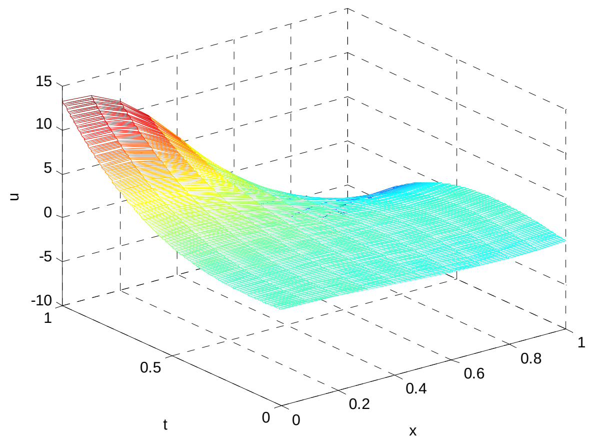


Fig.3. Variation of approximate solution for different values of x and t

Table 4 Example 4

$h_x = 0.1, h_t = 0.004 \quad 3 - Iterates$

x_i	u_{ex}	u_{hpm}	$\lvert u_{ex} - u_{hpm}\rvert$
0.0	0;9805	0.9805	0.0
0.1	0.9429	0.9429	0.0
0.2	0.8340	0.8340	0.0
0.3	0.6675	0.6675	0.0
0.4	0.4646	0.4646	0.0
0.5	0.2520	0.2520	0.0
0.6	0.0594	0.0594	0.0
0.7	-0.0835	-0.0835	0.0
0.8	-0.1500	-0.1500	0.0
0.9	-0.1189	-0.1189	0.0

IV. CONCLUSION

The aim of this paper has been to construct an approximate solution to the heat conduction problem with Dirichlet and Neumann boundary conditions using homotopy perturbation method (HPM). The problems solved using the (HPM) gave satisfactory results in comparison to those recently obtained by other searchers using finite difference schemes. The case studies gave sufficiently good agreements with the exact solutions. These results are obtained without using linearization, discretization, transformation or restrictive assumptions. The results demonstrate the stability and convergence of the method.The obtained solutions are shown graphically. Moreover, the method is easier to implement than the traditional techniques. It is worth mentioning that the technique and ideas presented in this paper can be extended for finding the analytic solution of the obstacle, unilateral and contact problems encountered in mathematical and engineering sciences.

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