The Hybrid Extragradient Method for the Split Feasibility and Fixed Point Problems

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***Abstract*—In this paper, we suggest a hybrid extragradient method for finding a common element of the set of fixed point sets of an infinite family of nonexpansive mappings and the solution set of the split feasibility problem (SFP) in real Hilbert spaces.**

***Index Terms*—Fixed point problems, Split feasibility problem, CQ method, Projection, Strong convergence, Hybrid extragradient method**

I. Introduction

THROUGHOUT this paper, let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C and Q be a nonempty closed convex subset of infinite-dimensional real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point x^* with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1}$$

where $A \in B(H_1, H_2)$ and $B(H_1, H_2)$ denotes the family of all bounded linear operators from H_1 to H_2.

We use Γ to denote the solution set of the (SFP), i.e.,

$$\Gamma = \{x^* \in C : Ax^* \in Q\}.$$

In 1994, the SFP was introduced by Censor and Elfving [1], in finite dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction and many researches; see, e.g., [2–5].

A special case of the SFP is the following convex constrained linear inverse problem [6] of finding an element x such that

$$x \in C \quad \text{such that} \quad Ax = b. \tag{2}$$

This problem, due to its applications in many applied disciplines, has extensively been investigated in the literature ever since Lanweber [7] introduced his iterative method in 1951.

In 2002, Byrne [2] proposed his CQ algorithm to solve (1). The sequence $\{x_n\}$ is generated by the following iteration scheme:

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \quad n \in \mathbb{N}, \tag{3}$$

where $\gamma \in (0, \frac{2}{\lambda})$, with λ being the spectral radius of the operator A^*A.

The variational inequality problem $VI(C, A)$ is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{4}$$

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see, e.g., [8–10]. Let us start with Korpelevich's extragradient method which was introduce by Korpelevich [10] in 1976 and which generates a sequence $\{x_n\}$ via the recursion;

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \quad n \geq 0, \end{cases} \tag{5}$$

where P_C is the metric projection from $\mathbb{R}^n$ onto $C, A : C \to H$ is a monotone operator and λ is a constant. Korpelevich [10] prove that the sequence $\{x_n\}$ converges strongly to a solution of $VI(C, A)$. Note that the setting of the problems in the Euclidean space $\mathbb{R}^n$.

We note that Nadezhkina and Takahashi [11] employed the monotonicity and Lipschitz-continuity of A to define a maximal monotone operator T as follows:

$$Tv = \begin{cases} Av + N_C v & \text{if} \quad v \in C, \\ \emptyset & \text{if} \quad v \notin C. \end{cases} \tag{6}$$

where $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$ is the normal cone to C at $v \in C$ (see, [12]). However, if the mapping A is a pseudomonotone Lipschitz-continuous, then T is not necessarily a maximal monotone operator.

Yu, Yao and Liou [13] introduced a new iterative method as follows:

$$\begin{cases} x_1 = x_0 \in C, \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n)W_n P_C(x_n - \lambda_n Ay_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{7}$$

under their condition, they proved that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to the same point $P_{\cap_{n=1}^{\infty} Fix(S_n)\cap\Omega} x_0$.

Ceng, Ansari and Yao [14] introduce an extragradient method for solving split feasibility and fixed point problems. They propose the following method:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n). \end{cases} \tag{8}$$

They prove that the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same elemaent $\hat{x} \in Fix(S) \cap \Gamma$.

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In 2013, Ceng, Wong and Yao [15] investigate the hybrid extragradient-like iteration algorithm with regularization:

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ z_n = (1-\beta_n-\gamma_n)x_n + \beta_n y_n \\ +\gamma_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 \\ +2\alpha_n \lambda_n k(k + \|y\|)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{9}$$

They prove the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to $q = P_{Fix(S)\cap\Gamma} x_0$.

Motivated and inspired by the works of Nadezhkina and Takahashi [11], Yu, Yao and Liou [13] and Ceng, Ansari and Yao [14], and Ceng, Wong and Yao [15], in this paper we suggest a hybrid shrinking method for finding a common element of the set of solution of the split feasibility problem and common fixed points of an infinite family of nonexpansive mappings.

II. Preliminaries

We write $x_n \to x$ (respectively, $x_n \rightharpoonup x$), the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. Recall that a mapping S defined on C of H is *nonexpansive* if there holds that $\|Sx - Sy\| \le \|x - y\|, \forall x, y \in C$ and the set of fixed points of S by $Fix(S)$.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in H$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \tag{10}$$

Proposition 1. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \le 0, \forall y \in C.$
(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2, \forall y \in C.$
(iii) $\langle P_C x - P_C y, x - y \rangle \ge \|P_C x - P_C y\|^2, \forall y \in H$, *which implies that P_C is nonexpansive and monotone.*

Definition 2. Let T be a nonlinear operator whose domain is $D(T) \subseteq H$ and whose range is $R(T) \subseteq H$.

(i) T is monotone if

$$\langle x - y, Tx - Ty \rangle \ge 0, \quad \forall x, y \in D(T).$$

(ii) Given a number $\beta > 0, T$ is said to be β-strongly monotone if

$$\langle x - y, Tx - Ty \rangle \ge \beta \|x - y\|^2, \quad \forall x, y \in D(T).$$

(iii) Given a number $\nu > 0, T$ is said to be ν-inverse strongly monotone (ν-ism) if

$$\langle x - y, Tx - Ty \rangle \ge \nu \|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

It can be easily seen that if S is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism.

A mapping $T : H \to H$ is said to be be *averaged* mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projection) are $\frac{1}{2}$-averaged maps.

Proposition 3. *[16] Let $T : H \to H$ be a given mapping. Then consider the following.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
(ii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1), T$ is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*
(iii) *The composite of finite many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^n$ is averaged, then so is the composite $T_1 \circ T_2 \circ \ldots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

Recall that a Banach space is said to satisfies the Opial condition [17] ; i.e., for any sequence $\{x_n\}$ in X the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

holds for every $y \in X$ with $y \ne x$. It is well-known that every Hilbert spaces satisfies the Opial condition.

Let $\{S_i\}_{i=1}^\infty$ be an infinite family of nonexpansive mappings of C into itself and let $\{\xi_i\}_{i=1}^\infty$ be real number sequences such that $0 \le \xi_i \le 1$ for every $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into itself as follows:

$$\begin{aligned} &U_{n,n+1} = I, \\ &U_{n,n} = \xi_n S_n U_{n,n+1} + (1 - \xi_n)I, \\ &U_{n,n-1} = \xi_{n-1} S_{n-1} U_{n,n} + (1 - \xi_{n-1})I, \\ &\vdots \\ &U_{n,2} = \xi_2 S_2 U_{n,3} + (1 - \xi_2)I, \\ &W_n = U_{n,1} = \xi_1 S_1 U_{n,2} + (1 - \xi_1)I. \end{aligned} \tag{11}$$

Such W_n us called the W-mapping generated by $\{S_i\}_{i=1}^\infty$ and $\{\xi_i\}_{i=1}^\infty$.

Lemma 4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^\infty Fix(S_n)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_1 \le b < 1$ for any $i \in \mathbb{N}$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Lemma 5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S_1, S_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^\infty Fix(S_n)$ is nonempty, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_1 \le b < 1$ for any $i \in \mathbb{N}$. Then, $Fix(W) = \cap_{n=1}^\infty (S_n)$.*

Lemma 6. *[18] Using Lemmas 4 and 5, one can define a mapping W of C into itself as: $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. If $\{x_n\}$ is a bounded sequence in C, then we have*

$$\lim_{n\to\infty} \|Wx_n - W_n x_n\| = 0.$$

We also need the following well-known lemmas for proving our main results.

Lemma 7. *([19], Demiclosedness Principle). Let C be a nonempty closed and convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping with $Fix(S) \neq \emptyset$. If the sequence $\{x_n\} \subseteq C$ converges weakly to x and the sequence $\{(I-S)x_n\}$ converges strongly to y, then $(I-S)x = y$; in particular, if $y = 0$, then $x \in Fix(S)$.*

Lemma 8. *[20] Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_\omega(x_n) \subset C$ and satisfies the condition $\|x_n - u\| \le \|u - q\|, \ \forall n \in \mathbb{N}$. Then $x_n \to q$.*

Lemma 9. *[21] Let H be a real Hilbert space. Then the following equations hold:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \ \forall x, y \in H;$
(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \ \forall x, y \in H;$
(iii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0,1]$ *and* $x, y \in H$.

Throughout this paper, we assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem:

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{12}$$

is ill-posed. Therefore, (see [5]) consider the following Tikhonov regularize problem:

$$\min_{x \in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{13}$$

where $\alpha > 0$ is the regularization parameter. The regularized minimization (13) has a unique solution which is denoted by x_α.

It is know that x^* is a solution of the SFP if and only if x^* solves the fixed point equation:

$$P_C(I - \lambda\nabla f)x^* = P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*. \tag{14}$$

It is proved in [5, Proposition 3.2].

Lemma 10. *[5] The following hold:*

(i) *$\Gamma = F(P_C(I - \lambda\nabla f)) = VI(C, \nabla f)$ for any $\lambda > 0$, where $F(P_C(I - \lambda\nabla f))$ and $VI(C, \nabla f)$ denoted the set of fixed point of $P_C(I - \lambda\nabla f)$ and the solution set of VIP;*
(ii) *$P_C(I - \lambda\nabla f_\alpha)$ is ξ-averaged for each $\lambda \in (0, \frac{2}{(\alpha + \|A\|^2)})$, where $\xi = \frac{(2 + \lambda(\alpha + \|A\|^2))}{4}$.*

Proposition 11. *[14] There hold the following statement:*

(i) *the gradient*
$$\nabla f_\alpha = \nabla f + \alpha I = A^*(I - P_Q)A + \alpha I$$
is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone;
(ii) *the mapping $P_C(I - \lambda\nabla f_\alpha)$ is a contraction with coefficient*
$$\sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)} (\le \sqrt{1 - \alpha\lambda} \le 1 - \frac{1}{2}\alpha\lambda),$$
where $0 < \lambda \le \frac{\alpha}{(\|A\|^2 + \alpha)^2}$;
(iii) *if the SFP is consistent, then the strong $\lim_{n\to\infty} x_\alpha$ exists and is the minimum norm solution of the SFP.*

III. Main results

Theorem 12. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}_{n=1}^\infty$ be an infinite family of nonexpansive mappings of C into itself such that $\cap_{n=1}^\infty Fix(S_n) \cap \Gamma \neq \emptyset$. Let $x_1 = x_0 \in C$. For $x_1 \in C, C_1 = C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated as*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ z_n = \beta_n x_n + (1-\beta_n) W_n P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 \le \|x_n - z\|^2 \\ +2\alpha_n\lambda_n k(k + \|y\|)\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \ge 1, \end{cases} \tag{15}$$

where $\{W_n : n \ge 1\}$ are W-mappings of (11), $\sup_{p \in \cap_{n=1}^\infty Fix(S_n)\cap\Gamma} \|p\| \le k$ for some $k \ge 0$, and the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{\|A\|^2})$;
(iii) $\{\beta_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$;

then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by (15) converge strongly to the same point $P_{\cap_{n=1}^\infty Fix(S_n)\cap\Gamma} x_0$.

Proof: By Lemma 10 (ii), we get $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged for each $\lambda_n \in (0, \frac{2}{\alpha + \|A\|^2})$, where $\zeta = \frac{2 + \lambda(\alpha + \|A\|^2)}{4} \in (0, 1)$. It is known that $P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \in [a, b]$ with $a, b \in (0, \frac{1}{\|A\|^2})$, $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is ζ_n-averaged with $\zeta_n = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0, 1)$. It is known that $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is nonexpansive for all $n \ge 0$.

Step 1. We will show
(1) Every C_n is closed and convex, $n \ge 1$;
(2) $\cap_{n=1}^\infty Fix(S_n) \cap \Gamma \subset C_{n+1}, \forall n \ge 1$;
(3) $\{x_{n+1}\}$ is well-defined.

First, we note that $C_1 = C$ is closed and convex. Assume that C_k is closed and convex. From (3) and since $C_{k+1} = \{z \in C_k : \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z\rangle \le 2\lambda_k\alpha_k(\|y_k\| + k)\}$. Thus, C_{k+1} is closed and convex. By induction, we deduce that C_n is closed and convex for all $n \ge 1$.

Next, we show that $\cap_{n=1}^\infty Fix(S_n) \cap \Gamma \subset C_{n+1}, \forall n \ge 1$.

Set $g_n = P_C(x_n - \lambda_n\nabla f_{\alpha_n} y_n)$ and $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is nonexpansive for each $n \ge 0$. Pick up $p \in \cap_{n=1}^\infty Fix(S_n) \cap \Gamma$. Then, we get $P_C(I - \lambda\nabla f)p = p$ for $\lambda \in (0, \frac{2}{\|A\|^2})$. From (15)

$$\begin{aligned} \|y_n - p\| &\le \|P_C(I - \lambda_n\nabla f_{\alpha_n})x_n - P_C(I - \lambda_n\nabla f_{\alpha_n})p\| \\ &\quad + \|P_C(I - \lambda_n\nabla f_{\alpha_n})p - P_C(I - \lambda_n\nabla f)p\| \\ &\le \|x_n - p\| + \|(I - \lambda_n\nabla f_{\alpha_n})p - (I - \lambda_n\nabla f)p\| \\ &\le \|x_n - p\| + \alpha_n\lambda_n\|p\|. \end{aligned}$$

Then, by Proposition 1 (ii), we have

$$\begin{aligned} & \|g_n - p\|^2 \\ \le\ & \|x_n - \lambda_n\nabla f_{\alpha_n} y_n - p\|^2 - \|x_n - \lambda_n\nabla f_{\alpha_n} y_n - g_n\|^2 \\ =\ & \|x_n - p\|^2 - \|x_n - g_n\|^2 + 2\lambda_n\langle\nabla f_{\alpha_n} y_n, p - g_n\rangle \\ \le\ & \|x_n - p\|^2 - \|x_n - g_n\|^2 \\ & +2\lambda_n(\langle\nabla f_{\alpha_n} p, p - y_n\rangle + \langle\nabla f_{\alpha_n} y_n, y_n - g_n\rangle) \\ =\ & \|x_n - p\|^2 - \|x_n - g_n\| \\ & +2\lambda_n[\langle(\alpha_n I + \nabla f)p, p - y_n\rangle + \langle\nabla f_{\alpha_n} y_n, y_n - g_n\rangle] \\ \le\ & \|x_n - p\|^2 - \|x_n - g_n\|^2 + 2\lambda_n[\alpha_n\langle p, p - y_n\rangle \\ & +\langle\nabla f_{\alpha_n} y_n, y_n - g_n\rangle] \end{aligned}$$

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$$\begin{aligned}
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - g_n\rangle \\
&\quad -\|y_n - g_n\|^2 + 2\lambda_n[\alpha_n\langle p, p - y_n\rangle \\
&\quad +\langle \nabla f_{\alpha_n} y_n, y_n - g_n\rangle] \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - g_n\|^2 \\
&\quad +2\langle x_n - \lambda_n \nabla f_{\alpha_n} y_n - y_n, g_n - y_n\rangle \\
&\quad +2\lambda_n\alpha_n\langle p, p - y_n\rangle.
\end{aligned}$$

Further, by Proposition 1 (i), we have

$$\begin{aligned}
&\langle x_n - \lambda_n \nabla f_{\alpha_n} y_n - y_n, g_n - y_n\rangle \\
&= \langle x_n - \lambda_n \nabla f_{\alpha_n} x_n - y_n, g_n - y_n\rangle \\
&\quad +\langle \lambda_n \nabla f_{\alpha_n} x_n - \lambda_n \nabla f_{\alpha_n} y_n, g_n - y_n\rangle \\
&\leq \langle \lambda_n \nabla f_{\alpha_n} x_n - \lambda_n \nabla f_{\alpha_n} y_n, g_n - y_n\rangle \\
&\leq \lambda_n \|\nabla f_{\alpha_n} x_n - \nabla f_{\alpha_n} y_n\|\|g_n - y_n\| \\
&\leq \lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|\|g_n - y_n\|.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\|g_n - p\|^2 \qquad (16)\\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - g_n\|^2 \\
&\quad +2\lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|\|g_n - y_n\| \\
&\quad +2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - g_n\|^2 \\
&\quad +\lambda_n^2(\alpha_n + \|A\|^2)^2\|x_n - y_n\|^2 + \|g_n - y_n\|^2 \\
&\quad +2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&= \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\quad +(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1)\|x_n - y_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\|.
\end{aligned}$$

Them, from Lemma 9 (iii), (15) and the last inequality, we conclude that

$$\begin{aligned}
&\|z_n - p\|^2 \\
&= \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|W_n g_n - p\|^2 \\
&\quad -\beta_n(1 - \beta_n)\|x_n - W_n g_n\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|g_n - p\|^2 \\
&\quad -\beta_n(1 - \beta_n)\|x_n - W_n g_n\|^2 \\
&\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)[\|x_n - p\|^2 \\
&\quad +2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\quad +(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1)\|x_n - y_n\|^2] \\
&\quad -\beta_n(1 - \beta_n)\|x_n - W_n g_n\|^2 \\
&= \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\quad -2\beta_n\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\quad +(1 - \beta_n)(\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1)\|x_n - y_n\|^2 \\
&\quad -\beta_n(1 - \beta_n)\|x_n - W_n g_n\|^2 \\
&\leq \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\quad -(1 - \beta_n)(1 - \lambda_n^2(\alpha_n + \|A\|^2)^2)\|x_n - y_n\|^2 \\
&\quad -\beta_n(1 - \beta_n)\|x_n - W_n g_n\|^2 \qquad (17)\\
&\leq \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - y_n\| \\
&\leq \|x_n - p\|^2 + 2\lambda_n\alpha_n k(\|y_n\| + k),
\end{aligned}$$

which implies that $p \in C_{n+1}$. Therefore $\cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma \subset C_{n+1}, \forall n \geq 1$. This implies that $\{x_{n+1}\}$ is well-defined.
Step 2. We will show that the sequences $\{x_n\}, \{z_n\}$ and $\{g_n\}$ are all bounded and $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

From $x_{n+1} = P_{C+1}x_0$ and Proposition 1 (i), we have

$$\langle x_0 - x_{n+1}, x_{n+1} - y\rangle \geq 0, \forall y \in C_{n+1}.$$

Since $\cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma \subset C_{n+1}$, we have

$$\langle x_0 - x_{n+1}, x_{n+1} - p\rangle \geq 0, \forall p \in \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma.$$

So, for $p \in \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma$, we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_{n+1}, x_{n+1} - p\rangle \\
&\leq -\langle x_0 - x_{n+1}, x_0 - x_{n+1}\rangle + \langle x_0 - x_{n+1}, x_0 - p\rangle \\
&\leq -\|x_0 - x_{n+1}\|^2 + \|x_0 - x_{n+1}\|\|x_0 - p\|,
\end{aligned}$$

hence

$$\|x_0 - x_{n+1}\| \leq \|x_0 - p\|, \forall p \in \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma. \quad (18)$$

Therefore $\{x_n\}$ is bounded and so $\{z_n\}$ and $\{g_n\}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0. \quad (19)$$

Hence

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&\leq -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|,
\end{aligned}$$

and therefore $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$. Thus the sequence $\{\|x_n - x_0\|\}$ is a bounded and nonincreasing sequence, so $\lim_{n\to\infty}\|x_n - x_0\|$ exists, that is $\lim_{n\to\infty}\|x_n - x_0\| = m$.

Step 3. We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|x_n - z_n\| = \lim_{n\to\infty}\|x_n - g_n\| = 0$ and $lim_{n\to\infty}\|x_n - W_n x_n\| = \lim_{n\to\infty}\|x_n - W x_n\| = 0$.

It is well know that in Hilbert spaces H, the following identity holds:

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \quad \forall x, y \in H.$$

Therefore,

$$\begin{aligned}
&\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle.
\end{aligned}$$

It follow from (19), we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, so we get $\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \quad (20)$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + 2\alpha_n\lambda_n k(k + \|y\|).$$

Since $\{y_n\}$ is bounded, $\lambda_n \subset [a, b]$ and $\lim_{n\to\infty}\alpha_n = 0$, we deduce from (20) that

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0. \quad (21)$$

Again from (20) and (21) it follows that

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \to 0. \quad (22)$$

For each $p \in \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma$, from (17), we get

$$\begin{aligned} & (1-\beta_n)(1-\lambda_n^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2 \\ & +\beta_n(1-\beta_n)\|x_n-W_ng_n\|^2 \\ \leq\ & \|x_n-p\|^2-\|z_n-p\|^2+2\lambda_n\alpha_n\|p\|\|p-y_n\|. \end{aligned}$$

So, we obtain

$$\begin{aligned} 0 \ <\ & (1-d)(1-b^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2 \\ & +c(1-d)\|x_n-W_ng_n\|^2 \\ \leq\ & (1-\beta_n)(1-\lambda_n^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2 \\ & +\beta_n(1-\beta_n)\|x_n-W_ng_n\|^2 \\ \leq\ & \|x_n-p\|^2-\|z_n-p\|^2+2\lambda_n\alpha_n\|p\|\|p-y_n\| \\ =\ & (\|x_n-p\|+\|z_n-p\|)\|x_n-z_n\| \\ & +2\lambda_n\alpha_n\|p\|\|p-y_n\|. \end{aligned}$$

Since $\|x_n-z_n\|\to 0$, $\alpha_n\to 0$, $[a,b]\in(0,\frac{1}{\|A\|^2})$, we have $1-b^2\|A\|^4>0$, $\{\beta_n\}\subset[c,d]$, we have $0<1-d\leq 1-\beta_n$ and $0<c(1-d)\leq\beta_n(1-\beta_n)$, it follows that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_n-W_ng_n\|=0. \tag{23}$$

Consider

$$\begin{aligned} & \|y_n-g_n\| \\ =\ & \|P_C(x_n-\lambda_n\nabla f_{\alpha_n}x_n)-P_C(x_n-\lambda_n\nabla f_{\alpha_n}y_n)\| \\ \leq\ & \|(x_n-\lambda_n\nabla f_{\alpha_n}x_n)-(x_n-\lambda_n\nabla f_{\alpha_n}y_n)\| \\ =\ & \lambda_n\|\nabla f_{\alpha_n}x_n-\nabla f_{\alpha_n}y_n\| \\ =\ & \lambda_n(\alpha_n+\|A\|^2)\|x_n-y_n\|. \end{aligned}$$

This together with (23) implies that

$$\lim_{n\to\infty}\|y_n-g_n\|=0. \tag{24}$$

From $\|x_n-g_n\|\leq\|x_n-y_n\|+\|y_n-g_n\|$, we also have from (23) and (24)

$$\lim_{n\to 0}\|x_n-g_n\|=0. \tag{25}$$

Since $z_n=\beta_nx_n+(1-\beta_n)W_ng_n$, we have

$$(1-\beta_n)(W_ng_n-g_n)=\beta_n(g_n-x_n)+(z_n-g_n).$$

Then

$$\begin{aligned} & (1-d)\|W_ng_n-g_n\| \\ \leq\ & (1-\beta_n)\|W_ng_n-g_n\| \\ \leq\ & \beta_n\|g_n-x_n\|+\|z_n-g_n\| \\ \leq\ & \beta_n\|g_n-x_n\|+\|z_n-x_n\|+\|x_n-g_n\| \\ =\ & (1+\beta_n)\|g_n-x_n\|+\|z_n-x_n\|, \end{aligned}$$

and from (22) and (25), hence

$$\lim_{n\to\infty}\|W_ng_n-g_n\|=0. \tag{26}$$

Observe that

$$\begin{aligned} & \|x_n-W_nx_n\| \\ \leq\ & \|x_n-g_n\|+\|g_n-W_ng_n\|+\|W_ng_n-W_nx_n\| \\ \leq\ & \|x_n-g_n\|+\|g_n-W_ng_n\|+\|g_n-x_n\| \\ \leq\ & 2\|x_n-g_n\|+\|g_n-W_ng_n\|, \end{aligned}$$

from (25) and (26), we have $\|x_n-W_nx_n\|\to 0$. On the other hand, since $\{x_n\}$ is bounded, from Lemma 6, we have $\lim_{n\to\infty}\|W_nx_n-Wx_n\|=0$. Therefore, we have

$$\lim_{n\to\infty}\|x_n-Wx_n\|=0. \tag{27}$$

Step 4. We claim that $\omega_\omega(x_n)\subset\cap_{n=1}^{\infty}Fix(S_n)\cap\Gamma$, where $\omega_\omega(x_n)$ denotes the ω-limit set of $\{x_n\}$, i.e., $\omega_\omega(x_n):=\{u\in H_1: x_{n_j}\rightharpoonup u$ for some subsequence $\{x_{n_j}\}$ of $\{x_n\}\}$.

Step. 4.1 We will show that $u\in Fix(W)$.

Indeed, since $\{x_n\}$ is bounded, it has a subsequence which converges weakly to some point in C and hence $\omega_\omega(x_n)\neq\emptyset$. Let $u\in\omega_\omega(x_n)$ be arbitrary. Then there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ which converges weakly to u. Since we also have $\lim_{j\to\infty}\|x_{n_j}-Wx_{n_j}\|=0$. Note that, from Lemma 7, it follows that $I-W$ is demiclosed at zero. Thus $u\in Fix(W)$.

Step. 4.2 We will show that $u\in\Gamma$.

Since $\|x_n-g_n\|\to 0$ and $\|y_n-g_n\|\to 0$, it is know that $g_{n_j}\rightharpoonup u$ and $y_{n_j}\rightharpoonup u$.

Let

$$Tv=\begin{cases}\nabla fv+N_Cv & \text{if } v\in C,\\ \emptyset & \text{if } v\notin C,\end{cases}$$

where $N_Cv=\{w\in H_1:\langle v-y,w\rangle\geq 0,\forall y\in C\}$. Then T is maximal monotone and $0\in Tv$ if and only if $v\in VI(C,\nabla f)$; (see [12]) for more details.

Let $(v,w)\in G(T)$, we have $w\in Tv=\nabla fv+N_Cv$, and hence $w-\nabla fv\in N_Cv$. So, we have $\langle v-y,w-\nabla fv\rangle\geq 0,\forall y\in C$. On the other hand, from $g_n=P_C(x_n-\lambda_n\nabla f_{\alpha_n}y_n)$ and $v\in C$, we have $\langle x_n-\lambda_n\nabla f_{\alpha_n}y_n-g_n,g_n-v\rangle\geq 0$, and hence, $\left\langle v-g_n,\frac{g_n-x_n}{\lambda_n}+\nabla f_{\alpha_n}y_n\right\rangle\geq 0$. Therefore, from $w-\nabla fv\in N_Cv$ and $g_{n_j}\in C$, it follows that

$$\begin{aligned} & \langle v-g_{n_j},w\rangle \\ \geq\ & \langle v-g_{n_j},\nabla fv\rangle \\ \geq\ & \langle v-g_{n_j},\nabla fv\rangle-\left\langle v-g_{n_j},\frac{g_{n_j}-x_{n_j}}{\lambda_{n_j}}+\nabla f_{\alpha_{n_j}}y_{n_j}\right\rangle \\ =\ & \langle v-g_{n_j},\nabla fv\rangle-\langle v-g_{n_j},\frac{g_{n_j}-x_{n_j}}{\lambda_{n_j}}+\nabla fy_{n_j}\rangle \\ & -\alpha_{n_j}\langle v-g_{n_j},y_{n_i}\rangle \\ =\ & \langle v-g_{n_j},\nabla fv-\nabla fg_{n_j}\rangle+\langle v-g_{n_j},\nabla fg_{n_j}-\nabla fy_{n_j}\rangle \\ & -\left\langle v-g_{n_j},\frac{g_{n_j}-x_{n_j}}{\lambda_{n_j}}\right\rangle-\alpha_{n_j}\langle v-g_{n_j},y_{n_j}\rangle \\ \leq\ & \langle v-g_{n_j},\nabla fg_{n_j}-\nabla fy_{n_j}\rangle-\left\langle v-g_{n_j},\frac{g_{n_j}-x_{n_j}}{\lambda_{n_j}}\right\rangle \\ & -\alpha_{n_j}\langle v-g_{n_j},y_{n_j}\rangle. \end{aligned}$$

So, we obtain $\langle v-u,w\rangle\geq 0$, as $j\to\infty$. Since T is maximal monotone, we have $u\in T^{-1}0$, and hence $u\in VI(C,\nabla f)$. Therefore, by Lemma 10 (i), it is clear that $u\in\Gamma$. Consequently $u\in\cap_{n=1}^{\infty}Fix(S_n)\cap\Gamma$. That is $\omega_\omega(x_n)\subset\cap_{n=1}^{\infty}Fix(S_n)\cap\Gamma$.

Step 5. We show that $\{x_n\},\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{\cap_{n=1}^{\infty}Fix(S_n)\cap\Gamma}x_0$.

In (18), if we take $p=P_{\cap_{n=1}^{\infty}Fix(S_n)\cap\Gamma}x_0$, we get

$$\|x_0-x_{n+1}\|\leq\|x_0-P_{\cap_{n=1}^{\infty}Fix(S_n)\cap\Gamma}x_0\|. \tag{28}$$

Notice that $\omega_\omega(x_n) \subset \cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma$. Then, (28) and Lemma 8 ensure the strong convergence of $\{x_{n+1}\}$ to $P_{\cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma} x_0$. Consequently, $\{y_n\}$ and $\{z_n\}$ also converge strongly to $P_{\cap_{n=1}^{\infty} Fix(S_n) \cap \Gamma} x_0$. This completes the proof. ■

Taking $W_n = S$, one finds the following result:

Corollary 13. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping of C into itself such that $Fix(S) \cap \Gamma \neq \emptyset$. Let $x_1 = x_0 \in C$. For $x_1 \in C, C_1 = C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences generated as*

$$\begin{cases} y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n} x_n), \\ z_n = \beta_n x_n + (1-\beta_n) S P_C(x_n - \lambda_n \nabla f_{\alpha_n} y_n), \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 \\ +2\alpha_n \lambda_n k(k + \|y\|)\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases} \tag{29}$$

where $\sup_{p \in \cap_{n=1}^{\infty} Fix(S) \cap \Gamma} \|p\| \leq k$ for some $k \geq 0$, and the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{\|A\|^2})$;
(iii) $\{\beta_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$;

then the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ generated by (29) converge strongly to the same point $P_{Fix(S) \cap \Gamma} x_0$.

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