On Positive Definite Solutions of the Linear Matrix Equation $X + A^*XA = I$

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Abstract—Two effective iterative methods are constructed to solve the linear matrix equation of the form $X + A^*XA = I$. Some properties of a positive definite solution of the linear matrix equation are discussed. Necessary and sufficient conditions for existence of a positive definite solution are derived for $\|A\| < 1$ and $\|A\| > 1$. Several numerical examples are given to show the efficiency of the presented iterative methods.


I. INTRODUCTION

Considering the linear matrix equation

$$X + A^*XA = I, \quad (1)$$

with unknown matrix $X$, where $A \in C^{m \times n}$. $I$ is the identity matrix of order $n$. The equation (1) could be viewed as a special case of the symmetric matrix equations

$$X ± A^*XA ± \cdots ± A^kXAk = Q. \quad (2)$$

Where $Q$ is a positive definite matrix [15]. There are many linear matrix equations which were studied by some authors [2],[3],[8],[11],[13],[15],[18],[21],[24]. Two effective iterative methods for computing a positive definite solution of this equation are proposed. The first one is fixed point iteration method and the second one is two sided iteration method of the fixed point iteration method. These two iterative methods are used for computing a positive definite solution of nonlinear matrix equations, see [1],[4]-[7],[12],[16],[17],[22],[23].

This paper aims to find the positive definite solution of the matrix equation (1) for all values of $\|A\| \neq 1$, for this purpose we investigated two iterative methods, the first one is based on fixed point iteration and the second is based on two sided iteration method, also to derive necessary and sufficient conditions for the existence of the solution of equation (1).

Section II describes some properties of positive definite solutions of the equation (1). Section III, presents a first iterative method (Fixed point iteration method) for obtaining the solution of our problem. Also, it presents theorems for obtaining the necessary and sufficient conditions for the existence of a solution of matrix equation (1). Section IV represents the second iterative method (Two sided iteration method of the fixed point iteration method) for obtaining the solution of the problem and theorems for the sufficient conditions for the existence of a positive definite solution of (1). Numerical examples in Section V illustrate the effectiveness of these methods. Conclusion drawn from the results obtained in this paper are in section VI.

The notation $X > 0$ means that $X$ is a positive definite Hermitian matrix and $A \succ B$ is used to indicate that $A - B \succ 0$. $A^*$ denotes the complex conjugate transpose of $A$. Finally, throughout the paper, $\|\|$ will be the spectral norm for square matrices unless otherwise noted.

II. SOME PROPERTIES OF THE SOLUTIONS

This section discusses some properties of positive definite solutions of the matrix equation (1).

1) Theorem

If $m$ and $M$ are the smallest and the largest eigenvalues of a solution $X$ of (1), respectively, and $\lambda$ is an eigenvalue of $A$, then $\sqrt{\frac{1-M}{M}} \leq |\lambda| \leq \sqrt{\frac{1-m}{m}}$.

Proof

Let $v$ be an eigenvector corresponding to an eigen-value $\lambda$ of the matrix $A$ and $\|v\| = 1$. Since the solution $X$ of (1) is a positive definite matrix, then $0 < m \leq M < 1$,

$$\langle (X + A^*XA)v, v \rangle = \langle A^*Av, v \rangle + \langle A^*XAv, v \rangle = \langle A^*Av, v \rangle + \langle A^*XAv, v \rangle = \|v\|^2,$$

Consequently, $\sqrt{\frac{1-M}{M}} \leq |\lambda| \leq \sqrt{\frac{1-m}{m}}$.

2) Theorem

If (1) has a positive definite solution $X$, then $A^*A + (AA^*)^T > I$

Proof

Since $X$ is a positive definite solution of (1), then $X < I, A^*XA < I$, i.e. $X < (AA^*)^T$. 

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Thus we have \((AA')^T > X = I - A'XA > I - A'A\).
Consequently, \(A'A + (AA')^T > I\).

III. THE FIRST ITERATION METHOD (FIXED POINT ITERATION METHOD)

This section establishes the first iterative method which is suitable for obtaining a positive definite solution of (1) when \(\|A\| < 1\).

**A. Algorithm**

Take \(X_0 = ad\). For \(k = 0, 1, 2, \ldots\), compute
\[X_{i+1} = I - A'X_iA.\] (3)

Our theorems give necessary and sufficient conditions for the existence of a positive definite solution of (1).

1) **Theorem**

Let the sequence \(\{X_i\}\) be determined by the Algorithm A and \(\|A\| < 1\) (4)

If (1) has a positive definite solution, then \(\{X_i\}\) converges to \(X\), which is a solution of (1) for all numbers \(\alpha > 1\). Moreover, if \(X_i > 0\) for every \(k\), then (1) has a positive definite solution.

**Proof.**

Let (1) have a positive definite solution. From Algorithm A, we have
\[X_0 = ad > I > X_1 = I - \alpha A'A,\]
\[X_1 = I - A'X_0A > I - A'XA = X_0,\]
\[X_0 > X_1 > X_2.\]

To prove \(X_i > X_j\) if \(X_i > X_j\) for all \(s\), we have
\[X_0 = I - A'X_1A > I - A'XA = X_1,\]
\[X_i > X_{i+1}.\]
We will find the relation between \(X_1, X_2, X_3, \ldots\). Since \(X_1 < X_2\), then \(X_2 = I - A'X_1A > I - A'XA = X_2\) and \(X_1 > X_2\).

From previous Theorem it follows that the sequence \(\{X_i\}\) converges to \(X\) with at least the linear convergence rate.

**Proof.**

We have \(\|X_{i+1} - X\| \leq \alpha \|X_i - X\|\). Choose a real number satisfying \(\|A\| < \alpha < 1\). Since \(X_i \to X\), there exists a \(N\) such that for every \(k \geq N\), \(\|X_i - X\| < \theta\). Hence \(\|X_{i+1} - X\| \leq \theta^k \|X_0 - X\|\).
\[ X_k + A^*X_k A - I = X_k + A^*X_k A - X_k - A^*X_{k-1} A = A^*(X_k - X_{k-1} A). \] (5)

Take the norms of both sides of (5)
\[
\|X_k + A^*X_k A - I| \leq \|A^*X_k A - I\| < \varepsilon A \|\].
For
\[
\|I - A^*X_k A\|, \quad X_k \rightarrow x \text{ as } k \rightarrow \infty.
\]

Consequently, \[
\|I - A^*X_k A\| \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ that is }, \|I - A^*X_k A\| < \varepsilon, \text{ for } \varepsilon > 0 \text{ and from theorem 1, } \|X_k\| \leq \alpha
\]
for every \( k \), thus \[
\|X_k + A^*X_k A - I\| < \varepsilon A \|\].

IV. THE SECOND ITERATION METHOD (TWO SIDED ITERATION METHOD OF THE FIXED POINT ITERATION METHOD)

This section establishes the second iterative method which is suitable for obtaining a positive definite solution of (1).

B. Algorithm

Take \( X_0 = aI, Y_0 = \beta I \). For \( k = 0, 1, 2, ..., \) compute \( X_{k+1} = I - A^*X_k A \) and \( Y_{k+1} = I - A^*Y_k A \) .

(6)

Next theorems provide necessary and sufficient conditions for the existence of a solution of (1) when \( \|A\| < 1 \).

1) Theorem

If (1) has a positive definite solution, the sequences \( \{X_k\} \) and \( \{Y_k\} \) are determined by Algorithm B and
\[
\|A\| < 1,
\]
then the two sequences \( \{X_k\} \), \( \{Y_k\} \) converge to the positive definite solution \( X \) for all real numbers \( a, \beta \) such that \( \beta > a > 0 \). On the other hand, if \( X_k, Y_k > 0 \) for every \( k \), \( \|A\| < 1 \) and \( \beta > a > 0 \), then (1) has a positive definite solution.

Proof

First, considering sequence (6), for \( X_k, X_{k+1} \) we have \( X_k = aI > I \), \( X_{k+1} = I - A^*X_k A > I \). So \( \|A\| < 1 \), \( \|X_k + A^*X_k A - I\| < 1 \).

Theorem 1: \( X_k \) and \( X_{k+1} \) are positive definite solutions of (1). Consequently, the subsequences \( \{X_k\}, \{Y_k\} \) are positive definite solutions of (1).

Finally, from (6) we have \( X_0 = aI > I \), \( Y_0 = \beta I \), and \( \|A\| < 1 \).

Therefore, \( X_k = I - A^*X_k A < I \), \( Y_k = I - A^*Y_k A > I \), for every \( k \).

Consequently, \( X_k, Y_k > 0 \) for every \( k \).

Finally, we can prove that \( X_k \) is the positive definite solution of (1). \( \square \)
2) Theorem.

For the Algorithm $B$, if there exist a positive real numbers $\alpha$ and $\beta$ such that $\beta > \alpha$ and $q = \|A\| < 1$, then

$$
\|X_k - X\| < q \cdot (2\alpha - 1), \quad \|Y_k - X\| < q \cdot (2\beta - 1)
$$

and

$$
\|X_k - X\| < \|Y_k - X\| < q \cdot (2\beta - 1),
$$

where $X$ is a positive definite solution of (1) and $X, Y, k = 0, 1, 2, ...$ are defined in (6).

Proof.

From previous Theorem it follows that the sequence (6) is convergent to a positive definite solution $X$ of (1). We will compute the norm of the matrices $X_k - X$ and $Y_k - X$.

$$
\|X_k - X\| \leq \|A\| \|X_k - X_{k-1}\| \leq q \|X_k - X_{k-1}\| \leq q \|X_0 - X\|
$$

From previous Theorem, we have

$$
\|X_k - X\| \leq q \|X_k - X\| < q \cdot (2\alpha - 1).
$$

Similarly, $\|Y_k - X\| < q \cdot (2\beta - 1)$. Also, we have $\|X_k - X\| < \|Y_k - X\|$. Therefore,

$$
\|X_k - X\| < \|Y_k - X\| < q \cdot (2\beta - 1).
$$

3) Corollary

Suppose that (1) has a solution. If $q = \|A\| < 1$, then $\{X_k\}$ and $\{Y_k\}$ converge to $X$ with at least the linear convergence rate.

4) Theorem

If the (1) has a solution and after $k$-iterative steps of the Algorithm $B$, we have $\|I - X_k^+I\| \leq \epsilon$ and $\|I - Y_k^+Y_k\| \leq \epsilon$. Then $X_k = A(X_k - A - I) < \alpha \epsilon \|A\|$ and $Y_k = A(Y_k - A - I) < \beta \epsilon \|A\|^2$.

Also, $X_k + A^2X_kA - I < Y_k + A^2Y_kA - I < \beta \epsilon \|A\|^2$, where $X_k, Y_k, k = 0, 1, 2, ...$ are defined in (6) and $\epsilon > 0$.

Proof

Since $X_k + A^2X_kA - I = X_k + A^2X_kA - X_k - A^2X_kA = A^2(X_k - X_kA)$, we take the norms of both sides

$$
\|X_k + A^2X_kA - I\| \leq \|A^2\| \|X_k - X_{k-1}\| \leq \alpha \epsilon \|A\|^2
$$

Similarly, $\|A^2Y_kA - Y_k - I\| < \beta \epsilon \|A\|^2$. From Theorem 1, we have $\|X_{k+1} - X_k\| < \|Y_{k+1} - Y_k\|$ and since

$$
\|X_{k+1} - X_k\| = \|A^2X_kA - X_k\| < \|X_k + A^2X_kA - I\|
$$

and

$$
\|Y_{k+1} - Y_k\| = \|A^2Y_kA - Y_k\| < \|Y_k + A^2Y_kA - I\|,
$$

we have $\|X_{k+1} - X_k\| < \beta \epsilon \|A\|^2$.

C. Algorithm

Take $X_0 = aY_0$. For $k = 0, 1, 2, ...$, compute

$$
X_{k+1} = B(I - X_k)B \quad \text{and} \quad Y_{k+1} = B(I - Y_k)B.
$$

Next theorem provides necessary and sufficient conditions for the existence of a solution of (1) when $\|A\| > 1$.

1) Theorem

If (1) has a positive definite solution, the sequences $\{X_k\}$ and $\{Y_k\}$ are determined by the Algorithm $C$ and the inequalities

(i) $B^\dagger B < \alpha \beta$ and $B^\dagger B < \beta l$, $0 < \alpha < \beta$,

(ii) $q = \|A\| < 1$,

are satisfied, then $\{X_k\}, \{Y_k\}$ converge to a positive definite solution $X$. Moreover, if $X_k > 0$ and $Y_k > 0$ for every $k$, $B^\dagger B < \alpha \beta$, $B^\dagger B < \beta l$ and $\alpha, \beta > 0$, then (1) has a positive definite solution.

Proof

First, from Algorithm $C$, we have

$$
X_0 = aY_0 > Y_0 > B^\dagger B = B^\dagger B - aB^\dagger B
$$

and

$$
X_0 = aY_0 > Y_0 > B^\dagger B > B^\dagger B - aB^\dagger B = X_1,
$$

i.e., $X_0 = X_1 > X_2 > X_3 > X_4$. To prove $X_k > Y_k$, if $X_k > X_k$, we will find the relation between $X_k, X_k, X_k$, since $X_k > X_k$, $X_k > X_k$, $X_k > X_k$, $X_k > X_k$.

Also since $X_k > X_k$, we get $X_k = B^\dagger B > B^\dagger B - B^\dagger B = X_k$. Thus, we have $X_k = aY_k > X_k > X_k > X_k > X_k > B^\dagger B - B^\dagger B$.

We will prove that $X_k > X_k$, $X_k$, if we have $X_k > X_k$, $X_k$, thus $X_k = aY_k > X_k > X_k > X_k > X_k > X_k$. Also, we will prove that $X_k > X_k$, $X_k$, if we have $X_k > X_k$, $X_k$, thus $X_k = aY_k > X_k > X_k > X_k > X_k > B^\dagger B - B^\dagger B$.

for every positive integers $r, s$. Consequently, the subsequences $\{X_{r,s}\}$ and $\{Y_{r,s}\}$ are monotone and bounded, and $\lim_{s \to \infty} X_{r,s}$, $\lim_{s \to \infty} Y_{r,s}$ exist.

For the sequence $\{Y_k\}$, similarly

$$
Y_0 > B^\dagger B > Y_0 > Y_0 > Y_0 > Y_0 = (1 - \beta)B^\dagger B,
$$

for every positive integers $r, s$. Consequently, the subsequences $\{Y_{r,s}\}$ and $\{Y_{r,s}\}$ are monotone and bounded, and $\lim_{s \to \infty} Y_{r,s}$, $\lim_{s \to \infty} Y_{r,s}$ exist.
Finally, from (8) we have \( Y_0 = \beta I > \alpha I = X_0 \) and
\[ Y_1 = B' Y_1 + B' Y_1 - B' X_1 B = X_1. \]
i.e. \( Y_1 > X_1 > X_1 > Y_1 \). Also,
\[ Y_1 = B' B' Y_1 + B' B' Y_1 - B' X_1 B = X_1. \]
Similarly,
\[ Y_1 = B' Y_1 > X_1 > Y_1. \]
Since \( Y_1 > X_1, X_1 > Y_1 \), then
\[ X_1 = B' B' Y_1 - B' Y_1 B = Y_1 \]
and
\[ Y_1 = B' B' Y_1 - B' Y_1 B = X_1. \]
Also, since \( Y_1 > X_1 \), then
\[ Y_1 = B' B' Y_1 - B' B' Y_1 B = X_1. \]
(From 8) we have \( X_0 = \alpha I > B' B' Y_1 - B' B' Y_1 B = Y_1 \). Therefore,
\[ X_1 = B' B' Y_1 - B' B' Y_1 B = Y_1 \]
and
\[ X_1 = B' B' Y_1 - B' B' Y_1 B = Y_1. \]
Consequently,
\[ X_1 = B' B' Y_1 - B' B' Y_1 B = Y_1. \]
We will prove that \( Y_0 > X_{i+1} > Y_i, \) if we have
\[ Y_0 > X_{i+1} > Y_i \]
and
\[ Y_1 = B' B' Y_1 - B' B' Y_1 B = Y_1. \]
Also, we can prove that \( Y_0 > X_1 > Y_0 \), if we have
\[ Y_0 > Y_0 > X_1 \]
and
\[ Y_1 = B' B' Y_1 - B' B' Y_1 B = Y_1. \]
From (9), let \( q = \| A \| < 1 \), and we get
\[ Y_0 > X_1 > Y_0 \]
and
\[ Y_1 = B' B' Y_1 - B' B' Y_1 B = Y_1. \]
Therefore, we have
\[ Y_0 > X_1 > Y_0 \]
and
\[ Y_1 = B' B' Y_1 - B' B' Y_1 B = Y_1. \]
Finally, to prove that the subsequence \( \{ Y_{i+1} \} \) has the same limit, we have
\[ Y_{i+1} - Y_{i+1} = B' B' Y_{i+1} - B' B' Y_{i+1} B - B' B' Y_{i+1} B = B' (Y_{i+1} - Y_{i+1}) B \]
and
\[ \| Y_{i+1} - Y_{i+1} \| < \| B' (Y_{i+1} - Y_{i+1}) B \| < \| B' \| \| Y_{i+1} - Y_{i+1} \| \]
Therefore, we have for the limit \( X \) of the subsequences \( \{ Y_{i+1} \} \),
\[ Y_{i+1} = B' B' Y_{i+1} - B' B' Y_{i+1} B = Y_{i+1}. \]
Taking the limit of (8) as \( s \rightarrow \infty \), we have
\[ X = B' (I - X) B. \]
If \( X_1 > 0 \) and \( Y_1 > 0 \) for every \( k \), we proved that the sequences have the same limit \( X \). Since
\[ X_{i+1} = B' (I - X) B > 0 \]
and
\[ X_{i+1} = B' (I - X) B > 0 \]
and hence \( X = B' (I - X) B \), equation (1) has a positive definite solution.

2) Theorem

For the Algorithm C, if there exist positive numbers \( \alpha \) and \( \beta \) such that \( 0 < \alpha < \beta \) and the following two conditions are hold
(i) \( B' B < \alpha I \) and \( B' B < \beta I \),
(ii) \( q = \| A \| < 1 \),
then
\[ \| X - X \| < q^2 (2 \alpha - 1), \]
\[ \| Y_i - X \| < q^2 (2 \beta - 1), \]
and
\[ \| X - X \| < q^2 (2 \beta - 1), \]
where \( X \) is a positive definite solution of (1) and \( X_i, Y_i, k = 0,1,2, \ldots \) is defined in Algorithm C.

Proof. From Theorem 1, it follows that the sequence (8) is convergent to a positive definite solution \( X \) of (1). We compute the norms of the matrix \( X_i - X \) and \( Y_i - X \). We obtain
\[ \| X_i - X \| = \| B' B - B' X_i B - B' B + B' X_i B \| \leq \| B' \| \| X_i - X \| \]
\[ < q^2 (2 \beta - 1). \]
Similarly, \( Y_i - X < q^2 (2 \beta - 1). \)

3) Corollary

Suppose that (1) has a solution. If \( q = \| A \| < 1 \), then \( X, Y_i \) converge to \( X \) with at least the linear convergence rate.

4) Theorem

If (1) has a positive definite Solution and after \( k \) iterative steps of the Algorithm C, we have
\[ \| X_i - X_i \| < \epsilon \]
and
\[ \| Y_i - X_i \| < \epsilon, \]
then
(i) \( \| X_i + B' X_i B - B' \| < \alpha \| B \| \]
and
\[ \| Y_i + B' Y_i B - B' \| < \beta \| B \| \]
(ii) \( \| X_i + B' X_i B - B' \| < \alpha \| B \| \]
and
\[ \| Y_i + B' Y_i B - B' \| < \beta \| B \| \]
otherwise.

Where \( X_i, Y_i, k = 0,1,2, \ldots \) are the iterates generated by Algorithm C and \( \epsilon > 0 \).

Proof

(i) Since,
\[ X_i + B' X_i B - B' B = X_i + B' X_i B - X_i - B' X_i B \]
\[ = B' (X_i - X_i) B \]
Take the norms of both sides,
\[ \| X_i + B' X_i B - B' B \| \leq \| B' \| \| X_i - X_i \| \]
\[ < \alpha \| B \| \]
Similarly, \( \| Y_i + B' Y_i B - B' B \| < \beta \| B \| \)

(ii) From Theorem 1, we have
\[ \| X_i - X_i \| < \| Y_i - X_i \| \]
Since
\[ \| X_i - X_i \| = \| B' B' X_i B - B' B' \| \]
\[ \| Y_i - X_i \| = \| B' B' Y_i B - B' B' \| \]
\[ \| Y_i - X_i \| = \| B' B' Y_i B - B' B' \| \]
\[ \| Y_i - X_i \| = \| B' B' Y_i B - B' B' \| \]
thus \( \| X_i + B' X_i B - B' B \| < \beta \| B \| \)

V. NUMERICAL EXPERIMENTS

In this section the numerical experiments are used to display the flexibility of the methods. The solutions are computed for some different matrices \( A \) with different sizes \( n \). For the following examples, practical stopping
criterion $\|X - X_0\| \leq 10^{-6}$ and obtains the maximal solution $X = X_{100}$.

### A. Numerical experiments for the first method (Algorithm A)

In the following tables we denote

$$q = \|A\|, \quad e_i(X) = \|X_i - X\|, \quad e_i'(X) = \|X_i + A'X_iA - I\|,$$

where $X$ is the solution which is obtained by the iterative method (Algorithm A).

#### I. Example

Let $\alpha = 10$ and

$$A = \begin{bmatrix}
1 & 0.5542 & 0.6684 & 0.9700 & 0.2218 & 0.1120 \\
0 & 1.1270 & 3.2320 & 1.1440 & 1.1880 & 0.00528161 \\
0 & 0.6684 & 1.1270 & 1.1570 & 2.3180 & 0.1430 \\
0 & 0.9700 & 3.2320 & 1.1570 & 1 & 0.8855 \\
0 & 0.2218 & 1.1440 & 2.3180 & 0.9990 & 0.1287 \\
0 & 0.1120 & 1.1880 & 0.1430 & 0.8855 & 0.1287 \\
\end{bmatrix}, \quad q = 0.121701, \quad q_1 = 0.0148111, \text{ see Table I.}
$$

### B. Numerical experiments for the second method (Algorithm B):

The following tables denotes

$$q = \|A\|, \quad e_i(X) = \|X_i - X\|, \quad e_i'(X) = \|X_i - Y_i\|,$$

where $X$ and $Y$ are the solutions which are obtained by the iterative method (Algorithm B).

#### II. Example

Let $\alpha = 2, \beta = 3$ and $A = \frac{1}{1000}$

$$A = \begin{bmatrix}
-1.274 & -0.5755 & 2.384 & 4.118 & -0.1482 \\
-0.5755 & -3.221 & -0.9663 & 5.737 & 6.286 \\
-0.8774 & 6.286 & -0.9663 & 5.737 & -3.221 \\
2.384 & -0.5755 & -0.1482 & 4.118 & 1.274 \\
0.8774 & 5.737 & -0.9663 & -0.5755 & -2.384 \\
\end{bmatrix}, \quad q = 0.0112205, \quad q_1 = 0.00012589 < 1, \text{ see Table II.}
$$

### C. Numerical experiments for the second method (Algorithm C)

The following tables denotes

$$q = \|A\|, \quad e_i(X) = \|X_i - X\|, \quad e_i'(X) = \|X_i - Y_i\|,$$

where $X$ and $Y$ are the solutions obtained by the iterative method (Algorithm C).

#### III. Example

Let $\alpha = 5, \beta = 7$ and

$$A = \begin{bmatrix}
-12.74 & 5.755 \\
-5.755 & -32.21 \\
\end{bmatrix}^*, \quad \|A\| = 32.9351, \quad q = 0.00528, \text{ see Table III.}
$$

### VI. CONCLUSIONS

In this paper, the positive definite solution of the linear matrix equation $X + A'XA = I$, which is a special case of the symmetric matrix equations (2) for $\|A\| \neq 1$ was obtained. Two effective iterative methods for computing a positive definite solution of this equation were proposed. The first one is fixed point iteration method when $\|A\| < 1$ and the second one is two sided iteration method of the fixed point iteration method when $\|A\| < 1$ and $\|A\| > 1$. By Algorithm A, for initial matrix $X_0 = \alpha I$ and Algorithms B and C, for initial matrices $X_0 = \alpha I, Y_0 = \beta I$ satisfying the hypothesis of theorems (A, I) in chapter III, (B, I) and (C, I) in chapter IV, a positive definite solution $X$ can be obtained in finite iteration, with at least the linear convergence rate. The given numerical examples show that the proposed iterative algorithms are efficient.

### REFERENCES


**TABLE I**

<table>
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<th>( k )</th>
<th>( \varepsilon_1 )</th>
<th>( \varepsilon_2 )</th>
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1) Date of modification: 14/03/2014
2) Brief description of the changes

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