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Space–Time Hotelling Model and Its Application to Retail Competition in a Duopoly

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***Abstract*—We propose a space–time Hotelling model that introduces a unit size of the vertical time axis in the classical Hotelling unit interval model. The proposed model allows explicit consideration of the probability that a consumer arrives at a retail store up to time t to purchase goods. The proposed model is useful in a variety of retailing problems. We briefly demonstrate an application of the proposed model to retail competition in a duopoly.**

***Index Terms*—space–time Hotelling model, retail competition, duopoly, business hours, departure time distribution, demand distribution up to time t.**

I. Introduction

SINCE Hotelling [1] introduced the classical $[0, 1]$ unit interval to represent the spatial conditions in his competition problem, the underlying idea of the $[0, 1]$ unit interval has been used in the literature of a wide variety of research topics (e.g., [2]–[11]) due to its simple structure and mathematical tractability. Among them, Graitson [2] presented a review of works up to 1982, while Biscaia and Mota [9] provided an extensive review of studies since the 1970s. The literature in these reviews is related to competition problems. The Hotelling model has also been applied to the stock of pollution [5], investigation of channel performance [10], facility location problems [7], evaluation of influences of store-brand introductions [6], and strategic outsourcing for supply chain management [4]. The Hotelling model also influences deriving demand functions in decision science [11] and the theory of product variety in economics [3].

When dealing with problems associated with retail stores under spatial conditions, it is important to consider the arrival times of individual consumers because consumers not arriving during business hours cannot purchase products. Moreover, late-arriving consumers might not be able to purchase products due to stock outage. This study proposes a space–time Hotelling model that introduces a unit vertical time axis in the classical Hotelling unit interval model. Some applications to retail competition in a duopoly are also discussed.

II. Notations and assumptions

The notations and assumptions in this study are as follows: 1)

1) Homogeneous consumers are uniformly distributed on the Hotelling unit interval $[0, 1]$.
2) Store $\mathscr{A}$ is located at 0, while $\mathscr{B}$ is at 1 on the horizontal unit interval $[0, 1]$.
3) Stores $\mathscr{A}$ and $\mathscr{B}$ sell an identical product at prices $p_A(> 0)$ and $p_B(> 0)$, respectively.
4) The time horizon of a single day is expressed by a vertical unit interval $[0,\ 1]$.
5) Individual consumers on the horizontal $[0, 1]$ interval depart only once per day to travel to either $\mathscr{A}$ or $\mathscr{B}$, if the departure time permits arrival at the store during its business hours.
6) Consumer departure times are independent and identically distributed (i.i.d.) random variables having a general distribution with cumulative distribution function $G(t)$.
7) We assume $G(0) \geq 0$ and $G(1) \leq 1$, indicating that a consumer cannot depart with probability $1 + G(0) - G(1)$ for some arbitrary reason.
8) Consumer traveling velocity is $\lambda\,(> 0)$ with travel cost c per unit of time.
9) Both $\mathscr{A}$ and $\mathscr{B}$ are open during $[t_o, t_c]$, where $0 \leq t_o < t_c \leq 1$. To avoid analytical intricacies, $\frac{1}{2\lambda} \leq t_o \leq \frac{1}{\lambda}$.
10) Consumer willingness to pay for a product is represented by $u\ (> 0)$.
11) Without loss of generality, we assume $p_A \geq p_B$.

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III. Model

A. Consumer best response

When a consumer at $x \in [0, 1]$ visits store $\mathscr{A}$ to purchase a product, her net utility is given by

$$U_A = u - p_A - \frac{cx}{\lambda},$$

while purchasing a product at store $\mathscr{B}$ gives net utility

$$U_B = u - p_B - \frac{c(1-x)}{\lambda}.$$

Hence, by letting $U_A = U_B$, the boundary point, $\tilde{x}$, is given by

$$\tilde{x} = \begin{cases} 0, & p_A \geq p_B + \dfrac{c}{\lambda} \\ \dfrac{1}{2} - \dfrac{\lambda(p_A - p_B)}{2c}, & p_B - \dfrac{c}{\lambda} < p_A < p_B + \dfrac{c}{\lambda} \\ 1, & p_A \leq p_B - \dfrac{c}{\lambda} \end{cases}. \tag{1}$$

Under the classical Hotelling model, a consumer on $[0, \tilde{x}]$ will visit store $\mathscr{A}$, while a consumer on $[\tilde{x}, 1]$ will travel to store $\mathscr{B}$. This is not necessarily the case in this study.

Under the space–time Hotelling model, consumer best responses are broadly classified into the two cases shown in Figs. 1 and 2.

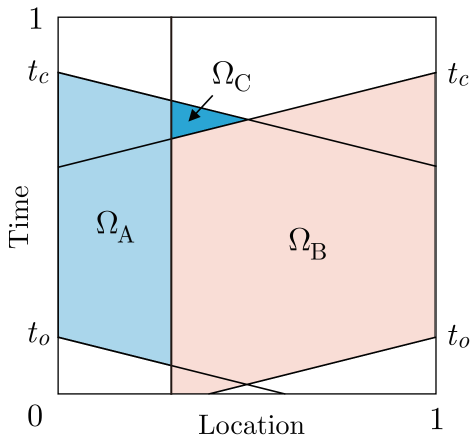

Fig. 1. Consumer best response in the case of $p_A > p_B$.

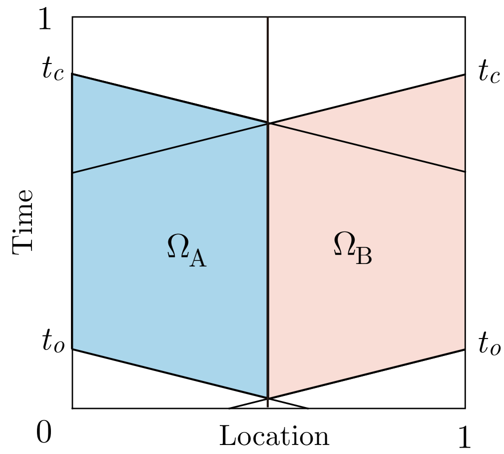

Fig. 2. Consumer best response for $p_A = p_B$.

Figure 1 shows the best response of the market in the case of $p_A > p_B$, where the regions of Ω_i $(i = A, B, C)$ are defined by

$$\Omega_A = \left\{(x,t); 0 \le x \le \frac{1}{2} - \frac{\lambda(p_A - p_B)}{2c},\right. \quad (2)$$

$$\left. \max\left(t_o - \frac{x}{\lambda}, 0\right) \le t \le t_c - \frac{x}{\lambda}\right\},$$

$$\Omega_B = \left\{(x,t); \frac{1}{2} - \frac{\lambda(p_A - p_B)}{2c} < x \le 1,\right. \quad (3)$$

$$\left. \max\left(t_o - \frac{1-x}{\lambda}, 0\right) \le t \le t_c - \frac{1-x}{\lambda}\right\},$$

$$\Omega_C = \left\{(x,t); \frac{1}{2} - \frac{\lambda(p_A - p_B)}{2c} \le x \le \frac{1}{2},\right. \quad (4)$$

$$\left. \max\left(t_c - \frac{1-x}{\lambda}, 0\right) \le t \le t_c - \frac{x}{\lambda}\right\}.$$

In Fig. 1, consumers with (x, t) in Ω_A travel to store $\mathscr{A}$, because store $\mathscr{B}$ is not close enough, while consumers in Ω_B visit store $\mathscr{B}$ owing to its lower price p_B. In Ω_C, however, it is too late for consumers to travel to $\mathscr{B}$, so they visit $\mathscr{A}$ because of its advantageous location.

It should be noted here that the classical Hotelling model cannot detect the existence of a region Ω_C in Fig. 1. Asian countries such as Japan, Korea and Republic of China have many convenience stores which correspond to $\mathscr{A}$. The convenience stores are small format and ubiquitous retail stores to sell items at higher prices than normal supermarkets. The region Ω_C explains the significances of convenience stores.

Figure 2 depict the best response of the market for $p_A = p_B$, defining Ω_i $(i = A, B)$ by

$$\Omega_A = \left\{(x,t) : 0 \le x \le \frac{1}{2},\right. \quad (5)$$

$$\left. \max\left(t_o - \frac{x}{\lambda}, 0\right) \le t \le t_c - \frac{x}{\lambda}\right\},$$

$$\Omega_B = \left\{(x,t) : \frac{1}{2} < x \le 1,\right. \quad (6)$$

$$\left. \max\left(t_o - \frac{1-x}{\lambda}, 0\right) \le t \le t_c - \frac{1-x}{\lambda}\right\}.$$

Consumer behavior in Ω_i $(i = A, B)$ is the same as that in Fig. 1.

Note that in both Figs. 1 and 2, consumers with (x, t) in the blank area below Ω_A (or Ω_B) can arrive at $\mathscr{B}$ (or $\mathscr{A}$) during business hours, but we have excluded them because (1) consumers in such cases would like to depart from their locations early in the morning, perhaps having plans in the daytime, and (2) if they travel to $\mathscr{B}$, they might arrive there in the daytime.

B. Probability of welcoming a single consumer

Let $\psi_i(t, x)$ $(i = A, B, C)$ be defined by

$$\psi_A(t;x) = G\left(t - \frac{x}{\lambda}\right) - G\left[\max\left(t_o - \frac{x}{\lambda}, 0\right)\right], \quad (7)$$

$$\psi_B(t;x) = G\left(t - \frac{1-x}{\lambda}\right) - G\left[\max\left(t_o - \frac{1-x}{\lambda}, 0\right)\right], \quad (8)$$

$$\psi_C(t;x) = \begin{cases} 0, & t_o \le t < t_c - \frac{1-2x}{\lambda} \\ G\left(t - \frac{x}{\lambda}\right) - G\left(t_c - \frac{1-x}{\lambda}\right), & t_c - \frac{1-2x}{\lambda} \le t \le t_c. \end{cases} \quad (9)$$

Then $\psi_A(t;x)$ signifies the probability that a consumer at $x \in [0, \tilde{x}]$ will arrive at $\mathscr{A}$ up to time t, while $\psi_B(t;x)$ indicates the probability that a consumer at $x \in [\tilde{x}, 1]$ will visit $\mathscr{B}$ up to time t. Finally, $\psi_C(t;x)$ represents the probability that a consumer at $x \in [\max(\frac{1}{2} - \frac{\lambda(p_- p_B)}{2c}, 0), \frac{1}{2}]$ reaches $\mathscr{A}$ up to time t.

We define $\rho_A(t)$ by

$$\rho_A(t) = \begin{cases} \int_0^{\frac{1}{2}} \psi_C(t;x)dx, & p_A \ge p_B + \frac{2c}{\lambda}, \\ \int_0^{\tilde{x}} \psi_A(t;x)dx + \int_{\tilde{x}}^{\frac{1}{2}} \psi_C(t;x)dx, & p_B \le p_A < p_B + \frac{2c}{\lambda}. \end{cases} \quad (10)$$

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Then, $\rho_A(t)$ expresses the probability that an arbitrary consumer on $[0, \frac{1}{2}]$ visits store $\mathscr{A}$ up to t. Likewise, let $\rho_B(t)$ be defined by

$$\rho_B(t) = \begin{cases} \int_0^1 \psi_B(t; x)dx, & p_A \geq p_B + \frac{c}{\lambda}, \\ \int_{\tilde{x}}^1 \psi_B(t; x)dx, & p_B < p_A < p_B + \frac{2c}{\lambda}. \end{cases} \tag{11}$$

These observations reveal that the share of store $\mathscr{A}$ at time t and that of store $\mathscr{B}$ are given by $\xi_A(t)$ and $\xi_B(t)$, respectively:

$$\xi_A(t) = \frac{\rho_A(t)}{\rho_A(t) + \rho_B(t)}, \tag{12}$$

$$\xi_B(t) = \frac{\rho_B(t)}{\rho_A(t) + \rho_B(t)}. \tag{13}$$

IV. APPLICATIONS

A. Demand distribution

One of the simplest applications of the proposed model is obtaining demand distributions at $\mathscr{A}$ and $\mathscr{B}$ up to time t. For this purpose, let us introduce i.i.d. random variables $Y_i(t)$ $(i = 1, 2, \cdots, n)$ having mass probabilities as follows:

$$\begin{aligned} \Pr[Y_i(t) = 0] &= 1 - \rho_A(t) - \rho_B(t), \\ \Pr[Y_i(t) = 1] &= \rho_A(t). \\ \Pr[Y_i(t) = 2] &= \rho_B(t), \end{aligned}$$

where n is the population size of consumers on $[0, 1]$. Note that $Y_i(t) = 0$ corresponds to an event where the ith consumer on $[0, 1]$ visits neither $\mathscr{A}$ nor $\mathscr{B}$ up to t. Likewise, $Y_i(t) = 1$ signifies a visit to $\mathscr{A}$, while $Y_i(t) = 2$ indicates reaching $\mathscr{B}$ up to time t.

Let indicator variables $\delta_i^{(A)}(t)$ and $\delta_i^{(B)}(t)$ be defined by

$$\begin{aligned} \delta_i^{(A)}(t) &= \begin{cases} 1, & Y_i(t) = 1, \\ 0, & Y_i(t) \neq 1, \end{cases} \\ \delta_i^{(B)}(t) &= \begin{cases} 1, & Y_i(t) = 2, \\ 0, & Y_i(t) \neq 2. \end{cases} \end{aligned}$$

Then $\delta_i^{(A)}(t)$ and $\delta_i^{(B)}(t)$ are i.i.d. random variables that follow a binomial distribution with parameters $\rho_A(t)$ and $\rho_B(t)$, respectively, where the number of trials is given by n. Moreover, the number of consumers visiting stores $\mathscr{A}$ and $\mathscr{B}$ up to t can be respectively expressed by $D_A(t)$ and $D_B(t)$, where

$$\begin{aligned} D_A(t) &= \sum_{i=1}^{n} \delta_i^{(A)}(t), \\ D_B(t) &= \sum_{i=1}^{n} \delta_i^{(B)}(t). \end{aligned}$$

When n is large, $D_i(t)$ asymptotically follows a normal distribution $N(\mu_i(t), \sigma_i^2(t))$, where

$$\mu_i(t) = n\rho_i(t), \tag{14}$$

$$\sigma_i^2(t) = n\rho_i(t)[1 - \rho_i(t)], \tag{15}$$

for $i = A$ and B.

B. Newsvendor problem

Another simple application of the proposed model is the newsvendor problem ([12]–[16]). Let p and w respectively denote the selling price and the raw price per unit of product, and let c_0 represent the opportunity loss per unit of product. Then the expected profit $\Pi_i(Q_i)$ of store i in the classical newsvendor problem is given by

$$\begin{aligned} \Pi_i(Q_i) = & -wQ_i + \int_0^{Q_i} pf(x)dx \\ & + \int_{Q_i}^{\infty} [pQ_i - c(x - Q_i)]f(x)dx, \ i = A, B, \end{aligned} \tag{16}$$

where Q_i denotes the stocking quantity at store i, and $f(x)$ is the probability density function of $N(\mu_i(t_c), \sigma_i^2(t_c))$. It is well known that the optimal stocking quantity $Q_i = Q_i^*$ is given by the solution to

$$\Phi\left(\frac{Q_i - \mu_i(t_c)}{\sigma_i(t_c)}\right) = \frac{p - w + c_0}{p + c_0}, \ i = A, B,$$

where $\Phi(\cdot)$ is the cumulative distribution function of $N(0, 1^2)$.

C. Business hours

Business hours have traditionally been regulated, particularly in many European countries, although the liberalization of business hours has generated debates over the past three decades (e.g., [17]–[20]). We can, however, discuss business hours competition between two stores within regulations, where store operation costs, such as labor costs, play an important role. This subsection briefly examines how to derive the best response of each store against its competitor in relation to business hours, which is the key to obtaining a Nash equilibrium. This is because the related mathematical analyses are very intricate.

Figures 3 and 4 show the best consumer response when store $\mathscr{B}$ in Fig. 1 changes its closing time t_c to $t_c + \Delta_t$. In Figs. 3 and 4, Ω_D shows the additional area of (x, t) where consumers visit $\mathscr{B}$. We have seen through Eqs. (14) and (15) that the probability of accepting a single consumer $\rho_B(t)$ influences the demand distribution at $\mathscr{B}$ through its mean and standard deviation.

In the following, we concentrate upon the best response to the closing time t_c of $\mathscr{B}$ against $\mathscr{A}$. The additional probability Δ_{ρ_B} of welcoming a single consumer for $\mathscr{B}$ is given by

(1) When $\Delta_t \leq \frac{p_A - p_B}{c}$, Δ_{ρ_B} is given by

$$\begin{aligned} \Delta_{\rho_B} = & \int_{\frac{1}{2} - \frac{\lambda(p_A - p_B)}{2c}}^{1} \left\{ G\left[\min\left(t_c + \Delta_t - \frac{1-x}{\lambda}, 1\right)\right] \right. \\ & \left. -G\left(t_c - \frac{1-x}{\lambda}\right) \right\}. \end{aligned} \tag{17}$$

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(2) In the case of $\Delta_t > \frac{p_A - p_B}{c}$, Δ_{ρ_B} becomes

$$\begin{aligned}
&\Delta_{\rho_B} \\
&= \int_{\frac{1}{2} - \frac{\lambda(p_A - p_B)}{2c}}^{1} \left\{ G\left[\min\left(t_c + \Delta_t - \frac{1-x}{\lambda}, 1\right)\right] \right. \\
&\qquad \left. - G\left(t_c - \frac{1-x}{\lambda}\right) \right\} \qquad (18) \\
&+ \int_{\frac{1-\Delta_t}{2}}^{\frac{1}{2}} \left\{ G\left[\min\left(t_c + \Delta_t - \frac{1-x}{\lambda}, 1\right)\right] \right. \\
&\qquad \left. - G\left(t_c - \frac{x}{\lambda}\right) \right\}.
\end{aligned}$$

Let c_1 and w, respectively, denote the store operation cost per unit of time and the raw price of a product. Then, if $n(p_B - w)\Delta_{\rho_B} > c_1 \Delta_t$, store $\mathscr{B}$ has an incentive to adopt $t_c + \Delta_t$ as its closing time; otherwise, t_c would not increase. Hence, the best response Δ_t^* of $\mathscr{B}$ against $\mathscr{A}$ is obtained as the solution to $n(p_B - w)\Delta_{\rho_B} = c_1 \Delta_t$.

The best response of $\mathscr{A}$ with respect to its closing time against $\mathscr{B}$ can be obtained in the same manner, and we may thereby obtain the Nash equilibrium based on the best responses of $\mathscr{A}$ and $\mathscr{B}$. The opening times can also be obtained in an analogous manner.

Under a monopoly, in contrast, Hosseinipour and Sandoh [16] have discussed the optimal number of business hours and optimal stocking quantity within the framework of the newsvendor problem.

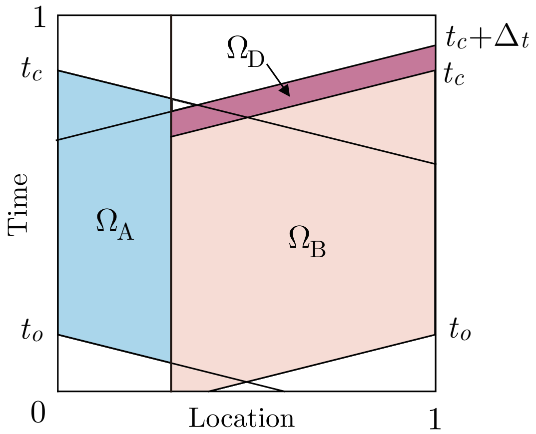

Fig. 3. Area associated with marginal probability ($\Delta_t \leq \frac{p_A - p_B}{c}$).

V. Concluding Remarks

We proposed a space–time Hotelling model, where a unit vertical axis is used to extend the classical Hotelling unit horizontal interval $[0, 1]$. Then, the departure time distribution of each individual consumer was introduced to express consumer random behavior for arriving at a store.

The proposed model is useful in considering a wide variety of competitions between retail stores under a duopoly because it can predict how many consumers each store can expect during its business hours. We also applied the proposed model to derive the demand distribution of each individual store, and then to discuss the newsvendor problem for retail stores. Competition in relation to business hours was also briefly considered.

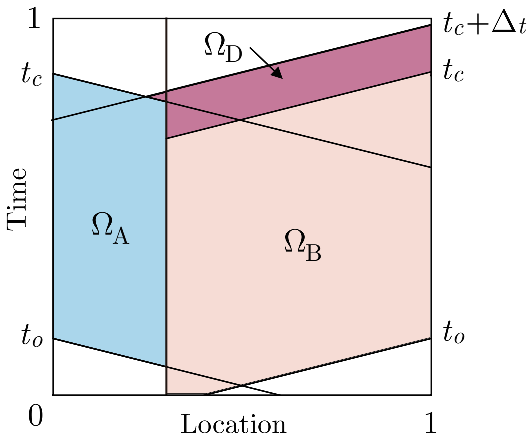

Fig. 4. Area associated with marginal probability ($\Delta_t > \frac{p_A - p_B}{c}$).

In the proposed model, the population size n of potential customers is assumed to be known. This may limit application of the proposed model in real circumstances because estimation of n is one of the most important problems. Bayesian approaches will be useful in coping with such a problem at the expense of complicating the model structure. A useful extension of our work would be to introduce direct marketers with the intent to comparing them and conventional retailers by regarding the distance to direct marketers and the lead time at conventional retailers to be negligible.

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