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Some Properties of Nonnegative Interval Matrices in Max Algebra

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***Abstract*—In this paper, we proposed the notion of max algebra of nonnegative interval matrices. The normalization of nonnegative interval matrices in max algebra are established. Some properties of max product of nonnegative interval matrices are derived as well.**

***Index Terms*—interval matrix; max algebra; maximum circuit geometric mean.**

I. Introduction

In the literature, the properties of interval matrix have been extensive studied (see, [1],[5-8], [10]). We refer to Alefeld and Herzberger [1] for the background materials of interval matrices. Real numbers are denoted by lowercase letters a, b. The $\overline{a}$ and $\underline{a}$ denote the upper and lower bounds of a real closed interval $[\underline{a}, \overline{a}]$, respectively. The set of all these closed intervals is denoted by $I(\mathcal{R})$. We may denote an interval $[\underline{a}, \overline{a}]$ by $[a] = [\underline{a}, \overline{a}]$. Let $* \in \{+, -, \times, \div\}$ be one of the usual binary operations on the set of real numbers. For $[a] = [\underline{a}, \overline{a}]$ $[b] = [\underline{b}, \overline{b}] \in I(\mathcal{R})$ the binary operation $[a] * [b] = \{a * b : a \in [a], b \in [b]\}$, is assumed that $0 \neq [b]$ in the case of division. For a nonnegative interval $[a] = [\underline{a}, \overline{a}]$, the width $d([\underline{a}, \overline{a}])$ and the absolute value $|[\underline{a}, \overline{a}]|$ are defined by

$$d([\underline{a}, \overline{a}]) = \overline{a} - \underline{a},$$

$$|[\underline{a}, \overline{a}]| = \max\{|\underline{a}|, |\overline{a}|\}, respectively.$$

We called $[a] = [\underline{a}, \overline{a}]$ a point interval if $\underline{a} = \overline{a}$. In this case, we say $[a] = [\underline{a}, \overline{a}]$ is degenerated to a point interval.

A matrix with entries belonging to $I(\mathcal{R})$ is called an interval matrix. The set of all real $n \times n$ interval matrices is denoted by $I(R^{n\times n})$. We denote an interval matrix $[A] \in (R^{n\times n})$ by $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = [\underline{a}_{ij}, \overline{a}_{ij}]$. Two interval matrices $[A]$ and $[B]$ are equal if and only if $([a]_{ij}) = ([b]_{ij})$ for all $i, j = 1, 2, \ldots, n$. That is $\underline{a}_{ij} = \underline{b}_{ij}$ and $\overline{a}_{ij} = \overline{b}_{ij}$ for all $i, j = 1, 2, \ldots, n$. For interval matrices $[A]$, $[B] \in I(R^{n\times n})$ and an interval $[x] = [\underline{x}, \overline{x}] \in I(\mathcal{R})$, the matrix operations $+, -, \times$ are formally defined as

$$[A] \pm [B] = ([a]_{ij} \pm [b]_{ij}),$$
$$[A] \times [B] = (\Sigma_{k=1}^{n} [a]_{ik} \times [b]_{kj}),$$
$$[x] \cdot [A] = ([x] \times [a]_{ij}).$$

Let I be an $n \times n$ identity matrix. The powers of interval matrix $[A]$ are defined as

$$[A]^0 = I,$$

$$[A]^k = [A]^{k-1} \times [A], \quad k = 1, 2, \ldots.$$

As noted by Mayer [6], the product of the interval matrices is not associative in general. Therefore, $([A] \times [B]) \times [C]$ may not be equal to $[A] \times ([B] \times [C])$. An interval $[a] = [\underline{a}, \overline{a}]$ is said to be nonnegative if $\underline{a} \geq 0$. The set of all nonnegative interval is denoted by $I(\mathcal{R}^+)$.

II. Maximum circuit geometric mean of a nonnegative real matrix

Let A be an $n \times n$ nonnegative matrix. A scalar λ is called a *max eigenvalue* of A if $A \otimes x = \lambda x$ for some nonnegative vector $x \neq 0$, namely,

$$\max_{1 \leq j \leq n} a_{ij} x_j = \lambda x_i \quad \text{for all } i = 1, 2, \ldots, n.$$

The vector x is called a corresponding *max eigenvector* of λ. The weighted directed graph $\mathcal{D}(A)$ associated with A has vertex set $\{1, 2, \ldots, n\}$ and an edge (i, j) from vertex i to vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path $L(i_1, i_2, \ldots, i_k, i_{k+1})$ of length k is a sequence of k edges $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$. The *weight* of a path $L(i_1, i_2, \ldots, i_{k+1})$, as denoted by $w(L(i_1, i_2, \ldots, i_{k+1}))$ or simply by $w(L)$, is defined by

$$w(L(i_1, i_2, \ldots, i_{k+1})) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_{k+1}}.$$

A *circuit* C of length $k \geq 2$ is a path $L(i_1, i_2 \ldots, i_{k+1})$ with $i_{k+1} = i_1$, and $i_1, i_2, \ldots, i_k$ are distinct. The class of circuits includes *loops*, ie., circuits of length 1. Associated with this circuit C is the *circuit geometric mean* known as $\hat{w}(C) = (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})^{1/k}$. The *maximum circuit geometric mean* in $\mathcal{D}(A)$ is denoted by $\mu(A)$. Note that we also consider *empty* circuits, namely, circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero. A circuit C with $W(C) = \mu(A)$ is called a critical circuit. Vertices on critical circuits are called critical vertices and edges on critical circuits are called critical edges.

Definition 1. Let $[A] = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix. The maximum circuit geometric mean of $[A]$ denoted by $\mu([A])$, is $\mu([A]) = \max\{\mu(A) : A \in [A]\}$.

As $\mu(A) \geq \mu(B)$ for all nonnegative matrices $A \geq B$, we see that $\mu([A]) = \mu(\overline{A})$. Let $[A] = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}]_{ij})$ be given. Recall that $r \cdot [A] = (r \cdot [a]_{ij}) = ([r\underline{a}_{ij}, r\overline{a}_{ij}])$, for all real number r. Suppose that $\mu([A]) \neq 0$. Set $k = \frac{1}{\mu([A])}$. It is easy to see that $\mu(k \cdot [A]) = 1$.

To prove Theorem 2, we need following theorem which was proved by Ludwig Elsner and P. van den Driessche.

Theorem 1 [4]. Let A be an $n \times n$ irreducible nonnegative real matrix with $\mu(A) \leq 1$. Let $x \geq 0, x \neq 0$ and $z = A^* \otimes x$. Then $z > 0$. If $D = diag(z_i)$, then $D^{-1}AD \leq J_n$, i.e., each entry has magnitude less than or equal to 1.

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Lemma 1. Let A be an $n\times n$ nonnegative real matrix with $\mu(A)\le 1$. Then there exists a diagonal real matrix D such that $D^{-1}AD\le J_n$, i.e., each entry has magnitude less than or equal to 1.

Theorem 2. Let $[A]=[\underline{A},\overline{A}]$ be an $n\times n$ nonnegative interval matrix with $\mu([A])\le 1$. Then there exists a diagonal real matrix D such that $D^{-1}\times[A]\times D\le[J_n]$, i.e., each entry has magnitude less than or equal to $[0,1]$.

III. Nonnegative interval matrices in max algebra

We refer to [2-4] and [9] for the study of nonnegative matrices in max algebra. In this section, we shall define the max algebra of nonnegative interval matrices. Let $[a]=[\underline{a},\overline{a}]$, $[b]=[\underline{b},\overline{b}]$ be two nonnegative intervals. Define the maximum of $[a]$ and $[b]$ by

$$[a]\vee[b]=\{a\vee b: a\in[a], b\in[b]\},$$

here $a\vee b=\max\{a,b\}$.

Theorem 3. Let $[a],[b]\in I(\mathcal{R})$. Then $[a]\vee[b]=[\max\{\underline{a},\underline{b}\},\max\{\overline{a},\overline{b}\}]$ is also an interval.

Proof. Let $a\in[a]=[\underline{a},\overline{a}]$ and $b\in[b]=[\underline{b},\overline{b}]$ be given. As $\underline{a}\le a\le\overline{a}$ we have

$$[a]\vee[b]\subset[\max\{\underline{a},\underline{b}\},\max\{\overline{a},\overline{b}\}]. \tag{1}$$

On the other hand, let $r\in[\max\{\underline{a},\underline{b}\},\max\{\overline{a},\overline{b}\}]$ be given. Without loss of generality we may assume that $\overline{a}\le\overline{b}$. Then $\underline{b}\le r\le\overline{b}$ and $\underline{a}\le r$. Hence $\underline{a}\vee r=r\in[a]\vee[b]$. This and (1) imply that $[a]\vee[b]=[\max\{\underline{a},\underline{b}\},\max\{\overline{a},\overline{b}\}]$. Therefore, $[a]\vee[b]$ is also an interval. This completes the proof.

Define $\max\{[a],[b]\}=[a]\vee[b]$. The max algebra interval system is defined as follow: Let $I(\mathcal{R}^+_{\max,\times})$ = $(I(\mathcal{R}^+),\oplus,\otimes)$ be consisted of the set of nonnegative interval numbers with sum $[a]\oplus[b]=[a]\vee[b]$ and the product of $[a]\otimes[b]$ is defined by $[a]\otimes[b]=[a]\times[b]$. The following theorem shows that the max algebra on interval is a semiring with identity element $[0]=[0,0]$. Moreover, $\oplus$ is idempotent, ie., $[a]\oplus[a]=[a]$. Let $\{[a_1],[a_2],\ldots,[a_k]\}$ be a finite set of nonnegative intervals. Define $\bigvee_{j=1}^{k}[a_j]=[a_1]\vee[a_2]\vee\cdots\vee[a_k]$.

Theorem 4. Let $I(\mathcal{R}^+_{\max,\times})$=$(I(\mathcal{R}^+),\oplus,\otimes)$ be the set of $n\times n$ nonnegative interval numbers with sum $[a]\oplus[b]=[a]\vee[b]$ and the product $[a]\otimes[b]=[a]\times[b]$. Then $I(\mathcal{R}^+_{\max,\times})$ is a semiring.

Proof. Let $[a],[b],[c]\in I(\mathcal{R}^+)$ be given. Then

$$([a]\vee[b])\vee[c]=[\max\{\underline{a},\underline{b},\underline{c}\},\max\{\overline{a},\overline{b},\overline{c}\}]=[a]\vee([b]\vee[c]),$$

$$[a]\vee[0]=[a]=[0]\vee[a]$$

and

$$[a]\vee[b]=[b]\vee[a].$$

This shows that $I(\mathcal{R}^+),\oplus)$ is a commutative monoid with identity element $[0]=[0,0]$. As $[a]\otimes[b]=[a]\times[b]=[\underline{ab},\overline{ab}]$, we see that

$$([a]\otimes[b])\otimes[c]=[a]\otimes([b]\otimes[c]).$$

It is easy to see that $[a]\otimes[1]=[1]\times[a]=[a]$. Thus $(I(\mathcal{R}^+),\otimes)$ is a monoid with identity element $[1]=[1,1]$. Observe that $[a]\otimes[0]=[0]\otimes[a]=[0]$. Now we claim that

$$([a]\oplus[b])\otimes[c]=([a]\otimes[c])\oplus([b]\otimes[c])$$

and

$$[c]\otimes([a]\oplus[b])=([c]\otimes[a])\oplus([c]\otimes[b]).$$

To see this, observe that

$$\begin{aligned}([a]\oplus[b])\otimes[c] &= [\max\{\underline{a},\underline{b}\},\max\{\overline{a},\overline{b}\}]\otimes[\underline{c},\overline{c}]\\ &= [\max\{\underline{ac},\underline{bc}\},\max\{\overline{ac},\overline{bc}\}]\\ &= [\underline{ac},\overline{ac}]\oplus[\underline{bc},\overline{bc}]\\ &= ([a]\otimes[c])\oplus([b]\otimes[c]).\end{aligned}$$

By the similarly argument, we have $[c]\otimes([a]\oplus[b])=([c]\otimes[a])\oplus([c]\otimes[b])$. This completes the proof.

It is easy to show the following result.

Theorem 5. Let $I(\mathcal{R}^+_{\max,\times})$=$(I(\mathcal{R}^+),\oplus,\otimes)$ be the set of $n\times n$ nonnegative intervals. Then the order $\le$ defined by $[a]\le[b]$ if $\underline{a}\le\underline{b}$ and $\overline{a}\le\overline{b}$ is a partial order.

Now we consider the max product of two nonnegative interval matrices in the max algebra interval system. Let $[A]$ and $[B]$ be two nonnegative interval matrices. The max-product $[A]\otimes[B]$ of $[A]$ and $[B]$ is defined by

$$([A]\otimes[B])_{ij}=\bigvee_{k=1}^{n}[a]_{ik}\otimes[b]_{kj}.$$

Let I be an $n\times n$ identity matrix. The powers of nonnegative interval matrix $[A]$ in the max algebra interval system are defined as

$$[A]^0_\otimes=I,$$

$$[A]^k_\otimes=[A]^{k-1}_\otimes\times[A],\quad k=1,2,\ldots.$$

Note that $[A]\otimes[B]$ may not be equal to $[B]\otimes[A]$. However, $([A]\otimes[B])\otimes[C]=[A]\otimes([B]\otimes[C])$.

Now we consider the max product of powers of a nonnegative interval matrix.

Theorem 6. Let $[A]=[\underline{A},\overline{A}]$ be an $n\times n$ nonnegative interval matrix. Then $[A]^k_\otimes=[\underline{A}^k_\otimes,\overline{A}^k_\otimes]$, for all $k\ge 1$.

For an $n\times n$ nonnegative interval matrix $[A]=([a]_{ij})$, the two nonnegative matrices $d([A])=(d([a]_{ij}))$ and $|[A]|=(|[a]_{ij}|)$, which are called the width and absolute value of $[A]$, respectively.

Let $A=(a_{ij})$ and $B=(b_{ij})$ be two $n\times n$ nonnegative matrices. Define $A\le B$ by $a_{ij}\le b_{ij}$ for all $1\le i,j\le n$. Let $[A]=([a]_{ij})$ and $[B]=([b]_{ij})$ be two $n\times n$ nonnegative interval matrices. Define $[C]=([c]_{ij})=[A]\vee[B]$ by $[c]_{ij}=[a]_{ij}\vee b]_{ij}$ for all $1\le i,j\le n$.

Theorem 7. Let $[A],[B]$ be two $n\times n$ nonnegative interval matrix. Then

(1). $d([A]\vee[B])\le\max\{d([A]),d([B])\}$.
(2). $|[A]\otimes[B]|=|[A]|\otimes|[B]|$.
(3). $|([A]\vee[B]|=\max\{|[A]|,|[B]|\}$.
(4). $d([A])\otimes|[B]|,|[A]|\otimes d([B])\le d([A]\otimes[B])\le d([A])\otimes|[B]|+|[A]|\otimes d([B])$.

IV. Conclusion

The max algebra system of nonnegative real numbers has been studied extensively in the literature. In this paper, we proposed the notion of max algebra system of nonnegative interval matrices which can be thought of as a generalization of the notion of the max algebra system of nonnegative matrices. Some properties of max algebra system of nonnegative interval matrices are established.

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