On Positive Definite Solutions of the Linear Matrix Equation $X + A^* XA = I$

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Abstract—**Two effective iterative methods are constructed to solve the linear matrix equation of the form $X + A^* XA = I$. Some properties of a positive definite solution of the linear matrix equation are discussed. Necessary and sufficient conditions for existence of a positive definite solution are derived for $\|A\| < 1$ and $\|A\| > 1$. Several numerical examples are given to show the efficiency of the presented iterative methods.**

Index Terms—**Algorithm, Fixed-Point-Iteration, linear-Matrix-Equation, Numerical-Analysis, Positive-Definite -Solutions, Two-Sided-Iteration**.

I. INTRODUCTION

Considering the linear matrix equation

$$X + A^* XA = I, \tag{1}$$

with unknown matrix X, where $A \in C^{n\times n}$, I is the identity matrix of order n. The equation (1) could be viewed as a special case of the symmetric matrix equations

$$X \pm A_1^* XA_1 \pm \cdots \pm A_m^* XA_m = Q. \tag{2}$$

Where Q is a positive definite matrix [15]. There are many linear matrix equations which were studied by some authors [2],[3],[8]-[11],[13]-[15],[18]-[21],[24]. Two effective iterative methods for computing a positive definite solution of this equation are proposed. The first one is fixed point iteration method and the second one is two sided iteration method of the fixed point iteration method. These two iterative methods are used for computing a positive definite solution of nonlinear matrix equations, see [1],[4]-[7],[12],[16],[17],[22],[23].

This paper aims to find the positive definite solution of the matrix equation (1) for all values of $\|A\| \neq 1$, for this purpose we investigated two iterative methods, the first one is based on fixed point iteration and the second is based on two sided iteration method, also to derive necessary and sufficient conditions for the existence of the solution of equation (1).

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Section II describes some properties of positive definite solutions of the equation (1). Section III, presents a first iterative method (Fixed point iteration method) for obtaining the solution of our problem. Also, it presents theorems for obtaining the necessary and sufficient conditions for the existence of a solution of matrix equation (1). Section IV represents the second iterative method (Two sided iteration method of the fixed point iteration method) for obtaining the solution of the problem and theorems for the sufficient conditions for the existence of a positive definite solution of (1). Numerical examples in Section V illustrate the effectiveness of these methods. Conclusion drawn from the results obtained in this paper are in section VI.

The notation $X > 0$ means that X is a positive definite Hermitian matrix and $A > B$ is used to indicate that $A - B > 0$. A^* denotes the complex conjugate transpose of A. Finally, throughout the paper, $\|.\|$ will be the spectral norm for square matrices unless otherwise noted.

II. SOME PROPERTTIES OF THE SOLUTIONS

This section discusses some properties of positive definite solutions of the matrix equation (1) .

1) Theorem

If m and M are the smallest and the largest eigen-values of a solution X of (1), respectively, and λ is an eigenvalue of A, then $\sqrt{\frac{1-M}{M}} \le |\lambda| \le \sqrt{\frac{1-m}{m}}$.

Proof

Let v be an eigenvector corresponding to an eigen-value λ of the matrix A and $\|v\| = 1$. Since the solution X of. (1) is a positive definite matrix, then $0 < m \le M < 1,$

$\langle (X + A^* XA)v, v\rangle = \langle Iv, v\rangle = \langle v, v\rangle$, $\langle Xv, v\rangle + \langle A^* XAv, v\rangle = \langle v, v\rangle$,

$\langle Xv, v\rangle + \langle XAv, Av\rangle = 1$ and $\langle Xv, v\rangle + |\lambda|^2 \langle Xv, v\rangle = 1.$

Consequenly, $\sqrt{\frac{1-M}{M}} \le |\lambda| \le \sqrt{\frac{1-m}{m}}$.

2) Theorem

If (1) has a positive definite solution X ,then $A^* A + (AA^*)^{-1} > I$

Proof

Since X is a positive definite solution of (1), then $X < I, A^* XA < I$, i.e. $X < (AA^*)^{-1}$.

Thus we have $(AA^*)^{-1} > X = I - A^*XA > I - A^*A.$
Consequently, $A^*A + (AA^*)^{-1} > I.$

III. THE FIRST ITERATION METHOD (FIXED POINT ITERATION METHOD)

This section establishes the first iterative method which is suitable for obtaining a positive definite solution of (1) when $\|A\| < 1$.

A. Algorithm

Take $X_0 = \alpha I.$ For $k = 0,1,2,\ldots,$ compute

$$X_{k+1} = I - A^* X_k A. \quad (3)$$

Our theorems give necessary and sufficient conditions for the existence of a positive definite solution of (1).

1) Theorem

Let the sequence $\{X_k\}$ be determined by the *Algorithm A* and $\|A\|^2 < 1$ (4)

If (1) has a positive definite solution, then $\{X_k\}$ converges to X, which is a solution of (1) for all numbers $\alpha > 1$. Moreover, if $X_k > 0$ for every k, then (1) has a positive definite solution.

Proof.

Let (1) has a positive definite solution. From *Algorithm A*, we have

$X_0 = \alpha I > I > X_1 = I - \alpha A^*A,$

$X_0 = \alpha I > I > X_2 = I - A^*X_1A > I - \alpha A^*A = X_1, i.e.,$

$X_0 > X_2 > X_1$. To prove $X_s > X_1$ if $X_{s-1} < X_0$ for all *s*, we have $X_s = I - A^*X_{s-1}A > I - A^*X_0A = I - \alpha A^*A = X_1$. We will find the relation between X_2, X_3, X_4, X_5. Since

$X_1 < X_2$, then $X_2 = I - A^*X_1A > I - A^*X_2A = X_3$ and

$X_4 = I - A^*X_3A > I - A^*X_2A = X_3$, since $X_3 > X_1$, then

$X_4 = I - A^*X_3A < I - A^*X_1A = X_2$ and

$X_5 = I - A^*X_4A > I - A^*X_2A = X_3.$

Also, since $X_4 > X_3$, then

$X_5 = I - A^*X_4A < I - A^*X_3A = X_4.$ Thus, we get

$X_0 = \alpha I > X_2 > X_4 > X_5 > X_3 > X_1 = I - \alpha A^*A..$

We will prove that $X_0 > X_{s+1} > X_s$, if we have

$X_0 > X_{s-1} > X_s$, thus

$X_0 = \alpha I > I > I - A^*X_{s-1}A < I - A^*X_sA \Rightarrow X_0 > X_{s+1} > X_s.$

Also, we can prove that $X_0 > X_s > X_{s+1}$, if we have

$X_0 > X_s > X_{s-1}$, $X_0 = \alpha I > I - A^*X_{s-1}A > I - A^*X_sA \Rightarrow$

$X_0 > X_s > X_{s+1}$. Therefore, we have

$X_0 = \alpha I > X_{2r} > X_{2r+2} > X_{2s+3} > X_{2s+1} > X_1 = I - \alpha A^*A,$ for every positive integers *r,s*. Consequently, the subsequences $\{X_{2r}\},\{X_{2s+1}\}$ are monotonic and bounded, and $\lim_{s\to\infty} X_{2s}$, $\lim_{s\to\infty} X_{2s+1}$ exist.

To prove these sequences have a common limit, we have

$$\|X_{2s} - X_{2s+1}\| = \|I - A^*X_{2s-1}A - I + A^*X_{2s}A\|$$
$$= \|A^*(X_{2s} - X_{2s-1})A\| \le \|A\|^2 \|X_{2s} - X_{2s-1}\|.$$

Let $q = \|A\|^2 < 1$, and we get

$$\|X_{2s} - X_{2s+1}\| \le q\|X_{2s} - X_{2s-1}\| \le q^2 \|X_{2s-2} - X_{2s-1}\|$$
$$\le \ldots. \le q^{2s}\|X_0 - X_1\| \le q^{2s}(2\alpha - 1).$$

Since $q < 1$ and $(2\alpha - 1) > 0$, $\|X_{2s} - X_{2s+1}\| \to 0$ as $s \to \infty$, that is, $\{X_{2r}\}$ and $\{X_{2s+1}\}$ have the same limit X and $X_{2s} > X > X_{2s+1}, s = 1,2,\ldots.$ Taking the limit of the sequence $\{X_k\}$ generated by *Algorithm* ***A*** leads to $X = I - A^*XA,$ which is a solution of (1). Assuming that $X_k > 0$ for every k. We proved that the sequences have a common limit X. Since $X_{k+1} = I - A^*X_kA > 0$, taking the limits of both sides as k approaches to ∞, we get $X = I - A^*XA, > 0.$ Hence (1) has a positive definite solution.

2) Theorem

Let X_k be the iterates in *Algorithm* ***A***. If $q = \|A\|^2 < 1$. Then $\|X_k - X\| < q^k(2\alpha - 1)$, for all real number $\alpha > 1$, where X is a positive definite solution of (1).

Proof

From previous Theorem it follows that the sequence (3) is convergent to a positive definite solution X of (1). From (3) and $X = I - A^*XA,$ we have

$X_k - X = A^*(X - X_{k-1})A$. Thus,

$$\|X_k - X\| = \|A^*(X - X_{k-1})A\| \le \|A\|^2 \|X - X_{k-1}\| \le q\|X - X_{k-1}\|$$
$$\le q^2 \|X - X_{k-2}\|, \quad \textit{after } k-\textit{steps}$$
$$\le q^k \|X_0 - X\|.$$

From Theorem ***1***, we have

$\|X_k - X\| \le q^k \|X_0 - X\| < q^k(2\alpha - 1).$

3) Corollary

Suppose that (1) has a solution. If $q = \|A\|^2 < 1$, then $\{X_k\}$ converges to X with at least the linear convergence rate.

Proof

We have $\|X_{k+1} - X\| \le \|A\|^2 \|X_{k-1} - X\|$. Choose a real number satisfying $\|A\| < \theta < 1.$ Since $X_k \to X$, there exists a *N* such that for any $k \ge N.$ $\|A\| \le \theta.$ Hence

$\|X_{k+1} - X\| \le \theta^2 \|X_k - X\|.$

4) Theorem

If (1) has a positive definite solution and after k iterative steps of *Algorithm* ***A***, we have $\|I - X_k^{-1}X_{k-1}\| < \varepsilon,$ then $\|X_k + A^*X_kA - I\| < \alpha\varepsilon\|A\|^2$, where X_k is the iterates in *Algorithm* ***A***.

Proof

$$X_k + A^* X_k A - I = X_k + A^* X_k A - X_k - A^* X_{k-1} A$$
$$= A^*\left(X_k - X_{k-1}\right)A. \qquad (5)$$

Take the norms of both sides of (5)
$\left\|X_k + A^* X_k A - I\right\| \le \|A\|^2 \|X_k\| \left\|I - X_k^{-1} X_{k-1}\right\| < \alpha\varepsilon \|A\|^2$. For $\left\|I - X_k^{-1} X_{k-1}\right\|$, $X_k \to X$ as $k \to \infty$.

Consequently, $\left\|I - X_k^{-1} X_{k-1}\right\| \to 0$ as $k \to \infty$, that is, $\left\|I - X_k^{-1} X_{k-1}\right\| < \varepsilon$, for $\varepsilon > 0$ and from *theorem* ***1***, $\|X_k\| \le \alpha$ for every k, thus $\left\|X_k + A^* X_k A - I\right\| < \alpha\varepsilon \|A\|^2$.

IV. THE SECOND ITERATION METHOD (TWO SIDED ITERATION METHOD OF THE FIXED POINT ITERATION METHOD)

This section establishes the second iterative method which is suitable for obtaining a positive definite solution of (1) .

B. Algorithm

Take $X_0 = \alpha I, Y_0 = \beta I$. For $k = 0,1,2,....$, compute
$$X_{k+1} = I - A^* X_k A \text{ and } Y_{k+1} = I - A^* Y_k A. \qquad (6)$$

Next theorems provide necessary and sufficient conditions for the existence of a solution of (1) when $\|A\| < 1$.

1) Theorem

If (1) has a positive definite solution, the sequences $\{X_k\}$ and $\{Y_k\}$ are determined by *Algorithm* ***B*** and
$$\|A\|^2 < 1, \qquad (7)$$
then the two sequences $\{X_k\}, \{Y_k\}$ converge to the positive definite solution X for all real numbers α, β such that $\beta > \alpha > 0$. On the other hand, if $X_k, Y_k > 0$ for every k, $\|A\|^2 < 1$ and $\beta > \alpha > 0$, then (1) has a positive definite solution.

Proof

First, considering sequence (6) , for X_1, X_2 we have
$X_0 = \alpha I > I > X_1 = I - \alpha A^* A = \alpha I > I > X_2 = I - A^* X_1 A >$
$I - \alpha A^* A = X_1$, *i.e.*, $X_0 > X_2 > X_1$. To prove $X_s > X_1$ if $X_{s-1} < X_0$ for all *s*, hence
$X_s = I - A^* X_{s-1} A > I - A^* X_0 A = I - \alpha A^* A = X_1$. We will find the relation between X_2, X_3, X_4, X_5. Since $X_1 < X_2$, then $X_2 = I - A^* X_1 A > I - A^* X_2 A = X_3$ and
$X_4 = I - A^* X_3 A > I - A^* X_2 A = X_3$, since $X_3 > X_1$, then
$X_4 = I - A^* X_3 A < I - A^* X_1 A = X_2$ and
$X_5 = I - A^* X_4 A > I - A^* X_2 A = X_3$. Also, since $X_4 > X_3$, then $X_5 = I - A^* X_4 A < I - A^* X_3 A = X_4$. Thus, we get
$X_0 = \alpha I > X_2 > X_4 > X_5 > X_3 > X_1 = I - \alpha A^* A$. We will prove that $X_0 > X_{s+1} > X_s$, if we have $X_0 > X_{s-1} > X_s$, thus
$X_0 = \alpha I > I > I - A^* X_{s-1} A < I - A^* X_s A \Rightarrow X_0 > X_{s+1} > X_s$.
Also, we can prove that $X_0 > X_s > X_{s+1}$, if we have $X_0 > X_s > X_{s-1}$, thus
$X_0 = \alpha I > I - A^* X_{s-1} A > I - A^* X_s A \Rightarrow X_0 > X_s > X_{s+1}$.

Therefore, we have
$X_0 = \alpha I > X_{2r} > X_{2r+2} > X_{2s+3} > X_{2s+1} > X_1 = I - \alpha A^* A$, for every positive integers *r,s* . Consequently, the subsequences $\{X_{2r}\}, \{X_{2s+1}\}$ are monotonic and bounded, and $\lim_{s\to\infty} X_{2s}$ and $\lim_{s\to\infty} X_{2s+1}$ exist.

For the sequence $\{Y_k\}$, similarly, we can prove that $Y_0 = \beta I > Y_{2r} > Y_{2r+2} > Y_{2s+3} > Y_{2s+1} > Y_1 = I - \beta A^* A$, for every positive integers *r,s*.
Consequently, the subsequences $\{Y_{2r}\}, \{Y_{2s+1}\}$ are monotonic and bounded, and $\lim_{s\to\infty} Y_{2s}$ and $\lim_{s\to\infty} Y_{2s+1}$ exist.

Finally, from (6) we have $Y_0 = \beta I > \alpha I = X_0$ and
$Y_1 = I - A^* Y_0 A < I - A^* X_0 A = X_1$, i.e. , $Y_0 > X_0 > X_1 > Y_1$.
Also, $Y_2 = I - A^* Y_1 A > I - A^* X_1 A = X_2$.
Similarly, $Y_3 < X_3, Y_4 > X_4$. Since $Y_0 > X_1, X_0 > Y_1$, then
$X_1 = I - A^* X_0 A < I - A^* Y_1 A = Y_2$ and
$Y_1 = I - A^* Y_0 A < I - A^* X_1 A = X_2$. Also, since $Y_2 > X_1$, then
$Y_3 = I - A^* Y_2 A < I - A^* X_1 A = X_2$. From (6) we have
$X_0 = \alpha I > I > I - A^* Y_1 A = Y_2$. Therefore,
$X_1 = I - A^* X_0 A < I - A^* Y_2 A = Y_3$ and
$X_2 = I - A^* X_1 A > I - A^* Y_3 A = Y_4$. Consequently,
$X_1 = I - A^* X_0 A < I - A^* Y_2 A = Y_3$ and
$X_2 = I - A^* X_1 A > I - A^* Y_3 A = Y_4$. Consequently,
$Y_0 > X_0 > Y_2 > X_2 > Y_4 > X_4 > X_3 > Y_3 > X_1 > Y_1$.
We will prove that $Y_0 > X_{s+1} > Y_s$, if we have $Y_0 > Y_{s-1} > X_s$, thus $Y_s = I - A^* Y_{s-1} A < I - A^* X_s A = X_{s+1} < I < \beta I = Y_0$.
Also, we can prove that $Y_0 > X_s > Y_{s+1}$, if we have $Y_0 > Y_s > X_{s-1}$, thus
$Y_{s+1} = I - A^* Y_s A < I - A^* X_{s-1} A = X_s < I < \beta I = Y_0$.
Therefore, we have
$Y_0 > X_0 > Y_{2r} > X_{2r} > Y_{2r+2} > X_{2r+2} > X_{2s+3} > Y_{2s+3} > X_{2s+1} > Y_{2s+1}$
$> X_1 > Y_1$, for every positive integers *r,s*.
Finally, to prove that the subsequences $\{Y_{2r}\}, \{Y_{2s+1}\}$ have the same limit, we have
$$\left\|Y_{2s} - Y_{2s+1}\right\| = \left\|I - A^* Y_{2s-1} A - I + A^* Y_{2s} A\right\| =$$
$$\left\|A^*\left(Y_{2s} - Y_{2s-1}\right)A\right\| \le \|A\|^2 \left\|Y_{2s} - Y_{2s-1}\right\|,$$
from (7) , it follows that $q = \|A\|^2 < 1$, and we get
$$\left\|Y_{2s} - Y_{2s+1}\right\| \le q\left\|Y_{2s-1} - Y_{2s}\right\| \le .... \le q^{2s}\left\|Y_0 - Y_1\right\| < q^{2s}(2\beta - 1).$$
Therefore, we have for the limit X of the subsequences $\{Y_{2r}\}, \{Y_{2s+1}\}$, $Y_{2s} > X > Y_{2s+1}$, $s = 1,2,....$.
Therefore, also we have for the limit X of the subsequences $\{X_{2r}\}, \{X_{2s+1}\}$
$Y_{2s} > X_{2s} > X > X_{2s+1} > Y_{2s+1}$, $s = 1,2,....$.
Taking limit in the *Algorithm B* leads to $X = I - A^* X A$.
If $X_k, Y_k > 0$ for every k. We proved that the sequences have the same limit X. Since $X_{k+1} = I - A^* X_k A > 0$ and $Y_{k+1} = I - A^* Y_k A > 0$, hence $X = I - A^* X A$ and (1) has a positive definite solution.

2) Theorem.

For the *Algorithm B*, if there exist a positive real numbers α and β such that $\beta > \alpha$ and $q = \|A\|^2 < 1$, then $\|X_k - X\| < q^k(2\alpha - 1)$, $\|Y_k - X\| < q^k(2\beta - 1)$ and $\|X_k - X\| < \|Y_k - X\| < q^k(2\beta - 1)$, where X is a positive definite solution of (1) and $X_k, Y_k, k = 0,1,2,...$ are defined in (6).

Proof.

From previous Theorem it follows that the sequence (6) is convergent to a positive definite solution X of (1). We will compute the norm of the matrices $X_k - X$ and $Y_k - X$,

$$\|X_k - X\| = \|I - A^* X_{k-1} A - I + A^* X A\| = \|A^*(X - X_{k-1})A\|$$
$$\le \|A\|^2 \|X - X_{k-1}\| \le q\|X - X_{k-1}\| \le ... \le q^k \|X_0 - X\|.$$

From previous Theorem, we have

$\|X_k - X\| \le q^k \|X_0 - X\| < q^k(2\alpha - 1)$.

Similarly, $\|Y_k - X\| \le q^k \|Y_0 - X\| < q^k(2\beta - 1)$. Also, we have $\|X_k - X\| < \|Y_k - X\|$. Therefore,

$\|X_k - X\| < \|Y_k - X\| < q^k(2\beta - 1)$.

3) Corollary

Suppose that (1) has a solution. If $q = \|A\|^2 < 1$, then $\{X_k\}$ and $\{Y_k\}$ converge to X with at least the linear convergence rate.

4) Theorem

If the (1) has a solution and after k - iterative steps of the *Algorithm B*, we have $\|I - X_k^{-1} X_{k-1}\| < \varepsilon$ and $\|I - Y_k^{-1} Y_{k-1}\| < \varepsilon$. Then $\|X_k + A^* X_k A - I\| < \alpha\varepsilon \|A\|^2$ and $\|Y_k + A^* Y_k A - I\| < \beta\varepsilon \|A\|^2$.

Also, $\|X_k + A^* X_k A - I\| < \|Y_k + A^* Y_k A - I\| < \beta\varepsilon \|A\|^2$, where $X_k, Y_k, k = 0,1,2,...$ are defined in (6) and $\varepsilon > 0$.

Proof

Since, $X_k + A^* X_k A - I = X_k + A^* X_k A - X_k - A^* X_{k-1} A$ $= A^*(X_k - X_{k-1})A$. Take the norms of both sides

$$\|X_k + A^* X_k A - I\| \le \|A\|^2 \|X_k - X_{k-1}\|$$
$$\le \|A\|^2 \|X_k\| \; \|I - X_k^{-1} X_{k-1}\| < \alpha\varepsilon \|A\|^2.$$

Similarly, $\|A^* Y_k A - Y_k - I\| < \beta\varepsilon \|A\|^2$. From *Theorem **1***, we have $\|X_{k+1} - X_k\| < \|Y_{k+1} - Y_k\|$ and since

$\|X_{k+1} - X_k\| = \|I - A^* X_k A - X_k\| = \|X_k + A^* X_k A - I\|$ and

$\|Y_{k+1} - Y_k\| = \|I - A^* Y_k A - Y_k\| = \|Y_k + A^* Y_k A - I\|$, then

$\|X_k + A^* X_k A - I\| < \|Y_k + A^* Y_k A - I\| < \beta\varepsilon \|A\|^2$.

C. Algorithm

Take $X_0 = \alpha I, Y_0 = \beta I$. For $k = 0,1,2,....$, compute

$$X_{k+1} = B^*(I - X_k)B \text{ and } Y_{k+1} = B^*(I - Y_k)B. \quad (8)$$

where $B = A^{-1}, B^* = A^{*-1}$.

Next theorems provide necessary and sufficient conditions for the existence of a solution of (1) when $\|A\| > 1$.

1) Theorem

If (1) has a positive definite solution, the sequences $\{X_k\}$ and $\{Y_k\}$ are determined by the *Algorithm C* and the inequalities

(i) $B^* B < \alpha I$ and $B^* B < \beta I$, $0 < \alpha < \beta$,

(ii) $q = \|B\|^2 < 1$, (9)

are satisfied, then $\{X_k\}$, $\{Y_k\}$ converge to a positive definite solution X. Moreover, if $X_k > 0$ and $Y_k > 0$ for every k, $B^* B < \alpha I$, $B^* B < \beta I$ and $\alpha, \beta > 0$, then (1) has a positive definite solution.

Proof

First, from *Algorithm C*, we have

$X_0 = \alpha I > B^* B > X_1 = B^* B - \alpha B^* B$ and

$X_0 = \alpha I > B^* B > X_2 = B^* B - B^* X_1 B > B^* B - \alpha B^* B = X_1$. i.e., $X_0 > X_2 > X_1$. To prove $X_s > X_1$ if $X_{s-1} < X_0$ for all *s*, from *Algorithm C*.

$X_s = B^* B - B^* X_{s-1} B > B^* B - B^* X_0 B = B^* B - \alpha B^* B = X_1$. We will find the relation between X_2, X_3, X_4, X_5. Since $X_1 < X_2$, then $X_2 = B^* B - B^* X_1 B > B^* B - B^* X_2 B = X_3$ and $X_4 = B^* B - B^* X_3 B > B^* B - B^* X_2 B = X_3$, since $X_3 > X_1$, then $X_4 = B^* B - B^* X_3 B < B^* B - B^* X_1 B = X_2$ and $X_5 = B^* B - B^* X_4 B > B^* B - B^* X_2 B = X_3$. Also since $X_4 > X_3$, then $X_5 = B^* B - B^* X_4 B < B^* B - B^* X_3 B = X_4$. Thus, we get $X_0 = \alpha I > X_2 > X_4 > X_5 > X_3 > X_1 = B^* B - \alpha B^* B$.

We will prove that $X_0 > X_{s+1} > X_s$ if we have $X_0 > X_{s-1} > X_s$, thus

$X_0 = \alpha I > B^* B > B^* B - B^* X_{s-1} B < B^* B - B^* X_s B$. i.e., $X_0 > X_{s+1} > X_s$. Also, we will prove that $X_0 > X_s > X_{s+1}$ if we have $X_0 > X_s > X_{s-1}$, thus

$X_0 = \alpha I > B^* B - B^* X_{s-1} B > B^* B - B^* X_s B$. i.e., $X_0 > X_s > X_{s+1}$. Therefore, we have

$$X_0 = \alpha I > X_{2r} > X_{2r+2} > X_{2s+3} > X_{2s+1} > X_1 = B^* B - \alpha B^* B = (1 - \alpha) B^* B,$$

for every positive integers *r,s*. Consequently, the subsequences $\{X_{2r}\}, \{X_{2s+1}\}$ are monotonic and bounded, and $\lim_{s\to\infty} X_{2s}$, $\lim_{s\to\infty} X_{2s+1}$ exist.

For the sequence $\{Y_k\}$, similarly,

$Y_0 = \beta I > Y_{2r} > Y_{2r+2} > Y_{2s+3} > Y_{2s+1} > Y_1 = (1 - \beta) B^* B$, for every positive integers *r,s*. Consequently, the subsequences $\{Y_{2r}\}, \{Y_{2s+1}\}$ are monotonic and bounded, and $\lim_{s\to\infty} Y_{2s}$, $\lim_{s\to\infty} Y_{2s+1}$ exist.

Finally, from (8) we have $Y_0 = \beta I > \alpha I = X_0$ and

$Y_1 = B^*B - B^*Y_0B < B^*B - B^*X_0B = X_1$.

i.e. $Y_0 > X_0 > X_1 > Y_1$. Also,

$Y_2 = B^*B - B^*Y_1B > B^*B - B^*X_1B = X_2$. Similarly,

$Y_3 < X_3, Y_4 > X_4$. Since $Y_0 > X_1, X_0 > Y_1$, then

$X_1 = B^*B - B^*X_0B < B^*B - B^*Y_1B = Y_2$ and

$Y_1 = B^*B - B^*Y_0B < B^*B - B^*X_1B = X_2$.

Also, since $Y_2 > X_1$,then

$Y_3 = B^*B - B^*Y_2B < B^*B - B^*X_1B = X_2$. From (8) we have

$X_0 = \alpha I > B^*B > B^*B - B^*Y_1B = Y_2$. Therefore,

$X_1 = B^*B - B^*X_0B < B^*B - B^*Y_2B = Y_3$ and

$X_2 = B^*B - B^*X_1B > B^*B - B^*Y_3B = Y_4$. Consequently,

$Y_0 > X_0 > Y_2 > X_2 > Y_4 > X_4 > X_3 > Y_3 > X_1 > Y_1$.

We will prove that $Y_0 > X_{s+1} > Y_s$, if we have

$Y_0 > Y_{s-1} > X_s$,

$Y_s = B^*B - B^*Y_{s-1}B < B^*B - B^*X_sB = X_{s+1} < B^*B < \beta I = Y_0$.

Also, we can prove that $Y_0 > X_s > Y_{s+1}$ .if we have

$Y_0 > Y_s > X_{s-1}$,

$Y_{s+1} = B^*B - B^*Y_sB < B^*B - B^*X_{s-1}B = X_s < B^*B < \beta I = Y_0$.

Therefore, we have

$Y_0 > X_0 > Y_{2r} > X_{2r} > Y_{2r+2} > X_{2r+2} > X_{2s+3} > Y_{2s+3} > X_{2s+1} > Y_{2s+1} > X_1 > Y_1$,

for every positive integers r,s .

Finally, to prove that the subsequences $\{Y_{2r}\},\{Y_{2s+1}\}$ have the same limit, we have

$\|Y_{2s} - Y_{2s+1}\| = \|B^*B - B^*Y_{2s-1}B - B^*B + B^*Y_{2s}B\| = \|B^*(Y_{2s} - Y_{2s-1})B\| \le \|B\|^2 \|Y_{2s} - Y_{2s-1}\|$.

From (9) , let $q = \|B\|^2 < 1$, and we get

$\|Y_{2s} - Y_{2s+1}\| \le q\|Y_{2s} - Y_{2s-1}\| \le \ldots \le q^{2s}\|Y_0 - Y_1\| < q^{2s}(2\beta - 1)$.

Therefore, we have for the limit X of the subsequences $\{Y_{2r}\},\{Y_{2s+1}\}$,

$Y_{2s} > X > Y_{2s+1}, \qquad s = 1,2,\ldots.$.

Therefore, also we have for the limit X of the subsequences $\{X_{2r}\},\{X_{2s+1}\}$

$Y_{2s} > X_{2s} > X > X_{2s+1} > Y_{2s+1}, \qquad s = 1,2,\ldots.$.

Taking the limit of (8) as $s \to \infty$, we have

$X = B^*(I - X)B$.

If $X_k > 0$ and $Y_k > 0$ for every k . We proved that the sequences have the same limit X . Since

$X_{k+1} = B^*(I - X_k)B > 0$ and $Y_{k+1} = B^*(I - Y_k)B > 0$ and

hence $X = B^*(I - X)B$, equation (1) has a positive definite solution.

2) Theorem

For the *Algorithm C*, if there exist positive numbers α and β such that $0 < \alpha < \beta$ and the following two conditions are hold

(i) $B^*B < \alpha I$ and $B^*B < \beta I$,

(ii) $q = \|B\|^2 < 1$,

then $\|X_k - X\| < q^k(2\alpha - 1)$, $\|Y_k - X\| < q^k(2\beta - 1)$ and $\|X_k - X\| < \|Y_k - X\| < q^k(2\beta - 1)$,. where X is a positive definite solution of (1) and $X_k.Y_k, k = 0,1,2,\ldots$ is defined in *Algorithm C*.

Proof. From *Theorem 1,* it follows that the sequence (8) is convergent to a positive definite solution X of (1). We compute the norms of the matrix $X_k - X$ and $Y_k - X$. We obtain

$$\|X_k - X\| = \|B^*B - B^*X_{k-1}B - B^*B + B^*XB\| = \|B^*(X - X_{k-1})B\|$$
$$\le \|B\|^2\|X - X_{k-1}\| \le q\|X - X_{k-1}\| \le \ldots \le q^k\|X_0 - X\| < q^k(2\alpha - 1).$$

Similarly, $\|Y_k - X\| \le q^k\|Y_0 - X\| < q^k(2\beta - 1)$. Also, we have $\|X_k - X\| < \|Y_k - X\|$. Therefore,

$\|X_k - X\| < \|Y_k - X\| < q^k(2\beta - 1)$.

3) Corollary

Suppose that(1) has a solution. If $q = \|B\|^2 < 1$, then $\{X_k\}$ and $\{Y_k\}$ converge to X with at least the linear convergence rate.

4) Theorem

If (1) has a positive definite Solution and after k iterative steps of the *Algorithm C*, we have $\|I - X_k^{-1}X_{k-1}\| < \varepsilon$ and $\|I - Y_k^{-1}Y_{k-1}\| < \varepsilon$, then

(i) $\|X_k + B^*X_kB - B^*B\| < \alpha\varepsilon\|B\|^2$ and

$\|Y_k + B^*Y_kB - B^*B\| < \beta\varepsilon\|B\|^2$.

(ii) $\|X_k + B^*X_kB - B^*B\| < \|Y_k + B^*Y_kB - B^*B\|$

$< \beta\varepsilon\|B\|^2$.

Where $X_k, Y_k, k = 0,1,2,\ldots$ are the iterates generated by *Algorithm C* and $\varepsilon > 0$.

Proof

(i) Since,

$$X_k + B^*X_kB - B^*B = X_k + B^*X_kB - X_k - B^*X_{k-1}B$$
$$= B^*(X_k - X_{k-1})B.$$

Take the norms of both sides,

$$\|X_k + B^*X_kB - B^*B\| \le \|B\|^2\|X_k - X_{k-1}\|$$
$$\le \|B\|^2\|X_k\|\|I - X_k^{-1}X_{k-1}\| < \alpha\varepsilon\|B\|^2.$$

Similarly, $\|Y_k + B^*Y_kB - B^*B\| < \beta\varepsilon\|B\|^2$.

(ii) From *Theorem 1*, we have $\|X_{k+1} - X_k\| < \|Y_{k+1} - Y_k\|$.

Since

$$\|X_{k+1} - X_k\| = \|B^*B - B^*X_kB - X_k\| = \|X_k + B^*X_kB - B^*B\|$$
$$\|Y_{k+1} - Y_k\| = \|B^*B - B^*Y_kB - Y_k\| = \|Y_k + B^*Y_kB - B^*B\|,$$

thus $\|X_k + B^*X_kB - B^*B\| < \|Y_k + B^*Y_kB - B^*B\| < \beta\varepsilon\|B\|^2$.

V. NUMERICAL EXPERIMENTS

In this section the numerical experiments are used to display the flexibility of the methods. The solutions are computed for some different matrices A with different sizes n. For the following examples, practical stopping criterion $\|X - X_k\| \le 10^{-9}$ and obtains the maximal solution $X = X_{500}$.

A. Numerical experiments for the first method (Algorithm A)

In the following tables we denote

$q=\|A\|^2$, $\varepsilon_1(X)=\|X-X_k\|$, $\varepsilon_2(X)=\|X_k+A^*X_kA-I\|$, where X the solution which is obtained by the iterative method *(Algorithm A)*

I. Example

Let $\alpha = 10$ and

$$A=\frac{1}{100}\begin{pmatrix} 1 & 0.5542 & 0.6684 & 9.9700 & 0.2218 & 0.1120 \\ 0.5542 & 1 & 1.1270 & 3.2320 & 1.1440 & 1.1880 \\ 0.6684 & 1.1270 & 1 & 1.1570 & 2.3180 & 0.1430 \\ 9.9700 & 3.2320 & 1.1570 & 1 & 0.9990 & 0.8855 \\ 0.2218 & 1.1440 & 2.3180 & 0.9990 & 1 & 0.1287 \\ 0.1120 & 1.1880 & 0.1430 & 0.8855 & 0.1287 & 1 \end{pmatrix}$$

$\|A\| = 0.121701$, $q = 0.0148111 < 1$, see Table I.

B. Numerical experiments for the second method (Algorithm B):

The following tables denotes

$q=\|A\|^2, \varepsilon_1(X)=\|X-X_k\|, \varepsilon_2(X)=\|Y-Y_k\|, \varepsilon_3(X)=\|X_k-Y_k\|,$

$\varepsilon_4(X)=\|X_k+A^*X_kA-I\|$, $\varepsilon_5(X)=\|Y_k+A^*Y_kA-I\|$, X and Y are the solutions which are obtained by the iterative method *(Algorithm B)*.

II. Example

Let $\alpha = 2$, $\beta = 3$ and

$$A=\frac{1}{1000}\begin{pmatrix} -1.274 & -0.5755 & 2.384 & 4.118 & -0.1482 \\ -0.5755 & -3.221 & -0.9663 & 5.737 & 6.286 \\ -0.8774 & 6.286 & -0.9663 & 5.737 & -3.221 \\ 2.384 & -0.5755 & -0.1482 & 4.118 & 1.274 \\ 0.8774 & 5.737 & -0.9663 & -0.5755 & -2.384 \end{pmatrix}$$

$\|A\| = 0.0112205$, $q = 0.000125899 < 1$, see Table II.

C. Numerical experiments for the second method (Algorithm C)

The following tables denotes

$q=\|B\|^2, \varepsilon_1(X)=\|X-X_k\|, \varepsilon_2(X)=\|Y-Y_k\|, \varepsilon_3(X)=\|X_k-Y_k\|,$

$\varepsilon_4(X)=\|X_k+B^*X_kB-B^*B\|$, $\varepsilon_5(X)=\|Y_k+B^*Y_kB-B^*B\|$, X and Y are the solutions obtained by the iterative method *(Algorithm C)*.

III. Example

Let $\alpha = 5$, $\beta = 7$ and

$$A=\begin{pmatrix} -12.74 & 5.755 \\ -5.755 & -32.21 \end{pmatrix},$$

$\|A\| = 32.9351$, $\|B\| = 0.0726747$ and $q = 0.0052816 < 1$, see Table III.

VI. CONCLUSIONS

In this paper, the positive definite solution of the linear matrix equation $X + A^*XA = I$, which is a special case of the symmetric matrix equations (2) for $\|A\| \ne 1$ was obtained. Two effective iterative methods for computing a positive definite solution of this equation were proposed. The first one is fixed point iteration method when $\|A\| < 1$ and the second one is two sided iteration method of the fixed point iteration method when $\|A\| < 1$ and $\|A\| > 1$. By *Algorithm A*, for initial matrix $X_0 = \alpha I$ and *Algorithms B and C*, for initial matrices $X_0 = \alpha I, Y_0 = \beta I$ satisfying the hypothesis of *theorems (A.1)* in chapter III, *(B.1)* and *(C.1)* in chapter IV, a positive definite solution X can be obtained in finite iteration, with at least the linear convergence rate. The given numerical examples show that the proposed iterative algorithms are efficient.

REFERENCES

[1] Anderson, W., Morley, T., and Trapp, G. Positive solution to $X = A - BX^{-1}B^*$. Linear Algebra and its Applications, 134 (1990) 53-62.

[2] Bing, Z., Hu Lijuan, Y., and Dragana, S. The congruence class of the solutions of the some matrix equations. Computers and Mathematics with Applications. 57 (2009) 540-549.

[3] Bouhamidi, A., and Jbilou, K. Anote on the numerical approximate solutions for generalized Sylvester matrix equations with applications. Applied Mathematics and Computation. 206 (2008) 687-694.

[4] El-Sayed ,M. S. An Algorithm for Computing the Extremal Positive Definite Solutions of a Matrix Equation $X + A^TX^{-1}A = I$. International Journal of Computer Mathematics. 80 (2003) 1527-1534.

[5] El-Sayed ,M. S. On the positive definite solutions for a nonlinear matrix equation. International Mathematical Journal,4 (2003) 27-42.

[6] Engwerda, J. On the existence of the positive definite solution of the matrix equation $X + A^TX^{-1}A = I$. Linear Algebra and its Applications. 194 (1993) 91-108.

[7] Guo, C., and Lancaster, P. Iterative solution of two matrix equations. Mathematics of Computations. 68 (1999) 1589-1603.

[8] Hua, D., and Lancaster, P. Linear Matrix Equation From an Inverse Problem of Vibration Theory. Linear Algebra and its Applications. 246 (1996) 31-47.

[9] Hung, L. The explicit solution and solvability of the linear matrix equations. Linear Algebra and its Applications. 311 (2000) 195-199.

[10] Jiang, J., Liu, H., and Yuan, Y. Iterative solutions to some linear matrix equations. World Academy of Science, Engineering and Technology 76 (2011), 925-930.

[11] Konghua, G., Hu, X., and Zhang, L. Anew iteration method for the matrix equation $AX = B^*$. Applied Mathematics and Computation. 178 (2007) 1434-1441.

Meini, B. Efficient computation of the extreme solutions of $X + A^*X^{-1}A = Q$ and $X - A^*X^{-1}A = Q$ .2002. Mathematics of Computation, 239 (2002) 1189-1204.

[12] Mirko, D. On minimal solution of the matrix equation $AX - YB = 0$. Linear Algebra and its Applications. 325 (2001) 81-99.

[13] Li, Z.-Y., Zhou, B., Wang, Y., and Duan, G.-R. Numerical solution to linear matrix equation by finite steps iteration. IET Control Theory & Application, 4 (7) (2010) 1245 – 1253.

[14] Peng, X., Hu, Y. and Zhang, L. An iteration method for symmetric solutions and the optimal approximation solution of the matrix equation $AXB = C^*$. Applied Mathematics and Computation, 160 (2005) 763-777.

[15] Ran, A. and Beurings, M. The symmetric linear matrix equation. Electronic Journal of Linear Algebra. 9 (2002) 93–107.

[16] El-Sayed, M. S. Two sided iteration methods for computing positive definite solutions of a nonlinear matrix equation. Journal of Australian Mathematical Society, Series B, 44 (2003), 145-152.

[17] Stykel, T. Stability and Inertia Theorems for Generalized Lyapunov

equations. Linear Algebra and its Applications. 355 (2002) 297-314.

[18] Wang, M., Cheng, X., and Wei, M . Iterative algorithms for solving the matrix equation $AXB + CX^T D = E^*$. Applied Mathematics and Computation. 187 (2007) 622-629.

[19] Vorontsov, Yu. O. Numerical algorithm for solving the matrix equation $AX + X^* B = C$. Moscow University Computational Mathematics and Cybernetics. 37(1) (2013), 1-7.

[20] Vorontsov, Yu. O. Numerical algorithm for solving the matrix equation $AX + X^* B = C$. Moscow University Computational Mathematics and Cybernetics. 37(1) (2013), 1-7.

[21] Yong, H. Ranks of solutions of the linear matrix equation $AX + YB = C$. Computers and Mathematics with Applications. 52 (2006) 861-872.

[22] Zarea, S.A., and El-Sayed , M. S. On the Matrix Equation X + A* X^{-a} A = I, International Journal of Computational Mathematics and Numerical Simulation, **1(1) (2008)** 89-97.

[23] Zarea, S.A., and El-Sayed , M. S., and Al-Eidan, A.A. Iterative Algorithms for the Nonlinear Matrix Equation $X + A^* X^{-r} A = I$. Assiut University Journal of Mathematics & Computer Science. 38(1) (2009) 11 – 24.

[24] Zhang, X. The general common Hermitian nonnegative-definite solution to the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$ with Applications in statistics. Journal of Multivariate Analysis,93 (2005) 257-266.

TABLE I
EXAMPLE I

k	ε_1	q^k	$q^k(2\alpha-1)$	ε_2
0	9.01459	1.	19.	9.11397
1	1.33516E-01	1.48111E-02	2.81411E-01	1.04486E-01
2	1.97752E-03	2.19369E-04	4.16801E-03	1.35647E-03
3	2.92893E-05	3.2491E-06	6.17329E-05	1.90955E-05
4	4.33808E-07	4.81228E-08	9.14333E-07	2.77617E-07
5	6.42517E-09	7.12752E-10	1.35423E-08	4.08379E-09
6	9.5164E-11	1.05567E-11	2.00577E-10	6.03296E-11

TABLE II
EXAMPLE II

k	ε_1	q^k	$q^k(2\alpha-1)$	ε_4
0	1.00013	1.0	3.0	1.00025
1	1.25911E-04	1.25899E-04	3.77698E-04	1.25922E-04
2	1.39267E-08	1.58507E-08	4.7552E-08	1.39278E-08
3	1.07697E-12	1.99559E-12	5.98677E-12	1.07704E-12
k	ε_2	ε_3	$q^k(2\beta-1)$	ε_5
0	2.00013	1.0	5.0	2.00038
1	2.5181E-04	1.25899E-04	6.29497E-04	2.51832E-04
2	2.78524E-08	1.39257E-08	7.92533E-08	2.78545E-08
3	2.15383E-12	1.07685E-12	9.97795E-12	2.15399E-12

TABLE III
EXAMPLE III

k	ε_1	q^k	$q^k(2\alpha-1)$	ε_4
0	4.99918	1.0	9.0	5.02206
1	2.63485E-02	5.28161E-03	4.75344E-02	2.6672E-02
2	1.38995E-04	2.78954E-05	2.51058E-04	6.65329E-04
3	7.33231E-07	1.47332E-07	1.32599E-06	7.33359E-04
4	3.86797E-09	7.78151E-10	7.00336E-09	7.32986E-04
5	2.04045E-11	4.10989E-12	3.6989E-11	7.32988E-04
k	ε_2	ε_3	$q^k(2\beta-1)$	ε_5
0	6.99918	2.0	13.0	7.03309
1	3.6899E-02	1.05505E-02	6.86609E-02	3.72743E-02
2	1.94651E-04	5.56563E-05	3.6264E-04	6.40508E-04
3	1.02683E-06	2.93601E-07	1.91532E-06	7.33508E-04
4	5.41679E-09	1.54881E-09	1.0116E-08	7.32985E-04
5	2.85749E-11	8.17037E-12	5.34285E-11	7.32988E-04

1) Date of modification: 14/03/2014

2) Brief description of the changes

Page	Column	Line	Change: From	Change: To
1	1	22	[16]	[15]
1	1	24	[2],[3],[8]-[11],[13]-[16],[18]-[21],[24].	[2],[3],[8]-[11],[13]-[15],[18]-[21],[24].
1	1	31	[7],[12],[17],[22],[23].	[7],[12],[16],[17],[22],[23].
2	1	10	*B. Algorithm*	*A. Algorithm*
3	1	7	*theorem B*	*theorem 1*
3	1	20	1) *Theorem*	1) *Theorem (Shiftting)*
4	2	10	$B^* B < \alpha I, \beta I$	$B^* B < \alpha I$

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