(Q, r) Inventory Model under Supply Disruptions and General Repairs

Thanawath Niyamosoth, Ya-Xiang Yuan

***Abstract*—Most of the inventory models under supply disruptions are based on the assumption that either the failure process is time-independent or the system is perfectly repaired after each failure. However, in the real situation, the failure process can be time-dependent and the supplier can also perform imperfect repairs. In this work, we study deterministic demand (Q, r) inventory models under time-dependent disruption due to machine breakdowns at the supplier with the possibility that a repair after each failure is imperfect.**

***Index Terms*—Inventory model, imperfect repair**

I. Introduction

Inventory problem under supply disruptions is widely studied in recent years. The reasons why this problem is important is because in reality the suppliers can be unavailable from time to time due to several disruptions such as natural disasters, machine breakdowns, labor strikes, terrorisms, etc. In these situations, the firm, the customer, need effective strategies to cope this problem in order that the business can run smoothly. Most of the inventory models under supply disruptions are based on the assumption that the failure rate and the repair rate are constant and independent of time. Under this assumption, the time to 1st failure and the repair time are exponentially distributed random variables. This assumption is convenient for the model formulation and the computation, for the Markov property is conserved. By contrast, time dependent problems are often non-Markovian, which are more complicated than the Markovian counterparts. However, the assumption of time-independence is not practical for many situations. One of such situations is that when the disruptions occur due to machine breakdowns. There are three main reasons that make the exponential model impractical in generalization disruptions due to machine breakdowns. First, normally, machines deteriorate with time. Older machines incline to fail more often than newer machines. Second, the repair time are practically not exponentially distributed. Third, the exponential model is based on the assumption that the repair is perfect, that is, each repair brings a machine back to the brand-new condition. Generally, this assumption is seldom true, for the general repair of machine is often imperfect. Namely, after each repair, a machine becomes younger, in terms of failure rate, but not brand-new. In our work, by applying the theory of non-homogeneous Markov chain, we propose a model for the inventory problem under time-dependent disruptions due to machine breakdowns that the Markov property is conserved. Moreover, we also consider the effect of imperfect repairs, which is overlooked by other publications, in our model as well. The results show that the assumptions about the failure rate functions, the repair rate functions, and the degree of imperfect repairs have significant impact on the optimal solutions of the model.

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II. MODEL FORMULATION

In our work, we use the following notation:

Q = order quantity
r = reorder point
t = time to 1st occurrence of failure or repair
$\lambda(t)$ = failure rate function
$\mu(t)$ = repair rate function
c = corrective maintenance coefficient
D = demand rate
K = fixed ordering cost per order
h = holding cost per unit per time
π = backorder cost per unit backordered
$\hat{\pi}$ = backorder cost per unit per time
y = repair time
$g(y)$ = probability density function of y

A. Failure and Repair Rate Function

In our work, we define failures as non-homogeneous Poisson process (NHPP), which is a Markov process, with Weibull failure rate as the following.

$$\lambda(t)=\frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1};\beta>0,\theta>0,t\geq 0 \tag{1}$$

β is called the shape parameter, and θ is called the scale parameter respectively. Note that when $\beta=1$, the failure rate is constant as in the exponential case. When β>1, the model becomes an increasing failure rate case, generalizes the deterioration process of the machine. The repair rate function is defined as:

$$\mu(t)=\frac{\psi}{\varphi}\left(\frac{t}{\varphi}\right)^{\psi-1};\psi>0,\varphi>0,t\geq 0 \tag{2}$$

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Similar to the failure rate function, ψ is the shape parameter, and φ is the scale parameter respectively. When $0<\psi<1$, the repair rate is an decreasing function, when $\psi=1$, the repair rate is constant as in the exponential case, and when $\psi>1$, the repair rate function is decreasing. Note that even though the exponential repair is applied widely in many publications; however, in reality, the repair time is seldom exponentially distributed. In the next section, we develop the two-state non-homogeneous Markov chain of the problem.

B. Two-State Non-Homogeneous Markov Chain of ON/OFF Period

The system is in ON period when the machine is operating and OFF when the machine is down. If a perfect repair at each failure is assumed, when the system is ON, it will move to OFF period with rate $\lambda(t)$, and when the system is OFF, it will move to ON with rate $\mu(t)$. If at each failure a repair is perfect (as good as new) with probability p, and with probability $1-p$ a repair is minimal (as bad as old), then the resulting failure rate due imperfect repairs is:

$$\lambda_I(t)=\frac{\lambda(t)}{c};\ 0<c<1, t\geq 0 \tag{3}$$

Proof.

Assume that a machine has been used for t_1, then it fails. If a repair is perfect, then the failure rate function after a perfect repair will be $\lambda(t)$. If a repair is minimal, then the failure rate after a minimal repair will be $\lambda(t+t_t)$. Thus, the failure rate after an imperfect repair is defined as $\lambda_I(t)=p\lambda(t)+(1-p)\lambda(t+t_1)$. So, $\lambda_I(t)$ is on the line joining $\lambda(t)$ and $\lambda(t+t_1)$. We can rewrite $\lambda_I(t)$ as $\lambda_I(t)=\lambda(t)+\alpha(\lambda(t+t_1)-\lambda(t));\ 0<\alpha<1$. Divide $\lambda_I(t)$ by $\lambda(t)$, we get $\frac{\lambda_I(t)}{\lambda(t)}=1+\frac{\alpha(\lambda(t+t_1)-\lambda(t))}{\lambda(t)}$. Because $\lambda(t+t_1)-\lambda(t)>0$, then $\lambda_I(t)=a\lambda(t);\ a>1$. Let $c=\frac{1}{a}$, then we have $\lambda_I(t)=\frac{\lambda(t)}{c};\ 0<c<1$. Proved. Note that if we relax the condition $0<c<1$ by allowing $c=1$, then the repair will be perfect. In our work, we define the failure and the repair process as a two-state non-homogeneous Markov chain as in Fig. 1

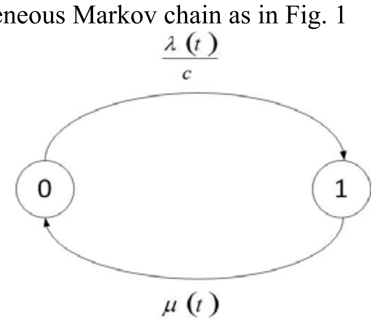

Fig. 1 Two-state non-homogeneous Markov chain for imperfect repair

Let $p_{01}(v,t)$ be the transition probability that the system is in state 0 (ON) at time v and will move to state 1 (OFF) at time t. By applying Kolmogorov's forward equation, $p_{01}(v,t)$ can be found by solving the following differential equation.

$$\frac{\partial p_{01}(v,t)}{\partial t}=-p_{01}(v,t)\left(\frac{\lambda(t)}{c}+\mu(t)\right)+\frac{\lambda(t)}{c};\ 0<c\leq 1, t\geq 0 \tag{4}$$

Because t = time to 1st failure, then we let $v=0$. Thus, by standard calculus methods, we have

$$p_{01}(0,t)=\frac{\int_0^t\left(\frac{\lambda(x)}{c}e^{\int_0^x\left(\frac{\lambda(x)}{c}+\mu(x)\right)dx}\right)dx}{e^{\int_0^t\left(\frac{\lambda(x)}{c}+\mu(x)\right)dx}};\ 0<c\leq 1, t\geq 0 \tag{5}$$

Thus, we can find the probability that the system is in state 0 at time 0 and in state 1 at time t by calculating $p_{01}(0,t)$. Note that when $\beta=1$, the failure rate is constant, that is $\lambda(t)=\lambda$. In this case the failure distribution reduces itself into the exponential case with $\lambda=\frac{1}{\theta}$. Likewise, when $\psi=1$, the repair rate is constant, that is $\mu(t)=\mu=\frac{1}{\varphi}$.

Therefore, the transition probability, $p_{01}(0,t)$, reduces itself into

$p_{01}(t)=\frac{\lambda}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)t}\right);\ \lambda>0, \mu>0, t\geq 0$ as in the case of the standard two-state homogeneous Markov chain. However, we can see that, in the non-homogeneous case, the calculation of $p_{01}(0,t)$ by solving Kolmogorov's forward equation is burdensome. This obstacle can be overcome by using the approximation method as described in [1]. In this method, we define the reliability function $R(t)=\Pr\{T>t\};\ t\geq 0$ which represents the probability that a machine will not fail before time t. Let $f(t)$ be a probability density function of t, the we can define the failure rate function $\lambda(t)$, for $c=1$, as the following:

$$\lambda(t)=\lim_{\Delta t\to 0}\frac{\Pr\{t\leq T<t+\Delta t/T\geq t\}}{\Delta t}=\frac{f(t)}{R(t)}$$

Let $F(t)$ be a cumulative distribution function of t, the $R(t)=1-F(t)$. Thus, $\lambda(t)=\frac{F'(t)}{R(t)}=-\frac{R'(t)}{R(t)}$, then we have $R'(t)=-\lambda(t)R(t)$, which leads to

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$R(t)=\exp\left(-\int_0^t \lambda(x)dx\right)$. Let $\Lambda(t)=\int_0^t \lambda(x)dx$ be the cumulative failure intensity function, then we have $R(t)=\exp(-\Lambda(t));\ t\geq 0$. By definition, $R(t)$ is the probability that starting from time 0 a machine will not fail at least t. Thus, $R(t)$ is the probability that starting from ON state at time 0, the system is still ON at time t. Let $p_{00}(0,t)$ be the probability that at time 0 the system is ON and is still ON at time t. Then, $p_{00}(0,t)=R(t)=\exp(-\Lambda(t))$. From Markov chain, we know that $p_{00}(0,t)+p_{01}(0,t)=1$. Thus, we can find the transition probability $p_{01}(0,t)$ from the following equation:

$$p_{01}(0,t)=1-\exp(-\Lambda(t));\ t\geq 0 \qquad (6)$$

Where $\Lambda(t)=\int_0^t \frac{\lambda(x)}{c}dx;\ 0<c\leq 1$ for imperfect repair.

C. The Objective Function

In this section, we continue to the formulating of the inventory model. The objective of this work is to find the order quantity Q and the reorder point r that minimize the objective function in, the average annual cost, which can be found by applying the renewal reward theorem as in the following equation.

$$\text{Average Annual Cost}=\frac{\text{E}[\text{Total costs per cycle}]}{\text{E}[\text{Cycle length}]} \qquad (7)$$

D. Cycle Length

Assuming that lead time is zero, the cycle starts when the system is ON and the inventory level is $Q+r$. When the inventory level hits the reorder point, r, at time Q/D, with probability $p_{01}(0,Q/D)$ the system will be OFF for the average duration of $MTTR$ where $MTTR$ = mean time to repair which can be found from $MTTR=\Gamma\left(1+\frac{1}{\psi}\right)\varphi$ where $\Gamma(x)$ is Gamma function of x, and φ is the scale parameter of Weibull repair time. Therefore, we can calculate the expected cycle length, $E[CL]$, by the following equation.

$$\text{E}[\text{CL}]=\frac{Q}{D}+p_{01}(0,Q/D)MTTR \qquad (8)$$

This means the cycle lasts at least Q/D and with probability $p_{01}(0,Q/D)$ it will last for another $MTTR$. Fig. 2 illustrates the cycle length of the model.

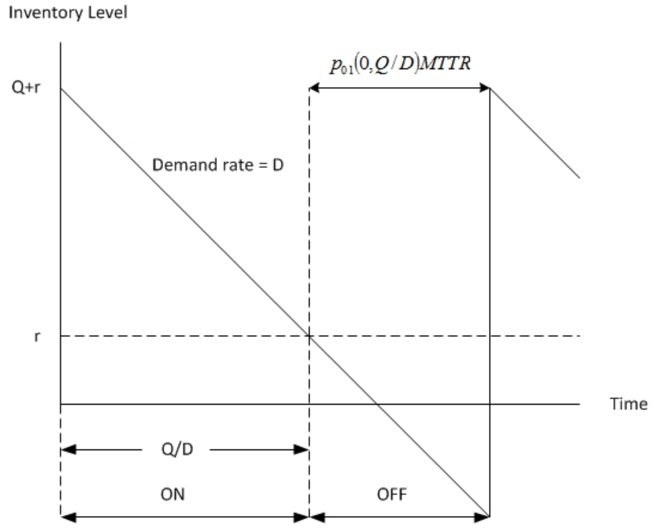

Fig. 2 Cycle length

D. Total Costs Per Cycle

The total costs in a cycle can be divided into two period (i) costs occurring in ON period (ii) costs occurring in OFF period.

(i) Expected Costs in ON Period

There are two costs in ON period, first fixed ordering cost and second holding cost. The expected costs in ON period can be calculated from the following equation.

$$E[\mathit{Costs\ in\ ON\ period}]\ =\ K+\frac{hQ^2}{2D}+\frac{hQr}{D} \qquad (9)$$

(ii) Expected Costs in OFF period

There are two costs in OFF period, first holding cost and second backorder cost. The amount of costs in this period depends on two factors. The first factor is the reorder point. The second factor is the time used for repairing the machine. Let r be the reorder point, and y be the Weibull repair time with pdf.

$$g(y)=\frac{\psi}{\varphi}\left(\frac{y}{\varphi}\right)^{\psi-1}e^{-(y/\varphi)^{\psi}};\ \psi>0,\varphi>0,y\geq 0 \qquad (10)$$

Let $C_{10}(r,y)$ be the random variable representing the costs occurring in OFF period which is the function of r and y and can be defined as:

$$C_{10}(r,y)=\begin{cases} h\left(ry-\frac{y^2D}{2}\right);\ 0<y<\frac{r}{D},r\geq 0 \\ \frac{hr^2}{2D}+\pi(yD-r)+\frac{\hat{\pi}(yD-r)^2}{2D};\ y\geq\frac{r}{D},r\geq 0 \end{cases} \qquad (11)$$

Hence, we can calculate the expected costs in OFF period from the following equation.

$$E[C_{10}(r,y)]=\int_0^{\frac{r}{D}} h\left(ry-\frac{y^2D}{2}\right)g(y)dy + \int_{\frac{r}{D}}^{\infty}\left(\frac{hr^2}{2D}+\pi(yD-r)+\frac{\hat{\pi}(yD-r)^2}{2D}\right)g(y)dy \qquad (12)$$

This cost occurs with the probability $p_{01}(0,Q/D)$, then by combining all costs, we obtain the expected total costs per cycle, $E[TC]$, as follows.

$$E[TC]=K+\frac{hQ^2}{2D}+\frac{hQr}{D}+p_{01}(0,Q/D)E[C_{10}(r,y)] \quad (13)$$

E. Optimization Problem

By applying the renewal reward theorem, we obtain the objective function, the average annual costs, $AC(Q,r)$, as follows.

$$AC(Q,r)=\frac{E[TC]}{E[CL]};\ Q\geq 0, r\geq 0 \quad (14)$$

Hence, the optimization problem in our research is:

$$Min\ \ AC(Q,r)$$
$$Q\geq 0, r\geq 0 \quad (15)$$

III. NUMERICAL EXAMPLE

In this section, we provide the results of the problem and the sensitivity analyses. The software that we use is Mathematica 9. The test parameters are borrowed from [11] except we use D = 10 in our test. Other parameters are K = \$10/order, h = \$5/unit/year, π = \$250/unit, $\hat{\pi}$ = \$25/unit/year, EOQ = $\sqrt{2KD/h}$ = 6.32456 units, R_{EOQ} = 0. The test results are shown in the following tables.

Table. 1 (a) Test results 1

θ = 4, c = 1 ψ = 1, φ = 0.4	β = 1	β = 1.5	β = 2
AC(Q*,r*)	95.3531	80.2664	59.303
r*	9.61317	5.8488	0.546079
Q*	6.57972	4.8404	3.54486
AC(EOQ, 0)	249.072	126.933	71.1124
AC(Q_{EXP}, r_{EXP})	95.3531	86.7072	86.6718

Table. 1 (b) Test results 1 (cont.)

θ = 4, c = 1 φ = 0.4	β = 2.5	β = 3	β = 4
AC(Q*,r*)	42.993	36.4991	32.5219
r*	0	0	0
Q*	4.2252	4.94934	5.85912
AC(EOQ, 0)	47.5907	38.0151	32.6373
AC(Q_{EXP}, r_{EXP})	80.9344	80.2122	79.7958

The results show that the shape parameter β has a significant impact on the optimal solution. Most publications in the literature assume that the failure rate is constant, that is β = 1 in our model; however, we can see that this assumption leads to great error. For example, it the failure rate is not constant but increasing with β = 2, the assumption of constant failure rate incurs 46% cost error. Another observation is that when β increases, the reliability of the system improves. We can see that the optimal safety stock r* and the optimal annual costs reduce significantly. Also, when β is large the problem approaches EOQ case. The reason for this result is that for larger β, the failure rate function increases from zero slower than that of the smaller β as shown in Fig. 3.

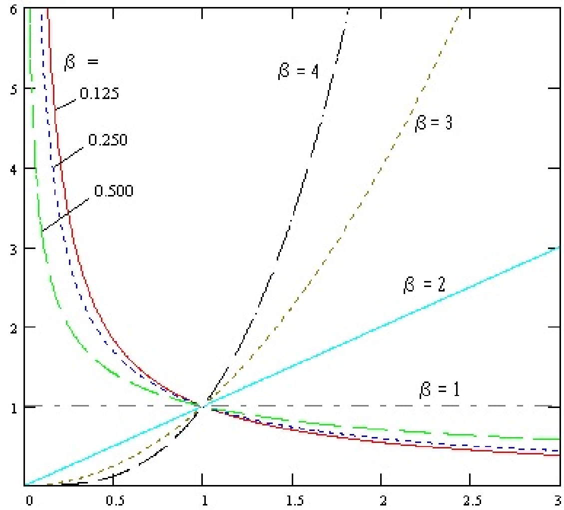

Fig. 3 Weibull Failure Rate
(http://www.mathpages.com/home/kmath122/kmath122.htm)

Table. 2 Test results 2

β =1.5 c = 1 ψ = 1 φ = 0.4	θ = 2	θ = 3	θ = 4	θ = 5	θ = 6
AC (Q*,r*)	96.3861	87.46	80.2664	74.3496	69.3547
r*	9.74226	7.5	5.8488	4.5817	3.47682
Q*	5.20167	4.94994	4.8404	4.78193	4.74654

Table. 2 shows the effect of θ on the optimal solutions. We can see that as θ increases AC(Q*,r*) and r* decrease significantly while when θ decreases the effect is opposite. The reason for this phenomenon is that increasing of θ reduces mean time between failure (MTBF). Because the failure process in our work is based on time to first failure, so in this case MTBF = MTTF = Γ(1+1/β)θ. Therefore, when θ increases, MTTF also increases leading to the increasing of the reliability of the system.

Table. 3 Test result 3

β = 1, θ = 4 c = 1, φ = 0.4	AC(Q*,r*)	r*	Q*	r*/Q*
ψ = 0.5	171.598	5.53715	2.03447	2.7217
ψ = 0.6	127.708	6.16996	2.92583	2.1088
ψ = 0.7	106.442	6.26103	3.63021	1.7247
ψ = 0.8	94.0086	6.16223	4.15774	1.4821
ψ = 0.9	85.9175	6.00834	4.54824	1.3210
ψ = 1.0	80.2664	5.8488	4.8404	1.2083

In general, the repair rate of a repairable unit is a decreasing function of repair time, y [3]. Based on Weibull repair in our work, we investigate the effect of ψ, the repair rate, on the optimal solution. When 0 < ψ <1, the repair rate is decreasing while when ψ = 1 the failure rate is constant as in the exponential case. The results in Table. 3 show that AC(Q*,r*) decreases as ψ increases. Because normally the repair rate is not often constant as in the exponential case,

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failure to consider time-dependent repair rate leads to suboptimal solution. For example, if we assume ψ = 1 when it is in fact 0.5, then $AC(Q_{EXP}, r_{EXP})$ = 194.35 which is larger than the optimal solution of 171.598. Another observation is that when ψ increases, the r*/Q* ratio reduces. This implies that when ψ increases the firm tends to use running stocks instead of safety stocks to fulfill customer demands.

Table. 4 Test results 4

β = 1.5, θ = 4, ψ = 0.5 c = 1	AC(Q*,r*)	r*	Q*	r*/Q*
φ = 0.1	68.3704	2.09849	4.43573	0.4731
φ = 0.2	103.709	3.5781	3.24959	1.1011
φ = 0.3	137.709	4.67139	2.51461	1.8577
φ = 0.4	171.598	5.53715	2.03447	2.7217
φ = 0.5	204.817	6.26318	1.70161	3.6807
φ = 0.6	237.843	6.8987	1.45894	4.7286
φ = 0.7	270.809	7.4731	1.27477	5.8623
φ = 0.8	303.806	8.00509	1.13046	7.0813
φ = 0.9	336.899	8.50723	1.01444	8.3861
φ = 1.0	370.133	8.98834	0.919199	9.7784

The results in Table. 4 show the effect of φ, the scale parameter of the repair time, on the optimal solutions. The conclusion we deduce from the test is that when φ increases, the optimal average annual costs, AC(Q*,r*), and the optimal safety stock, r*, also increase because mean time to repair (MTTR) increase. Also, the r*/Q* increases as φ increase. This implies that if the mean time to repair is long, the firm tends to fulfill the demand by using safety stock.

Table. 5 Test results 5

β = 1.5 θ = 4 ψ = 0.5 φ = 0.4	AC(Q*,r*)	r*	Q*
c = 1.0	171.598	5.53715	2.03447
c = 0.9	180.732	6.45996	1.97788
c = 0.8	191.273	7.57848	1.91961
c = 0.7	203.617	8.95754	1.85958
c = 0.6	218.347	10.6958	1.79776
c = 0.5	236.356	12.9524	1.7344

From the results in Table. 5, we can see that as c, the corrective maintenance coefficient, decreases, the average annual costs increase. By contrast, the average annual costs decrease when c increases. Note that the repair at each failure is perfect when c = 1. The conclusion for these results is that not only the failure and repair rate that have a significant impact on the optimal solutions but also the corrective maintenance action performed at each failure as well. If the supplier inclines to perform perfect maintenance more often than minimal repair, c will be closer to 1. On the other hand, if the supplier inclines to perform minimal repair more often and perform perfect repair less often, the coefficient c will more divert from 1. Failure to consider the degree of corrective maintenance action at each failure leads to significant loss. For example, if the firm, the customer, assumes that a perfect repair is performed by the supplier at each failure when it is in fact that the supplier performs imperfect repair with c = 0.5 at each failure, AC(Q, r) will be 252.99 instead of 171.598 as the firm first expects. Another observation is that when c decreases, r* increases significantly. This implies that the firm tends to fulfill the demand by using safety stocks when repairs divert from perfect repairs. One the other hand, if repairs incline to be perfect, that is c increases, the firm tends to reduce the safety stock; thus, r* decreases. The effect of c on the optimal solution can be illustrated in Fig. 4 and Fig. 5.

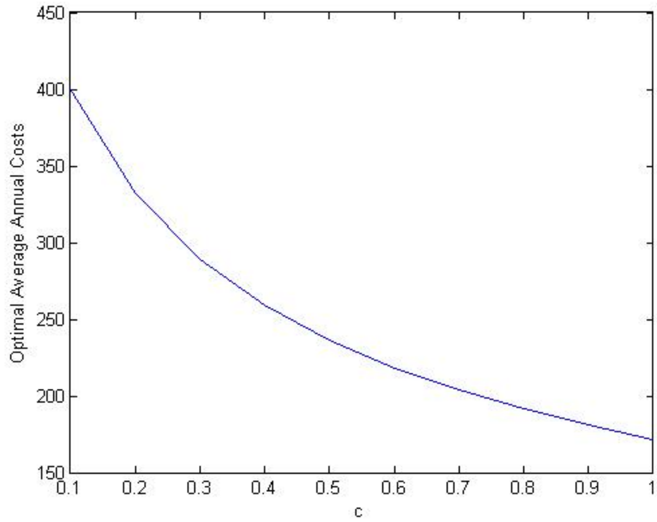

Fig. 4 c versus AC(Q*,r*)

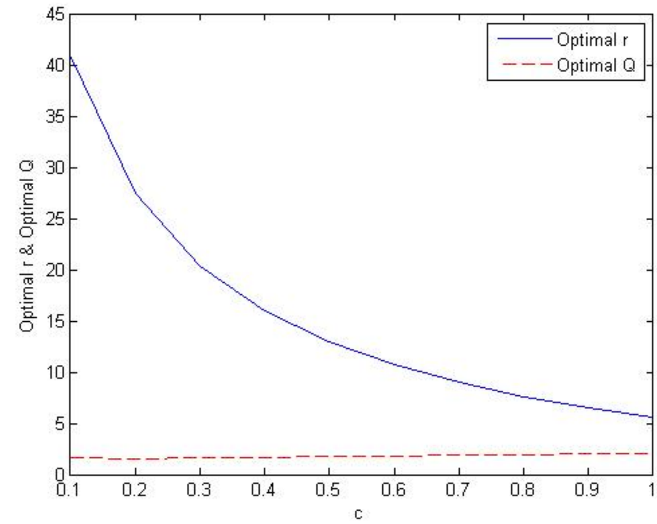

Fig. 5 c versus r* and Q*

IV. CONCLUSION

In this research, we develop an inventory model under disruptions due to machine breakdown. Instead of assuming that failure rate and repair rate are constant, we formulate a time-dependent model based on two-state non-homogeneous Markov process. We also study the effect of imperfect corrective maintenance on the optimal solutions. Numerical tests show that the assumptions about failure and repair rates as well as the degree of imperfect corrective maintenance at each failure have significant impacts on the optimal solutions.

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