The Calculation of Axisymmetric Duct Geometries for Incompressible Rotational Flow Using a Differential Equation Approach and a Boundary Integral Formula based on Green's Theorem

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***Abstract*— In this paper a numerical algorithm is described for solving the boundary value problem associated with axisymmetric, inviscid, incompressible, rotational (and irrotational) flow in order to obtain duct wall shapes from prescribed wall velocity distributions. The governing equations are formulated in terms of the stream function $\psi(x, y)$ and the function $\varphi(x, y)$ as independent variables where for irrotational flow $\varphi(x, y)$ can be recognized as the velocity potential function, for rotational flow $\varphi(x, y)$ ceases being the velocity potential function but does remain orthogonal to the stream lines. A numerical method based on finite differences solving a Poisson type equation on a uniform mesh is employed. The technique described is capable of tackling the so-called inverse problem where the velocity wall distributions are prescribed from which the duct wall shape is calculated, as well as the direct problem where the velocity distribution on the duct walls are calculated from prescribed duct wall shapes. Results for the case of prescribing the radius i.e. the so called Dirichlet boundary conditions are given. A downstream condition is prescribed such that cylindrical flow, that is flow which is independent of the axial coordinate, exists. An alternative formulation is also derived based on using Green's function for the Laplace equation on a rectangle.**

***Index Terms*—Irrotational Incompressible flow, Upstream conditions and Downstream Cylindrical flow condition, Adjoint equation, Green's function and Integral Formula.**

I. INTRODUCTION

Designers of ducts require numerical techniques for calculating wall shapes from a prescribed velocity distribution. The objective of the prescribed velocity is typically to avoid boundary layer separation. At inlet a velocity is prescribed to allow for a vorticity to be present calculated from $\underline{\omega} = \underline{\nabla} \wedge \underline{v}$ where the $\wedge$ denotes the usual cross product of vectors, $\underline{\omega}$ is the vorticity vector and $\underline{v}$ the velocity vector respectively.

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The objective of the present paper is to describe a numerical algorithm for solving the boundary value problem that arises when the independent variables are φ and ψ where φ may be identified as the velocity potential function (for irrotational flow only), for flow with vorticity φ ceases being the velocity potential function but does remain orthogonal to ψ which may be identified as the stream function. The dependent variable y, is the radial coordinate and x the axial coordinate. The numerical technique is based on the finite difference scheme on a uniform mesh.

II. THE DESIGN PLANE

As shown in Pavlika [4] when the independent variables are $\varphi(x, y)$ and $\psi(x, y)$ where the $\varphi(x, y)$ and $\psi(x, y)$ have been previously defined it can be shown that the governing partial differential equation that the radius satisfies is given by:

$$\frac{\partial}{\partial \varphi}\left(\frac{A}{B}\frac{\partial y}{\partial \varphi}\right) + \frac{\partial}{\partial \psi}\left(\frac{B}{A}\frac{\partial y}{\partial \psi}\right) = 0 \tag{1}$$

with the speed calculated from

$$\frac{1}{q^2} = \frac{1}{A^2}\left(\frac{\partial y}{\partial \psi}\right)^2 + \frac{1}{B^2}\left(\frac{\partial y}{\partial \varphi}\right)^2 \tag{2}$$

and completion of the physical coordinates provided from

$$dx = \frac{B}{A}\frac{\partial y}{\partial \psi}d\varphi - \frac{A}{B}\frac{\partial y}{\partial \varphi}d\psi$$

where x is the axial coordinate and A and B satisfy their own first order quasi-linear hyperbolic partial differential equations with characteristics parallel to the φ and ψ axes

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which maps the physical flow field into an infinite strip in the (φ,ψ) plane. In fact the A and B satisfy:

$$\frac{\partial}{\partial\varphi}(\log(A)) = \frac{\eta}{q^2}B \tag{3}$$

and

$$\frac{\partial}{\partial\psi}(\log(B)) = -\frac{\omega_\alpha}{q^2}A \tag{4}$$

Regarding temporarily η, ω_α and q as known functions of φ and ψ the system (3) and (4) as previously mentioned is quasi-linear hyperbolic with characteristics parallel to the φ and ψ axes which maps the physical flow field into an infinite strip in the (φ,ψ) plane. Bearing in mind the freedom available in the stream wise variation of φ and the cross stream variation of ψ, suitable values of A can be prescribed along one φ characteristic and those of B can be prescribed along one ψ characteristic.

III. The Numerical Algorithm in the Design Plane

Rewriting the partial differential equation that y satisfies i.e. equation (1) as:

$$\frac{\partial}{\partial\phi}(C\frac{\partial y}{\partial\phi}) + \frac{\partial}{\partial\psi}(\frac{1}{C}\frac{\partial y}{\partial\psi}) = c \tag{5}$$

where $C = \dfrac{A}{B}$, for problems posed in the design plane *c=0*, the value of C will vary depending on whether the flow field is irrotational or swirl free etc. Equation (5) will be re-written as a Poisson equation that is as:

$$\nabla^2 y = \frac{c}{C} + (1-\frac{1}{C^2})\frac{\partial^2 y}{\partial\psi^2} - \left(\frac{\partial}{\partial\varphi}\log_e|C|\right)\frac{\partial y}{\partial\varphi} - \frac{1}{C}\frac{\partial}{\partial\psi}\left(\frac{1}{C}\right)\frac{\partial y}{\partial\psi} \tag{6}$$

where ∇^2 is the usual two dimensional Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial\psi^2} + \frac{\partial^2}{\partial\varphi^2}$$

so that

$$\nabla^2 y = g(\frac{\partial^2 y}{\partial\psi^2}, \frac{\partial y}{\partial\varphi}, \frac{\partial y}{\partial\psi}, C, c)$$

where g is a function of the arguments shown as defined by equation (6). Writing in finite difference form using central differences gives:

$$\underline{Y}^{(i-1)} + A\underline{Y}^{(i)} + \underline{Y}^{(i+1)} = E^{(i)} \tag{7}$$

where

$$\underline{Y}^{(i-1)} = \begin{bmatrix} y_{i-1,1} \\ y_{i-1,2} \\ . \\ . \\ y_{i-1,N} \end{bmatrix}, \quad \underline{Y}^{(i)} = \begin{bmatrix} y_{i,1} \\ y_{i,2} \\ . \\ . \\ y_{i,N} \end{bmatrix}, \quad \underline{Y}^{(i+1)} = \begin{bmatrix} y_{i+1,1} \\ y_{i+1,2} \\ . \\ . \\ y_{i+1,N} \end{bmatrix}$$

$$\underline{E}^{(i)} = h^2 \begin{bmatrix} g_{i,1} - y_{i,0} \\ g_{i,2} \\ . \\ . \\ g_{i,N} - y_{i,N+1} \end{bmatrix}$$

and

$$A = \begin{bmatrix} -4 & 1 & 0 & 0 & . \\ 1 & -4 & 1 & & \\ 0 & 1 & -4 & . & . \\ & & & . & 1 \\ & & & 1 & -4 \end{bmatrix}$$

On a uniform mesh with $\Delta\varphi = \Delta\psi = h$.

IV. Direct Solution of the Difference Equations

The matrix-vector equation (7) is

$$\underline{Y}^{(i-1)} + A\underline{Y}^{(i)} + \underline{Y}^{(i+1)} = E^{(i)}$$

With all of order *(NXN)*, and column vectors $\underline{Y}^{(i)}$ and $\underline{E}^{(i)}$ of order *N*. To solve the vector recurrence relation a speculation is made that the $\underline{Y}^{(i-1)}$ vector can be related linearly to the $\underline{Y}^{(i)}$ vector as follows:

$$\underline{Y}^{(i-1)} = B^{(i)}\underline{Y}^{(i)} + \underline{K}^{(i)} \tag{8}$$

where the $B^{(i)}$ and the $\underline{K}^{(i)}$ are at present unknown matrices and column vectors respectively. Substituting (8) into (7) gives

$$(W^{(i)}B^{(i)} + A)\underline{Y}^{(i)} = E^{(i)} - W^{(i)}\underline{K}^{(i)} - E^{(i)}\underline{Y}^{(i+1)}$$
$$\Rightarrow \underline{Y}^{(i)} = -(W^{(i)}B^{(i)} + A)^{-1}E^{(i)}\underline{Y}^{(i+1)}$$
$$+ (W^{(i)}B^{(i)} + A)^{-1}(E^{(i)} - W^{(i)}\underline{K}^{(i)})$$

but

$$\underline{Y}^{(i)} = B^{(i+1)}\underline{Y}^{(i+1)} + \underline{K}^{(i+1)}$$

Thus equating coefficients implies

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$$B^{(i+1)} = -(W^{(i)}B^{(i)} + A)^{-1}\underline{E}^{(i)} \tag{9}$$

and

$$\underline{K}^{(i+1)} = (W^{(i)}B^{(i)} + A)^{-1}(\underline{E}^{(i)} - W^{(i)}\underline{K}^{(i)})$$

For $i=0$ this gives

$$\underline{Y}^{(0)} = B^{(1)}\underline{Y}^{(1)} + \underline{K}^{(1)} \tag{10}$$

To determine the $\underline{K}^{(1)}$, if the first iterate $B^{(1)} = \underline{0}$ then

$$\underline{K}^{(1)} = \underline{Y}^{(0)}$$

The matrix and vector sequences are now defined by equations (9) and (10) for $i=1$ *to* M. The $\underline{Y}^{(i)}$ vectors are now calculated starting from right to left (as $\underline{Y}^{(M+1)}$ is known) using

$$\underline{Y}^{(M)} = B^{(M+1)}\underline{Y}^{(M+1)} + \underline{K}^{(M+1)}$$

V. AXISYMMETRIC FLOW IN THE ABSENCE OF BODY FORCES

Here numerical solutions to inviscid axisymmetric flow with constant vorticity and a swirl velocity will be derived. The axial velocity component $u_x(y)$ at inlet will be chosen to be of the form $u_x(y) = \alpha y + \beta$, where α and β are constants chosen such that $u_x(y_1) = u_1$ and $u_x(y_2) = u_2$ where y_1 represents the inner radius and y_2 the outer radius at inlet. The swirl velocity $u_\alpha(y)$, will be of the form $u_\alpha(y) = ky + \dfrac{l}{y}$ where the k and l are constants with ky representing solid body rotation and l/y the so-called free vortex term respectively.

VI. THE FLOW EQUATIONS IN THE PHYSICAL PLANE (Y,α,X).

Adopting cylindrical polar coordinates with y being the radial coordinate, α the circumferential and x the axial coordinate, defining velocity components u_y , u_α and u_x with corresponding vorticity components ω_y , ω_α, ω_x in the direction of increasing y, α and x respectively, then the equation of motion with unit density becomes:

$$\frac{D\underline{u}}{Dt} = -\underline{\nabla}.\underline{p} \tag{11}$$

Where $\dfrac{D}{Dt}$ is the material derivative. Equation (11) can be written using well known vector identities as:

$$\begin{aligned}
&\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} - \frac{u_\alpha^2}{y} = -\frac{\partial p}{\partial y}\\
&\frac{\partial u_\alpha}{\partial t} + u_x\frac{\partial u_\alpha}{\partial x} + u_y\frac{\partial u_\alpha}{\partial y} - \frac{u_\alpha u_y}{y} = 0\\
&\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x}
\end{aligned} \tag{12}$$

Furthermore

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{u} = -\underline{\nabla}.\underline{p}$$

can be written (once again using an appropriate vector identity as)

$$\frac{\partial \underline{u}}{\partial t} + (\underline{\omega} \wedge \underline{u}) = -\underline{\nabla}(\underline{p} + \frac{1}{2}q^2)\text{. Thus}$$

for steady flow Crocco's form of the equation of motion is obtained, i.e.

$$(\underline{u} \wedge \underline{\omega}) = \underline{\nabla}H \tag{13}$$

where H is the total head defined by $H = p + \frac{1}{2}q^2$.

Calculating the cross product on the left hand side of equation (13), gives

$$\begin{aligned}
&\frac{\partial H}{\partial y} = u_\alpha\omega_x - u_x\omega_\alpha\\
&0 = u_x\omega_y - u_y\omega_x\\
&\frac{\partial H}{\partial x} = u_y\omega_\alpha - u_\alpha\omega_x
\end{aligned} \tag{14}$$

In addition for axisymmetric flow the vorticity vector $\underline{\omega}$ becomes

$$\underline{\omega} = \underline{\nabla} \wedge \underline{u} = \left\{-\frac{\partial u_\alpha}{\partial x}\right\}\underline{y} + \left\{\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right\}\underline{\alpha} + \left\{\frac{1}{y}\frac{\partial(yu_\alpha)}{\partial y}\right\}\underline{x} \tag{15}$$

The equation of continuity becomes

$$\underline{\nabla}.\underline{u} = \frac{\partial(yu_x)}{\partial x} + \frac{\partial(yu_y)}{\partial y} = 0$$

VII. THE DESIGN PLANE COUNTERPARTS

In order to compute numerical solutions in the design plane, expressions are required for the terms A, B and ω_α , thus

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$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=-\frac{1}{y}\left(u_x\frac{\partial y}{\partial x}+u_y\right)$$
$$=-q\frac{\partial}{\partial s}(\log(y))$$

or

$$\eta=-\frac{q^2}{B}\frac{\partial}{\partial\varphi}(\log(y)),$$

but

$$\eta=\frac{q^2}{B}\frac{\partial}{\partial\varphi}(\log(A))$$

thus $Ay = f(\psi)$, that is $\frac{\partial\psi}{\partial n}=\frac{yq}{f(\psi)}$. The arbitrary function $f(\psi)$ represents the freedom in the cross stream distribution of ψ and choosing $f(\psi)$ to be unity everywhere $f(\psi)$ can be identified as the usual Stokes stream function given by

$$\frac{\partial\psi}{\partial x}=-yu_y;\frac{\partial\psi}{\partial y}=yu_x$$

Equation (12), (circumferential component) gives

$$0=u_x\frac{\partial(yu_\alpha)}{\partial x}+u_y\frac{\partial(yu_\alpha)}{\partial y}$$

Referring to the meridian plane figure 1, it may be deduced that:

$$u_x=q\frac{\partial x}{\partial s};u_y=q\frac{\partial y}{\partial s}$$
$$\Rightarrow\frac{\partial}{\partial s}(yu_\alpha)=0$$
$$\therefore yu_\alpha=C(\psi)$$

where $q = \frac{ds}{dt}$. In terms of $C(\psi)$ the vorticity vector (expression (15)) becomes

$$\underline{\omega}=\underline{\nabla}\wedge\underline{u}=\left\{-\frac{1}{y}\frac{\partial C}{\partial x}\right\}\underline{y}+\left\{\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right\}\underline{\alpha}+\left\{\frac{1}{y}\frac{\partial C}{\partial y}\right\}\underline{x}$$

$= \omega_y\underline{y} + \omega_\alpha\underline{\alpha} + \omega_x\underline{x}$, by definition.

An expression for ω_α is required as this appears in the expression for B, so using the radial component of equation (14) gives

$$\omega_\alpha=\frac{u_x}{u_\alpha}\left(\frac{1}{y}\frac{\partial C}{\partial y}\right)-\frac{1}{u_x}\frac{\partial H}{\partial y}$$

using the Stokes' stream function this becomes

$$\omega_\alpha=\frac{C(\psi)}{y}\left(\frac{dC}{d\psi}\right)-y\frac{dH}{d\psi}$$

which is the required expression to be used in calculation of B according to definition (4). If far upstream the flow is assumed to be cylindrical so that all quantities are independent of x, then with unit density the equation of motion and the Stokes' Stream function give

$$u_y=0;\frac{\partial p}{\partial x}=0;\frac{\partial p}{\partial y}=\frac{u_\alpha^2}{y};\frac{\partial\psi}{\partial x}=0;\frac{\partial\psi}{\partial y}=yu_x$$

giving

$$\omega_\alpha=\frac{C(\psi)}{y}\left(\frac{dC}{d\psi}\right)-\frac{y}{2}\frac{d}{d\psi}(u_x^2+u_\alpha^2)-\frac{u_\alpha^2}{u_xy}$$

With $u_x(y)=\alpha y+\beta$ and $u_\alpha(y)=ky+\frac{l}{y}$ as previously defined. Once $\frac{dH}{d\psi}$ has been calculated upstream it takes this value throughout the (φ,ψ) since as is self evident the expression is independent of φ. This last expression for ω_α is required in the calculation of B and numerical coupling with equation (1) gives the numerical solution in the design plane.

VIII. Downstream Conditions

Downstream a cylindrical flow condition as discussed below will be prescribed. Defining the pressure function $H(\psi)$ and the function $C(\psi)$ as

$$H(\psi)=\frac{1}{2}(u_x^2+u_\alpha^2)+\frac{p}{\rho}\text{ and }C(\psi)=yu_\alpha$$

for cylindrical flow radial equilibrium (from equation (12)) radial component gives

$$\frac{1}{\rho}\frac{dp}{dy}=\frac{u_\alpha^2}{y}$$

Integrating gives

$$\frac{1}{\rho}(p-p_{y-inner})=\int_{y-inner}\frac{u_\alpha^2}{y}dy=\int_{y-inner}\frac{C^2(\psi)}{y^3}dy$$

Which gives $H(\psi)$ as

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$$H(\psi)=\frac{1}{2}(u_x^2+u_\alpha^2)+\frac{p_{y-inner}}{\rho}$$
$$+\int_{y-inner}\frac{C^2(\psi)}{y^3}dy$$

Now $\displaystyle\int_{y-inner}\frac{C^2(\psi)}{y^3}dy=-\frac{1}{2}\int_{y-inner}C^2 d(1/y^2)$
$$=-\frac{1}{2}\left[\frac{C^2}{y^2}-\left(\frac{C^2}{y^2}\right)_{y-inner}\right]+\frac{1}{2}\int_{y-inner}\frac{1}{y^2}\frac{dC^2}{dy}dy$$

Therefore

$$H(\psi)=\frac{1}{2}u_x^2+\frac{p_{y-inner}}{\rho}+\frac{1}{2}(u_\alpha^2)_{y-inner}$$
$$+\int_{\psi=0}\frac{1}{y^2}\frac{dC^2}{d\psi}d\psi$$

Suppose $u_{x,1}=u_{x,1}(\psi)$ and $u_{\alpha,1}=u_{\alpha,1}(\psi)$, where the subscript 1 denotes upstream conditions, then $u_{x,2}=u_{x,2}(\psi)$ and $u_{\alpha,2}=u_{\alpha,2}(\psi)$ are required as functions of ψ, where the subscript 2 similarly denoting downstream conditions, so that

$$\frac{1}{2}u_{x,2}^2=H(\psi)-\frac{p_{2,inner}}{\rho}-\frac{1}{2}(u_{\alpha,2}^2)_{inner}$$
$$-\frac{1}{2}\int_{\psi=0}\frac{1}{y_1^2}\frac{dC^2}{d\psi}d\psi \qquad (16)$$

and

$$\int_{\psi=0}\frac{d\psi}{u_{x,2}}d\psi=\frac{1}{2}\left(y_2^2-y_{2,inner}^2\right)$$

Furthermore $C(\psi)=y_1u_{\alpha,1}=y_2u_{\alpha,2}$, and equation (16) now gives

$$\frac{1}{2}u_{x,2}^2=\frac{1}{2}u_{x,1}^2+\frac{p_{1,inner}}{\rho}-\frac{p_{2,inner}}{\rho}+$$
$$\frac{1}{2}((u_{\alpha,1}^2)_{inner}-(u_{\alpha,2}^2)_{inner})$$
$$+\frac{1}{2}\int_{\psi=0}\left(\frac{1}{y_1^2}-\frac{1}{y_2^2}\right)d(C^2)$$

or

$$u_{x,2}^2=u_{x,1}^2+K+\int_{\psi=0}\left(\frac{1}{y_1^2}-\frac{1}{y_2^2}\right)d(C^2) \qquad (17)$$

where

$K=2(\frac{p_{1,inner}}{\rho}-\frac{p_{2,inner}}{\rho})+(u_{\alpha,1}^2)_{inner}-(u_{\alpha,2}^2)_{inner}$ and

$$y_2^2=y_{2,inner}^2+2\int_{\psi=0}\frac{d\psi}{u_{x,2}} \qquad (18)$$

with $u_{x,2}$ in this case given by (17).

IX. Calculation Procedure

The calculation of the downstream radii $y_2(\psi)$ follow from equation (18) with $u_{x,2}$ given by equation (17), which can be written as

$u_{x,2}^2=g(\psi)+K$, where

$$g(\psi)=u_{x,1}^2+\int_{\psi=0}\left(\frac{1}{y_1^2}-\frac{1}{y_2^2}\right)\frac{d(C^2)}{d\psi}d\psi \qquad (19)$$

In order to calculate the $(n+1)^{th}$ iterate it is known that:

$$\frac{\partial}{\partial K}(y_{2,outer}^2)=2\int_{\psi=0}\frac{\partial}{\partial K}\left(\frac{d\psi}{\sqrt{g(\psi)+K}}\right)$$
$$=-\int_{\psi=0}^{\Psi}\frac{d\psi}{\left(u_{x,2}^3\right)^{(n)}}$$

but

$$\left(\frac{\partial}{\partial K}(y_{2,outer}^2)\right)^{(n)}=\frac{\left(y_{2,outer}^2\right)^{(n+1)}-(y_{2,outer}^2)^{(n)}}{K^{(n+1)}-K^{(n)}} \qquad (20)$$

from which as can be seen from equation (20) the $K^{(n)}$ must be calculated iteratively with $K^{(0)}=0$, Once the $K^{(n+1)}$ has been calculated it is introduced into equation (19), giving rise to a new $(u_{x,2}^2)^{(n+1)}$ which in turn gives a new $(y_{x,2}^2)^{(n+1)}$ from equation (18) and the process repeated until some convergence criteria is satisfied.

X. Prescription of Wall Geometries.

In this paper the Dirichlet boundary conditions will be prescribed on the wall boundaries so that it is the radii values, y that are given as a function of ϕ on the boundaries. The function chosen to give a y distribution is based on the hyperbolic tangent, choosing $y(\phi)=C\tanh(a\phi+b)+k$ where C, a, b and k are constants, applying the conditions that $y=y_u$ at $\phi=0$ and $y=y_d$ at $\phi=\Phi$ taking $a\Phi+b=3$(arbitrary) and $b=-3$, so that $\tanh(a\Phi+b)\approx 1$ and $\tanh(b)\approx -1$, then it follows that

$$y(\varphi)=\left(\frac{y_d-y_u}{2}\right)\tanh(a\varphi+b)+\left(\frac{y_d+y_u}{2}\right) \qquad (21)$$

replacing φ, by x in equation (21) gives a $y(x)$ distribution. The inner radius is prescribed to be equal to unity in this paper (arbitrary). The geometries produced are shown in figures 2, 3 and 4 respectively.

XI. Alternative Solution using an Integral formula based on Green's theorem.

Here a second method of solution is derived using an integral formula. Commencing with the generalized form of Green's theorem for the self adjoint elliptic operator E(t) in normal form given by:

$$\iint_R vE(t)-tE^{(A)}(v)d\varphi d\psi=\oint_C t\frac{\partial v}{\partial n}-v\frac{\partial t}{\partial n}ds$$

where t = y^2, $E(u)=E^{(A)}(u)$ where $E^{(A)}(u)$ is the adjoint of E and v is the fundamental solution to the adjoint equation. In this case the adjoint equation is given by $\nabla^2=0$ and E(t) = g as defined by equation (6). The contour C bounding the surface R is traversed in the counter clockwise sense. For a doubly connected region introducing a singularity at the point (φ_0,ψ_0) (inside or on the contour C) and assuming

$v(\varphi,\psi)=F(\varphi,\psi)\log_e|r|$ so that the distance r is given by: $r=\left((\varphi-\varphi_0)^2+(\psi-\psi_0)^2\right)^{1/2}$

with $F(\varphi,\psi)$ analytic, then it can be shown that

$$m\pi t(\varphi_0,\psi_0)F(\varphi_0,\psi_0)=\oint_C t\frac{\partial v}{\partial n}-v\frac{\partial t}{\partial n}ds$$

$$-\iint_R vg(\frac{\partial^2 t}{\partial\varphi^2},\frac{\partial^2 L}{\partial\varphi^2})d\varphi d\psi, m=1,2$$

with $L=\log_e t$. Now m = 2 if (φ_0,ψ_0) is within C and m=1 if (φ_0,ψ_0) is on C (the m=1 case can be shown using the appropriate Plemelj formulae or by indenting the contour at (φ_0,ψ_0)). For the Dirichlet case of boundary condition of $t(\varphi,\psi)$ the requirement is that $v(\varphi,\psi)=0$ on C in addition to $v(\varphi,\psi)$ being harmonic and for the Neumann conditions on t, the requirement is that $\frac{\partial v}{\partial n}$=0 and $v(\varphi,\psi)$ once again satisfying Laplace's equation. Much literature is available for the Green's function for the Laplace equation (see Williams [6]) and need not be mentioned here. Hence for the Dirichlet problem without loss of generality setting $F(\varphi,\psi)=1\,\forall\varphi,\psi$ and for interior points:

$$2\pi t(\varphi_0,\psi_0)=-\oint_C t\frac{\partial v_D}{\partial n}ds$$

$$-\iint_R v_D g(\frac{\partial^2 t}{\partial\varphi^2},\frac{\partial^2 L}{\partial\varphi^2})d\varphi d\psi \qquad (22)$$

i.e., the Green's function v_D satisfies Laplace's equation on $\partial\Omega$ where $\partial\Omega$ is defined by: $\varphi_0\le\varphi\le\varphi_{M+1},\ \psi_0\le\psi\le\psi_{N+1}$ and vanishes on C. For the Neumann problem

$$2\pi t(\varphi_0,\psi_0)=-\oint_C v_N\frac{\partial t}{\partial n}ds$$

$$-\iint_R v_N g(\frac{\partial^2 t}{\partial\varphi^2},\frac{\partial^2 L}{\partial\varphi^2})d\varphi d\psi$$

Which give integral formulae for the square of the radius t, from which the radius y can be determined., above Green's function v_N satisfies the Laplace equation on $\partial\Omega$ with $\frac{\partial v_N}{\partial n}$ vanishing on C. Knowledge of the derivatives $\frac{\partial t}{\partial\varphi}$ and $\frac{\partial t}{\partial\psi}$ are also required for the

determination of the speed q given by equation (2) hence differentiating under the integral sign above with respect to φ and ψ gives integral formulae for both $\frac{\partial t}{\partial\varphi}$ and $\frac{\partial t}{\partial\psi}$,

such that:

$$2\pi\frac{\partial}{\partial\psi}t(\varphi_0,\psi_0)=\oint_C\frac{\partial t}{\partial\psi}\frac{\partial v_D}{\partial n}+t\frac{\partial^2 v_D}{\partial\psi\partial n}ds$$

$$-\iint_R v_D\frac{\partial g}{\partial\psi}+g\frac{\partial v_D}{\partial\psi}d\varphi d\psi$$

and similarly for $\frac{\partial t}{\partial\varphi}$

$$2\pi\frac{\partial}{\partial\varphi}t(\varphi_0,\psi_0)=\oint_C\frac{\partial t}{\partial\varphi}\frac{\partial v_D}{\partial n}+t\frac{\partial^2 v_D}{\partial\varphi\partial n}ds$$

$$-\iint_R v_D\frac{\partial g}{\partial\varphi}+g\frac{\partial v_D}{\partial\varphi}d\varphi d\psi$$

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XII. Iterative Solution

To convert formula (22) to a system of linear algebraic equations the point $t(\varphi_0,\psi_0)$ inside C is related to its boundary values on C. To obtain the first iterates $t_i^{(1)}(\varphi_0,\psi_0)$, $g_i^{(0)}$ is set equal to zero, so that

$$2\pi t_i^{(1)} = \sum_j^{2N+2M+4} \left(\frac{\partial v_D}{\partial n}\right)_j t_j \Delta s_j \quad \iota = 0,1,2,...2\mathrm{N}+2\mathrm{M}+4$$

Using the trapezoidal rule

$$\pi t_i^{(1)} = \sum_j^{2N+2M+4} \frac{1}{4}\left(\frac{\partial v_D}{\partial n}\right)_j \left(s_{j+1}-s_{j-1}\right) t_j$$
$$\iota = 0,1,2,...2\mathrm{N}+2\mathrm{M}+4$$

$$\Rightarrow t_i^{(1)} = \sum_j^{2N+2M+4} K(v_D,s)_j\, t_j, \iota = 0,1,2,...2\mathrm{N}+2\mathrm{M}+4$$

Where $K(v_D,s) = \dfrac{1}{4\pi}\left(\dfrac{\partial v_D}{\partial n}\right)\left(s_{j+1}-s_{j-1}\right)$

Using this method there is a simple self-consistency check. i.e. the t_j are known upstream and downstream for j=0,1,2,…,N+1 and j=N+M+3,N+M+4,….2N+M+3, hence the first iteration may be written as:

$$\begin{bmatrix} 1-K_{N+2} & -K_{N+3} & . & . & -K_{2N+2M+4} \\ -K_{N+2} & 1-K_{N+3} & -K_{N+4} & . & -K_{2N+2M+4} \\ . & & . & . & . \\ & & & . & . \\ -K_{N+2} & . & . & . & -K_{2N+2M+4} \end{bmatrix} \begin{bmatrix} t_{N+2} \\ t_{N+3} \\ . \\ . \\ t_{2N+2M+4} \end{bmatrix} = \begin{bmatrix} \sum_j K_j t_j \\ \sum_j K_j t_j \\ . \\ . \\ \sum_j K_j t_j \end{bmatrix}$$

so that $A^{(1)}\underline{t}^{(1)} = \underline{b}^{(i)}$ (23)

where the summations on the right hand side are performed over j=0,1,…N+1 and j=M+N+3, M+N+4.,,,2M+N+3. Once the first iterate t_j has been calculated the field integral containing g is then computed, where the central difference approximation to the second derivative is used, this is then introduced into the right hand side of equation (23) and compute the second iterate $\underline{t}^{(2)}$. The procedure is repeated until some convergence criteria is satisfied e.g. $\|\underline{t}_i^{(k)} - \underline{t}_i^{(k-1)}\|_p < \varepsilon$, where ε is a constant and the p denotes the p-norm (p=1,2 or ∞).

XIII. Conclusions

As shown, geometries have been produced subject to given upstream and downstream conditions with prescribed Dirichlet boundary conditions. In this case vorticity at inlet has been specified by defining the axial velocity to be of the form $u_x(y) = \alpha y + \beta$, and the swirl velocity of the form $u_\alpha(y) = ky + \dfrac{l}{y}$, where the *k* and *l* are constants, defining the so-called free and forced vortex whirl respectively. The downstream conditions where such that: cylindrical flow was present, Dirichlet boundary conditions were prescribed, however the case with Neumann conditions can be accommodated using the algorithm, in addition so can the case with Robin boundary condition. Further examples of the algorithm with a combination of boundary condition is given in Pavlika [5]. It was found that at most eight iterations were required to achieve an acceptable level of convergence, with the technique accelerated using Aitken's Method.

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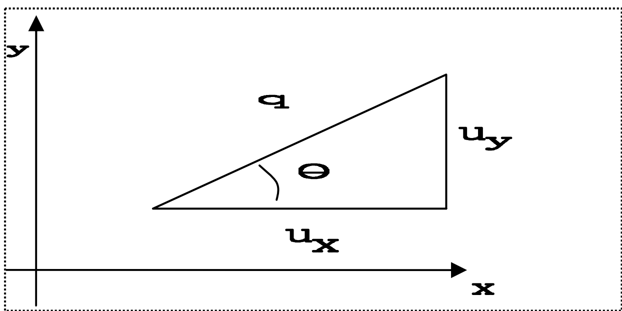

Fig. 1. The meridian plane.

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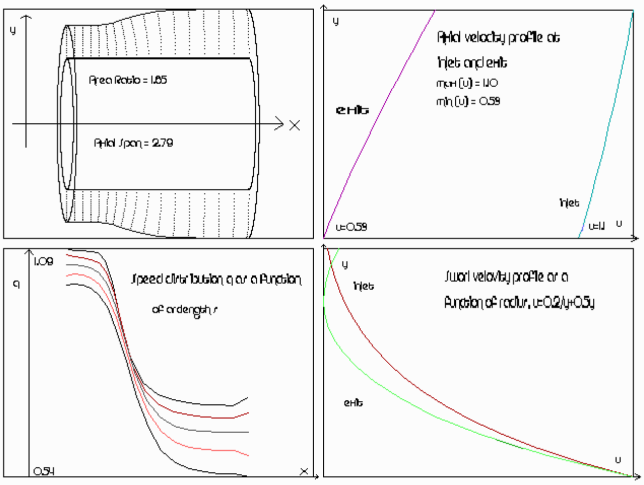

Fig.2. The geometry and speed distribution (along the top boundary) produced given a Swirl velocity

$u_\alpha = 0.5y + \dfrac{0.2}{y}$ and a velocity at inlet given

by $u_x(y) = \alpha y + \beta$.

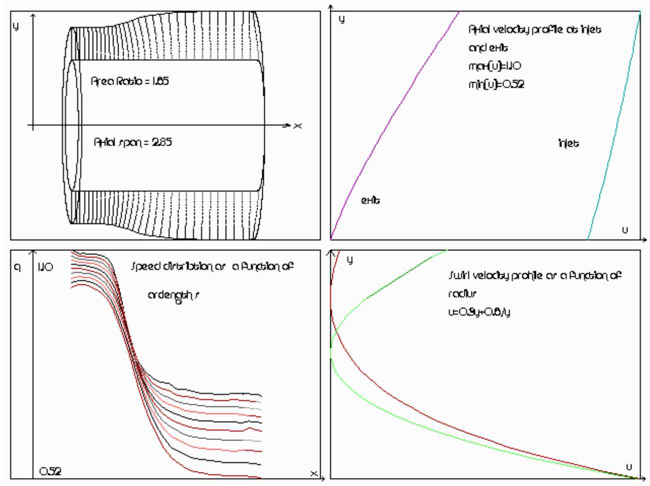

Fig. 3. The geometry and speed distribution (along the top boundary) produced given a Swirl velocity

$u_\alpha = 0.3y + \dfrac{0.6}{y}$ and an axial velocity at inlet given by

$u_x(y) = \alpha y + \beta$.

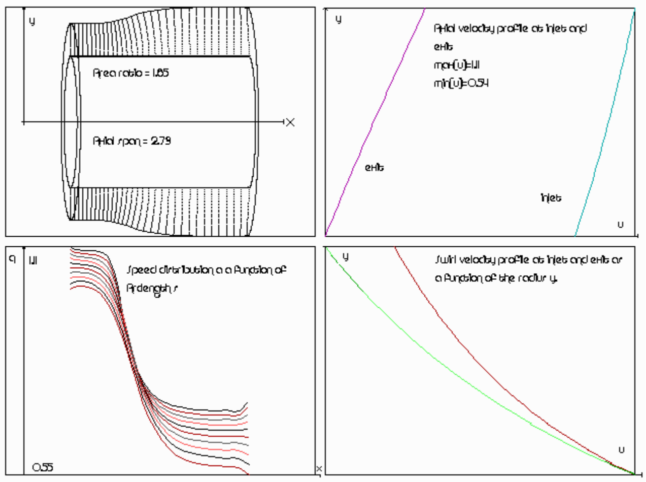

Fig. 4.The geometry and speed distribution (along the top boundary) produced given a Swirl velocity $u_\alpha = \dfrac{0.6}{y}$ and an axial velocity at inlet given by $u_x(y) = \alpha y + \beta$.

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