Review of Oscillator Phase Noise Models

Suhas Vishwasrao Shinde

Abstract— **Spectral purity of an oscillator, quantified in terms of phase noise is the most critical and important parameter for RF/ wireless communication systems and digital electronic systems. Over years most compact and insightful phase noise models have been developed. Some of them are pure mathematical physics, some are CAD oriented and some are design oriented. This paper reviews those phase noise models of oscillator and discusses historical challenges, similarities, differences and limitations. It starts with the earliest and most outstanding work by D. B. Leeson. Further it overviews phase noise analysis and modelling by B. Razavi, Ali Hajimiri, Alper Demir and Donhee Ham. This review adds to better understanding of phase noise phenomena from mathematics, physics, design, and simulation point of view.**

Keywords—**Q-factor, LTI, LTV, cyclostationarity, PSD, SNR, ISF, PPV, Injection locking, ergodicity.**

I. Introduction

Oscillators are the key building blocks in almost all of today's digital electronic systems and RF communication systems. More specifically used in phase locked loops, clock recovery circuits and frequency synthesizers. An oscillator being fundamentally nonlinear system their behavior is hard to analyze. One of the most important and stringent performance characteristics of an oscillator is the way its phase varies due to both deterministic and random noise sources. This characteristic is expressed in terms of timing jitter or phase noise. Phase noise and timing jitter are two equivalent definitions of oscillator's short term frequency instabilities. These two definitions are closely related and their values can sometimes be derived from each other. Phase noise is viewed as frequency domain counterpart of timing jitter and has stringent requirements in wireless communication. In recent years, much research has been devoted to the analysis of oscillator phase behavior [5-9][13].

Phase noise study of an oscillator considers its transfer function as linear with respect to infinitesimally small input noise perturbations to output phase. Furthermore transfer function varies based upon time instant of injected noise. This gives rise to more accurate linear time variant (LTV) approach. Traditional linear time invariant (LTI) analysis has some limitations for phase noise studies which are overcome by this new approach. In an oscillator, low frequency noise can be up-converted to the carrier vicinity and noise can also be aliased from the frequencies close to the harmonics of the carrier. This makes the phase noise study historically difficult and challenging. Additionally, the biasing conditions of devices change periodically, hence device noise, which contributes to circuit phase noise, are modulated in the same manner. Therefore it's no longer stationary. Its statistical properties repeat periodically and it becomes cyclostationary [11] further complicating the analysis.

All these difficulties, challenges, different historical phase noise models/theories motivate to write this survey based paper. In this paper existing phase noise models are reviewed. Some are LTI and others are more complicated models. We will discuss most well known Leeson's model in section II. B. Razavi's model in section III, Ali Hajimiri's model in section IV and most accurate Alper Demir's model in section V and Donhee Ham's model in section VI. Comparison of all these existing phase noise models for their similarities, differences and limitations is in table I. finally section VII gives brief summary.

Manuscript received December 17, 2013; revised January 27, 2014.
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II. Leeson's model

The most well-known phase noise model is Leeson's model which was proposed by D. B. Leeson in 1966 [1]. He presented a derivation of the expected spectrum of a feedback oscillator in terms of known oscillator parameters without any formal proof.

Leeson's model is expressed by equation (1) where $S_{\Delta\theta}(\Delta\omega)$ is the input phase noise spectrum and is given by equation (2). This input phase noise spectrum is expected to have two regions. One region is due to the additive white noise, $2FKT/P_s$ at frequencies around the oscillator frequency. The second region is due to $\alpha/\Delta\omega$ introduced by parameter variation at low frequencies. This includes both white noise and flicker noise which has a power spectral density inversely proportional to frequency. This can be seen from equation (2) by $\Delta\omega$ in denominator. In equations (1) and (2) ω_0 is the frequency of oscillation, Q is loaded quality factor such that $Q = \omega_0/2B$. Here B is half bandwidth of oscillator. $\Delta\omega$ is the offset frequency of interest, α is a constant determined by the flicker noise level, F is an empirical parameter (often called as excess noise number), K is the Boltzman's constant, T is the absolute temperature, and P_s is the signal power.

$$L(\Delta\omega) = 10 \cdot \log\left[S_{\Delta\theta}(\Delta\omega)\left[\left(\frac{\omega_0}{2Q\Delta\omega}\right)^2\right]\right] \quad (1)$$

$$S_{\Delta\theta}(\Delta\omega) = \frac{\alpha}{\Delta\omega} + \frac{2FKT}{P_s} \quad (2)$$

Although it seems intuitively true, Leeson's model was proposed without proof considering the effects of the input noise in two regions as shown by equations (1), (2) [1] and fig. 1a). When $\Delta\omega << \omega_0/2Q$, the input noise causes the same spectrum of the frequency variation with a multiplication factor of $(\omega_0/2Q)^2$. Hence the phase noise is multiplied by a factor of $(\omega_0/2Q\Delta\omega)^2$ where the $\Delta\omega$ in the denominator is

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)
(revised on 4 August 2014)

because the phase is the integral of frequency resulting, $S_\Phi'(\Delta\omega) = (\Delta\omega)^2 S_\Phi(\Delta\omega)$ where $S_\Phi'(\Delta\omega)$ is power spectrum of frequency and $S_\Phi(\Delta\omega)$ is power spectrum of phase. When $\Delta\omega >> \omega_0/2Q$, the phase noise is the same of the input noise. At resonance the impedance of the RLC tank is approximated by equation (3)[17]. As shown in fig. 1b) power spectral density (PSD) of resistor noise current is given by $\overline{i_n^2}/\Delta f = 4KT/R$. This noise current through the LC tank generates noise voltage v_n across it. The RMS value of the signal voltage is $V_{s,RMS}$ and its average power dissipation within one cycle is $P_s = V^2_{s,RMS}/R$. Therefore noise to signal ratio at an offset frequency of $\Delta\omega$ is given by equation (4). The quality of signal in communication systems is quantified by a signal to noise ratio (SNR). However for phase noise analysis noise to signal ratio is more relevant as seen from equation (5). Hence we drive equation (4) with linearity assumption where only additive noise in the vicinity of ω_0 is considered.

$$Z[j(\omega_0 + \Delta\omega)] = -jR \cdot \frac{\omega_0}{2Q\Delta\omega} \tag{3}$$

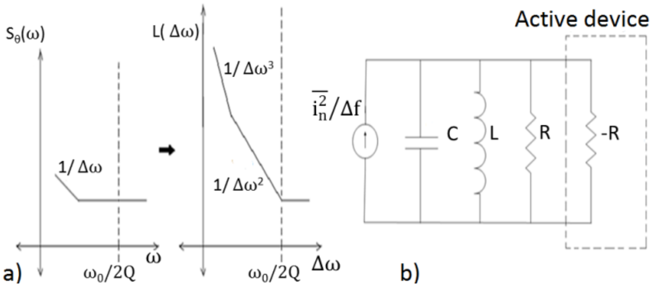

Fig. 1 a) Power spectral density plots for input device noise and phase noise and b) Equivalent one port circuit for LC oscillator [1][17].

$$\frac{\overline{v_n^2}/\Delta f}{v_{s,RMS}^2} = \frac{\left(\overline{i_n^2}/\Delta f\right) \cdot |Z\{j(\omega_0 + \Delta\omega)\}|}{v_{s,RMS}^2}$$

$$= \frac{4KT}{P_s}\left(\frac{\omega_0}{2Q\Delta\omega}\right)^2 \tag{4}$$

Carrier frequency, ω_0 for radio systems is usually very high hence for oscillators, flicker noise is negligible at ω_0, and only thermal noise has to be considered. Thus from derivation of equation (4)[17] the phase noise expression can be given by [17],

$$L(\Delta\omega) = 10 \cdot \log\left[\frac{1}{2} \cdot \frac{\overline{v_n^2}/\Delta f}{v_{s,RMS}^2}\right] = 10 \cdot \log\left[\frac{2KT}{P_s}\left(\frac{\omega_0}{2Q\Delta\omega}\right)^2\right] \tag{5}$$

The factor of ½ in the numerator of this equation arises from neglecting the contribution of amplitude noise, which is assumed to be suppressed. This is true in general for all CMOS oscillators.

In equation (5)[17] we have only considered the thermal noise generated by the resistor R. In practice there is also noise from the active devices which is modeled by '-R'. Hence combining all the noise sources in to one effective noise source, expressed in terms of the resistor noise with multiplicative factor, F, known as excess noise number, the close-in phase noise can be rewritten as [17],

$$L(\Delta\omega) = 10 \cdot \log\left[\frac{2FKT}{P_s}\left(\frac{\omega_0}{2Q\Delta\omega}\right)^2\right] \tag{6}$$

It is difficult to calculate F a priori because it is a posterior fitting parameter derived from measured data. Equation (6) [17] is consistent with equation (1) in Leeson's model for $\Delta\omega << \omega_0/2Q$. Within the bandwidth around oscillator frequency, the frequency selectivity of the LC tank is very weak hence any in-band noise can vary the frequency of oscillation. Since the phase is the integral of frequency over time, their relationship in the frequency domain is $\phi(s) = \Omega(s)/s$. Hence there is a region for the offset frequencies $\Delta\omega << \omega_0/2Q$ where the thermal noise induced phase noise power spectral density is proportional to $1/\Delta\omega^2$. The power spectral density plots for the input device noise and the phase noise are illustrated in fig. 1a). It should be noted that the flat portion does not extend forever; otherwise phase noise would have infinite mean-square. In practice the curve breaks at some cut-off frequency [12]. Due to frequency modulation within this region of $\Delta\omega << \omega_0/2Q$, the white thermal noise creates a phase noise slope of $1/\Delta\omega^2$, and the flicker noise creates a phase noise slope of $1/\Delta\omega^3$. In $\Delta\omega >> \omega_0/2Q$ region, the strong frequency selectivity of the LC tank prevents the oscillator from drifting, causing only instantaneous phase variation due to noise. Thus the phase noise follows the white spectrum of the thermal noise because of the phase modulation mechanism. Leeson's phase noise model predicts the phase noise spectrum which includes $1/\Delta\omega^3$, $1/\Delta\omega^2$ and the white region. One of the misconceptions here needs clarification that the device's 1/f to noise floor corner of $S_n(\omega)$ does not necessarily coincide with the actual $1/f^3$ corner of $L(\Delta\omega)$. This model is verified by numerous measurement results. It is written in a simple mathematical form easy to use and understand. One of the drawbacks of this model is that it contains an empirical factor F hence it cannot predict phase noise from circuit noise analysis. Thus pre-silicon it does not provide any clear direction for circuit improvement. However once the phase noise for one oscillator is characterized using calculated F factor from measured silicon data, phase noise for other oscillators of the same circuit topology can be calculated by applying the same F factor.

Leeson's model approach has been extended by accounting for the individual noise sources in the tuned tank oscillator model [2]. Unfortunately this approach also assumes linear time invariance and represents no fundamental improvement. Thus other drawback of this model is assumption of time invariance.

III. Razavi's model

Razavi's model for noise analysis considers an oscillator as two port LTI feedback system. This is in contrast with one port viewed in Leeson's model. Razavi's model is as shown in fig. 2a). Therefore the transfer function of this model is given by [4],

$$\frac{Y}{X}(j\omega) = \frac{H(j\omega)}{1 + H(j\omega)} \tag{7}$$

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)
(revised on 4 August 2014)

The only way the feedback system in fig. 2a) produces finite output without input is for the denominator to be zero. i.e. $H(j\omega_0) = -1$, also referred as Barkhausen's criteria. In LC oscillator, $H(j\omega)$ is the frequency selective block which consists of RLC network as shown in fig. 2b). The closed loop phase response of this tuned RLC tank feedback system is as shown in fig. 2c). Here the Q is proportional to the slope of the phase transfer function as shown in equation (9)[4]. From fig. 2a) and equation (7), total phase shift around the loop has to be integral multiple of 2π to maintain sustained oscillations. Due to various noise sources in the circuit, temporary phase lead or lag is introduced which needs to be compensated by change in frequency according to fig. 2c) and equation (9). In an oscillator with large Q, the required instantaneous change in frequency for given phase shift is smaller, thus giving better frequency stability. This observation is in perfect alignment with the tank's loaded Q in denominator of equation (6) and (10)[4].

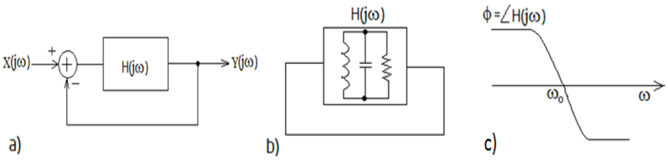

Fig. 2 a) Two port LTI feedback system model of an oscillator and b) Tuned tank feedback system and c) Phase response [4].

Modern communication systems prefer monolithic CMOS VCO design where spiral inductors are implemented in CMOS technologies. Drawback of such LC oscillators is that integrated inductors take large chip area, and have very low Q factor. Another drawback of LC oscillator is small frequency tuning range. Hence difficult to maintain the center frequency within the limited range across process, supply and temperature variations. Therefore, inductorless VCOs have attracted tremendous research in recent years. B. Razavi proposed a phase noise model [4] in 1996 for inductorless VCOs. This model is well suited for CMOS differential ring oscillators. In the case of LC resonator, the energy is stored as electric energy in the capacitor and the magnetic energy in the inductor. A part of the energy is dissipated in the parallel resistor in RLC tank. Thus the definition for Q factor, Q = 2π(energy stored)/ energy dissipated per cycle = $\omega_0/2B$. This is very basic definition valid for LC based oscillators. In a ring oscillator, there are no inductors. The load capacitors are charged and discharged periodically. Hence there is no energy storage within a clock cycle and from equation (7) closed loop transfer function approaches infinity at frequency of oscillation. Therefore this definition is not applicable for ring oscillators. Hence Razavi proposed a new universal definition for Q factor. This definition extends applicability of Leeson's model to a bigger class of inductorless oscillators. If an oscillator is modeled as in fig. 2a), and let $H(j\omega) = A(\omega)e^{j\phi(\omega)}$, then open loop Q factor is defined as shown in equation (8)[4].

$$Q = \frac{\omega_0}{2}\sqrt{\left(\frac{dA}{d\omega}\right)^2 + \left(\frac{d\Phi}{d\omega}\right)^2} \tag{8}$$

$$Q = \frac{\omega_0}{2}\left|\frac{d\Phi}{d\omega}\right| \tag{9}$$

$$\left|\frac{Y}{X}[j(\omega_0 + \Delta\omega)]\right|^2 = \frac{1}{4Q^2}\left(\frac{\omega_0}{\Delta\omega}\right)^2 \tag{10}$$

Due to circuit noise at an offset frequency of $\Delta\omega$ the transfer function varies instantaneously and the open loop transfer function deviates from Barkhausen's criteria. Proposed definition of Q factor in equations (8) is a measure of how sensitive the open loop transfer function for circuit parameters. For example, circuit noise causes instantaneous change in oscillation amplitude and phase. Hence open loop transfer function $H(j\omega)= A(\omega)e^{j\phi(\omega)}$, is changed. A larger Q factor means more deviation from Barkhausen's criteria. Therefore, for the same amount of circuit noise, there is a stronger feedback that brings the frequency back to its nominal frequency ω_0 so that conditions in Barkhausen's criteria are satisfied. The closed loop transfer function in equation (10) can easily be derived [4] and is familiar form of LC oscillators in equation (6). Although equation (8) defines the Q factor for a ring oscillator it is applicable for LC oscillators also due to inherent amplitude limiting mechanism in LC oscillators. Inherent amplitude limiting in LC oscillator implies $dA/d\omega = 0$ which gives Q factor as in equation (9). Thus definition in equation (8) is consistent with the existing definitions of Q factor for LC oscillators [1][4]. Leeson's phase noise model in equation (6) can be modified to account for N noise sources in N-stage ring oscillator. So the close in phase noise for an N-stage ring oscillator is given by equation (11)[17].

$$L(\Delta\omega) = 10 \cdot \log\left[\frac{2NFKT}{P_s}\left(\frac{\omega_0}{2Q\Delta\omega}\right)^2\right] \tag{11}$$

Razavi adopts LTI approach to model differential CMOS ring oscillators. Thus small signal single ended equivalent of 3 stage differential ring VCO looks like in fig. 3 with noise current sources of each stage included. To maintain sustained oscillations by Barkhausen's criteria, the total phase shift around the loop should be integral multiple of 2π with loop gain of each stage '1'. These two conditions translate to $\omega_0=\sqrt{3}/(RC)$ and $G_mR = 2$. The transfer function from one noise current source to the output voltage at the frequency offset of $\Delta\omega$ is given by equation (12)[4]. The expression for drain current noise is $\overline{i_n^2} = 4KT\gamma g_{ds0} \approx 8KT/R$, hence output power density is given by equation (13)[4]. There are three such noise sources in the circuit hence final phase noise expression becomes as shown in equation (14)[4] where carrier power is $V^2_{swing}/2$ and P_{load} is the power dissipated in the load device.

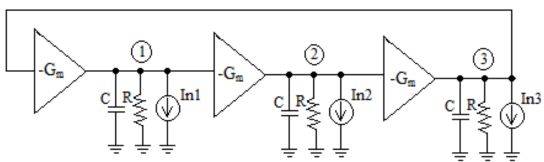

Fig. 3. Linearized single-ended equivalent model for differential ring CMOS VCO [4].

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)
(revised on 4 August 2014)

$$\left|\frac{V_1}{I_{n1}}[j(\omega_0+\Delta\omega)]\right|^2 = \frac{R^2}{27}\left(\frac{\omega_0}{\Delta\omega}\right)^2 \quad (12)$$

$$|V_{1tot}[j(\omega_0+\Delta\omega)]|^2 = 8KT\frac{R}{9}\left(\frac{\omega_0}{\Delta\omega}\right)^2 \quad (13)$$

$$L(\Delta\omega) = 10\cdot\log\left[\frac{16}{3}\frac{KTR}{V_{swing}^2}\left(\frac{\omega_0}{\Delta\omega}\right)^2\right]$$

$$= 10\cdot\log\left[\frac{8}{3}\frac{KT}{P_{load}}\left(\frac{\omega_0}{\Delta\omega}\right)^2\right] \quad (14)$$

A common misconception about digital, inverter based ring oscillators is that it is close to its balanced state during the transition and hence a linear model is a good approximation to analyze its phase noise. It is seen that the input and the output nodes of any of the delay stages never reach the balanced state together. Therefore, inverter based ring oscillator never act as a linear amplifier during the transitions. Hence its phase noise can not be analyzed with linearity assumption.

IV. Hajimiri's model

LTI model is inherently not capable to model phase noise of digital, inverter based ring VCO and cannot produce sideband noise components. Hence a more accurate linear time variant model is developed by A. Hajimiri and T. H. Lee in 1998 [5]. It introduces impulse sensitivity function (ISF) to consider the effects, time variance and cyclostationarity of noise. To understand the general theory of phase noise by Hajimiri and Lee, consider a current impulse injected in to a circuit node. It changes the voltage across load capacitor instantaneously. This change in voltage is given by [5],

$$\Delta V = \frac{\Delta q}{C_{total}} \quad (15)$$

Here Δq represents injected charge by current noise and C_{total} is the total capacitance at output node. A change in voltage after normalization is given by $\Delta V/V_{max}$ where V_{max} is peak amplitude of oscillation. Phase change is proportional to change in voltage hence can be written as [5],

$$\Delta\Phi \propto \frac{\Delta V}{V_{max}}$$

Therefore,

$$\Delta\Phi = \Gamma(\omega_0\tau)\,\frac{\Delta V}{V_{max}} = \Gamma(\omega_0\tau)\,\frac{\Delta q}{q_{max}} \quad (16)$$

Here q_{max} is maximum charge displacement and $\Gamma(\omega_0\tau)$ is the time-varying "proportionality constant" called as impulse sensitivity function. It should be noted that the voltage change settles back to its steady state however a permanent phase shift persists after noise impulse is injected. Since phase noise is the accumulation, or integral, of circuit noise over time it can be described as 1/s in frequency domain. This is consistent with our analysis in the previous sections that the white noise causes $1/\Delta\omega^2$ phase noise region. From equation (16)[5] impulse response can be written as [5],

$$h_\Phi(t,\tau) = \frac{\Gamma(\omega_0\tau)}{q_{max}}u(t-\tau) \quad (17)$$

where 'τ' is the time when impulse is injected. Phase noise depends on the time when the noise current is injected. This is illustrated [5] as shown in fig. 4.

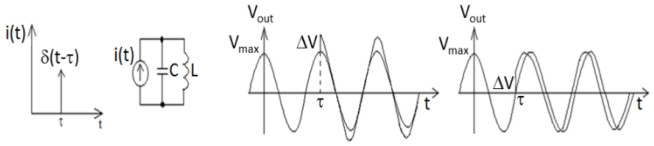

Fig. 4. Impulse response of an ideal LC oscillator [5].

Over a period, oscillator has different noise sensitivity at different time instants. An impulse at the zero crossing causes maximum phase change and it does not cause amplitude noise. An impulse at the peak of tank amplitude causes maximum amplitude noise without any phase deviation as shown in fig. 4. Thus ISF is a periodic function with the same period as the signal waveform.

The ISF provides way of analyzing oscillator phase noise by considering time variance. It accounts for cyclostationarity through modulated ISF. If a current impulse is injected into the circuit node in simulation, the ISF can be obtained by observing its phase shift after it settles to its steady state. Once ISF is obtained from circuit simulation, the output excess phase $\phi(t)$ can be calculated using the superposition integral [5],

$$\Phi(t) = \int_{-\infty}^{\infty} h_\Phi(t,\tau)i(\tau)d\tau = \frac{1}{q_{max}}\int_{-\infty}^{t}\Gamma(\omega_0\tau)\,i(\tau)d\tau \quad (18)$$

Since the ISF is periodic function at frequency ω_0, only noise close to DC, ω_0 and its harmonics will result in non-zero excess phase as seen from the integral in equation (18). Noise at all other frequencies will average out over time. Being periodic function, ISF can be expressed as Fourier series [5],

$$\Gamma(\omega_0\tau) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\cos(n\omega_0\tau + \theta_n) \quad (19)$$

The coefficients C_n represent how much noise is contributed from the vicinity around frequency $n\omega_0$ where n = 0, 1, 2,.... If the circuit has white noise with power spectral density of $\overline{i_n^2}/\Delta f$, its phase noise is given by equation (20) [5],

$$L(\Delta\omega) = 10\cdot\log\left[\frac{\Gamma_{rms}^2}{q_{max}^2}\cdot\frac{\overline{i_n^2}/\Delta f}{4\cdot\Delta\omega^2}\right] \quad (20)$$

The phase noise due to the flicker noise is given by equation (21)[5]. It indicates the $1/\Delta\omega^2$ region due to white noise and $1/\Delta\omega^3$ region due to flicker noise. This is consistent with the conclusions in Leeson's model and Razavi's model.

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)
(revised on 4 August 2014)

$$L(\Delta\omega) = 10 \cdot \log\left[\frac{c_0^2}{q_{max}^2} \cdot \frac{\overline{i_n^2}/\Delta f}{8 \cdot \Delta\omega^2} \cdot \frac{\omega_1/f}{\Delta\omega}\right] \qquad (21)$$

An equation (21) shows that the phase noise due to flicker noise is proportional to C_0, the DC component of ISF. A common belief is that the phase noise $1/\Delta\omega^3$ corner is the same as the 1/f corner for flicker noise, since the complete spectrum close to DC is up-converted to the vicinity of the carrier. However, Hajimiri's theory proves that the $1/\Delta\omega^3$ region can be reduced by minimizing C_0. In an ideal case where the waveform is symmetrical (equal rise and fall times), the $1/\Delta\omega^3$ is completely removed, since $C_0=0$. This is a major contribution of this work.

Reference [5] also considers cyclostationary noise by modulating $\Gamma(\omega_0\tau)$ by a factor $\alpha(\omega_0\tau)$. The factor $\alpha(\omega\tau)$ is different for thermal noise and flicker noise. In order to minimize $1/\Delta\omega^3$ phase noise, it is crucial to minimize the DC component for $\alpha_{flicker}(\omega_0\tau)\Gamma(\omega\tau)$.

Even though ISF is good way of modeling phase noise, computing ISF has some practical difficulties. An impulse is a 'δ' function of time which has to be injected into a circuit node to compute ISF. In theory, the value of amplitude in limit for delta function is infinity and duration in limit is zero. The strength of such delta function is an area under this curve. However, practically only a current with finite amplitude and time duration can be simulated. A better approximation can be achieved with a narrow impulse, small current amplitude and short duration. This results in a small amount of charge injection which is not sufficient to cause enough phase shift to be observed or subject to large numerical errors. This greatly limits the achievable accuracy of the computed ISF. On the other side, an impulse with large strength can drive the circuit away from its normal operating state and requires longer settling time. Another practical drawback for computing ISF is the long simulation time. In order to compute the ISF for a circuit node at time t, fine granularity on time scale is required to ensure accuracy and longer time is needed to allow the circuit to settle to its steady state after the impulse is injected. Therefore, computing ISF at a single node and at a single time instant needs a long transient simulation. This is to compute ISF over a complete period, over many time points for a single circuit node. As the circuit complexity grows, computation for all the ISFs becomes very time consuming and tedious job.

In conclusion, Hajimiri's model is a general theory of phase noise in electrical oscillators which is extensive simulation based and provides a comprehensive phase noise analysis. The advantage is that it covers all kinds of architectures of VCO. It has some practical limitations compared with Leeson's model and Razavi's theory, however ISF provides a more detailed analysis and design insight helpful for optimizing an oscillator phase noise.

V. Demir's model

Alper Demir's model is the most generic and most accurate treatment of noise in oscillators. It's purely mathematical and CAD oriented, suitable for simulator type application. However it lacks the circuit design insight and does not help designers to optimize oscillator for better phase noise performance. This model does not provide intuition in to circuits but is definitely remarkable to the simulation of phase noise in oscillators. This model establishes results about the dynamics of nonlinear oscillator in the presence of deterministic and random perturbations. Here unperturbed limit cycle represents oscillator without noise and when noise sources are considered, oscillator does not follow the original orbit but undergoes change in orbit. This model encompasses the decomposition of the perturbation in to two components. First component is phase deviation responsible for shifting the phase of oscillator and second component is orbital deviation responsible for momentarily disturbing limit cycle. The orbital deviation does not accumulate and its effect on oscillators limit cycle dies to zero if perturbation is removed. On the other side phase deviation keeps on accumulating as long as perturbation source is present. Once perturbation source is removed the phase error produced by it remains indefinitely. Demir's model represents oscillator by a group of equations in the form [8],

$$\frac{\partial x(t)}{\partial t} = f(x(t)) \qquad (22)$$

Where x(t) is the oscillator's output voltage. When the oscillator is perturbed by a small perturbation b(t), the output voltage takes the form $x(t+\theta(t)) + y(t)$, where y(t) is the orbital deviation. Finally the phase noise resulting from the voltage perturbation b(t) can be obtained by solving the following one dimensional differential equation as [8],

$$\frac{\partial\theta}{\partial t} = v(t+\theta(t))B(t+\theta(t))(b(t)) \qquad (23)$$

This is same as Hajimiri's phase noise modeling of oscillator where instead integral equation is used as below [5].

$$\theta(t) = \epsilon\int_0^t \Gamma(\tau)n(\tau)d\tau \qquad (24)$$

b(t) and n(t) in equation (23) and (24) respectively are noise sources, whereas in both equations, $\theta(t)$ is the oscillator phase and $\upsilon(t)$, $\Gamma(t)$ are the functions characterizing the oscillator topology. They are called as perturbation projection vector (PPV) and the impulse sensitivity function (ISF) respectively. Both methods produce equal results for stationary noise sources. However when analyzing injection locking phenomena in oscillators both models yield different results with Hajimiri's model unable to predict the locking behavior. This has been confirmed theoretically using averaging transformation method introduced in [15][16]. This method turns out to be a powerful method to deal with the type of equations arising while analyzing oscillator behavior. Although both modeling seems almost alike, Demir's model is mathematically more exact and Hajimiri's model is approximate [14]. Hajimiri's model is simpler, facilitating its use for purposes of analysis and design of electrical oscillators. Previous analysis based on LTI and LTV theories erroneously predict infinite noise power density at the carrier; hence infinite total integrated power.

On the other side, Demir's model accurately predicts results at frequencies closer to carrier frequency. Further Demir rules out cyclostationarity in the oscillator output because it would imply prefect time reference which noisy systems can not provide. Thus this model provides more accurate analysis of oscillator noise. The methods of this model are faster than the traditional brute force Monte Carlo approach of phase noise simulation. The only disadvantage of Demir's model is that it is mathematically complex, especially for hand calculations since one need to solve differential equation involving θ(t) on both sides of equation.

VI. HAM'S MODEL

Donhee Ham's model is one of the recent outstanding contributions which bridges the gap between fundamental physics of noise and the existing phase noise theories [13]. Through thought experiment he shows that virtual damping rate is a fundamental measure of phase noise. Further this model shows that the virtual damping rate can be obtained from the Einstein relation without resorting to specific circuit parameters. This model introduces virtual damping concept and puts the oscillator phase noise theory and well-known resonator theory under the same framework. Ham's thought experiment assumes ensemble of N identical oscillators. Assuming all oscillators start oscillating at the same time instant, it's found that the ensemble average over time exhibits exponential damping. From the ergodicity [11] of the system ensemble average of oscillator signal is equal to the time average hence time average also shows similar exponential damping. This conclusion is from the assumption that the system is ergodic which is fundamental assumption true in many measurement systems. The concept of phase diffusion due to noise is used to build this model. If the phase diffusion is due to white noise, the variance of ϕ, which signifies the width of the probability distribution is given by [13],

$$< \phi^2(t) > = 2Dt \qquad (25)$$

Here phase diffusion constant 'D' indicates how fast the phase diffusion occurs. 'D' is the reciprocal of the exponential time constant in damping of ensemble/time average and is also called as virtual damping rate. By this theory phase noise at given offset frequency Δω is given by [13],

$$L(\Delta\omega) \equiv \frac{S_V(\omega)}{\frac{V_0^2}{2}} = \frac{2D}{(\Delta\omega)^2 + D^2} \qquad (26)$$

If offset frequency is large enough compared to 'D', i.e. Δω >> D, equation (26) assumes familiar $1/f^2$ behavior [13].

$$L(\Delta\omega) \approx \frac{2D}{(\Delta\omega)^2} \qquad (27)$$

Where diffusion constant D can be evaluated using Einstein relation taking in to account sensitivity and loss. This model is valid for LTI analysis as well as LTV analysis. For LTV analysis time varying effects are taken in to account for evaluating diffusion constant. In this theory also loaded quality factor is redefined.

Table 1. Similarities, differences and limitations of different oscillator phase noise models.

Sr. No./Criteria	Leeson	Razavi	Hajimiri	Demir	Ham
1] Model complexity	Simple and intuitive model without formal proof.	Simple and intuitive.	Simple and intuitive.	Complex, pure mathematical and no circuit intuition.	Simple and intuitive.
2] Type of model	This model is one port, LTI feedback system.	This model is two-port, LTI feedback system.	This model is LTV and described by integral equation.	This model is non linear, mathematically involved and CAD oriented. It can be described by one dimensional differential equation.	Valid for both LTI and LTV modeling of phase noise.
3] Ability to produce sidebands and ability to take in to account cyclostationary noise sources	Being LTI model it's unable to produce sidebands and fails to take in to account cyclostationary noise sources.	Being LTI model it's unable to produce sidebands and fails to take in to account cyclostationary noise sources. It uses rather complex mathematical formulation to take in to account	Being LTV model it's able to produce side bands and explains up conversion and down conversion of noise in the vicinity of integral multiple of oscillation frequency. Cyclostationarity of certain noise sources	This model proves stationarity in the oscillator output. It rules out possibility of cyclostationarity in the oscillator output since it would imply a perfect time reference which noisy systems can not provide.	This model is capable to explain up conversion, down conversion of noise in the vicinity of integral multiple of oscillation frequency. It can take in to account cyclostationarity of certain noise sources.

		cyclostationary noise sources.	is taken in to account by introducing noise modulating function.		
4] Model applicability/ Validity	It's valid to all types of LC oscillators only. This model is not applicable to inductorless CMOS ring oscillators.	Valid to all types of LC oscillators as well as inductorless CMOS differential ring oscillators. However this model is not applicable to ring VCO with digital inverters as delay stages.	Valid to all classes of oscillators.	Valid to all classes of oscillators.	Valid to all classes of oscillators.
5] Q-factor definition	Basic Q-factor definition in the context of LC resonators.	Introduced new Q-factor definition to extend applicability of Leeson's model to inductorless CMOS differential ring oscillators.	Basic Q-factor definition used.	Q-factor definition is not utilized throughout analysis. It develops solid foundation of phase noise regardless of operating mechanism.	Basic Q-factor definition for LTI analysis is used. New definition of Q-factor introduced for LTV analysis.
6] Dependence on empirical parameter	This model depends on empirical parameter, F for phase noise prediction. This parameter is posterior parameter derived from measured data.	This model depends on empirical parameter, F for phase noise prediction. This parameter is posterior parameter derived from measured data.	This model does not depend on any empirical parameter. It depends on ISF which can be calculated through circuit simulations.	This model does not depend on any empirical parameter. It depends on PPV which is equivalent counterpart of ISF.	This model does not depend on any empirical parameter.
7] Circuit insight	Due to empirical parameter F, phase noise can not be predicted from circuit noise analysis. Hence no clear direction for circuit improvement.	Due to empirical parameter F, phase noise can not be predicted from circuit noise analysis. Hence no clear direction for circuit improvement.	No empirical parameter and clear direction for improving phase noise. E.g. Symmetry of rise and fall times minimizes fourier coefficient, C_0 which reduces $1/\Delta\omega^3$ corner. This in turn reduces integrated phase noise within given band of frequency.	No empirical parameter used but being mathematically involved lacks circuit insight. Good for simulator type of application.	No empirical parameter and bridges gap between fundamental physics of noise and oscillator phase noise. It provides circuit insight for improvement.
8] Simulation time and computational overhead	It's simple and does not require long circuit simulation time or complex computations.	It's simple and does not require long circuit simulation time or complex computations.	Long simulation time and tedious computations required to compute ISF and involves error due to approximate nature of impulse injected.	Requires solving of complex differential equations but more accurate.	Does not require long simulation time and does not have computational overhead.
9] Injection locking behavior	Unable to predict injection locking behavior of oscillator.	Unable to predict injection locking behavior of oscillator.	Unable to predict injection locking behavior of oscillator.	It's able to predict injection locking behavior of oscillator hence its universal model.	Unable to predict injection locking behavior of oscillator.

ISBN: 978-988-19253-3-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

10] Close-in phase noise prediction	Erroneously predicts infinite noise power at carrier hence infinite total integrated noise.	Erroneously predicts infinite noise power at carrier hence infinite total integrated noise.	It predicts accurate results even at frequencies closer to the carrier frequency [18].	It predicts accurate results even at frequencies closer to the carrier frequency.	It predicts accurate results even at frequencies closer to the carrier frequency.

VII. Summary

This paper reviews the existing phase noise models of an oscillator starting from earliest work by D. B. Lesson to the most popular models by Demir, Hajimiri, and Ham. This paper gives designer a guide and degree of freedom to choose most suitable model to optimize oscillator topology of his interest.

Acknowledgment

Author would like to thank to IMC WPRD ETS DSI PHY group, Germany for all the support during this survey work.

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Date of modification: 4th August 2014

Modifications:

1] Modification done to the last row under columns "Hajimiri" and "Ham" as follows.

"Erroneously predicts infinite noise power at carrier hence infinite total integrated noise."

Replaced with

"It predicts accurate results even at frequencies closer to the carrier frequency."

2] Added another reference [18]