Proceedings of the International MultiConference of Engineers and Computer Scientists 2015 Vol I,
IMECS 2015, March 18 - 20, 2015, Hong Kong

On Duality Concepts regarding Hypergraphs and Propositional Formulas

Stefan Porschen *

Abstract—**There is a well known duality concept in propositional logic (cf. e.g. [3, 14]) essentially relating CNF formulas to tautologically equivalent DNF formulas. In this paper we propose and discuss another duality principle in propositional logic based on a set theoretic duality interchanging *clauses* and *literals*, thus working for arbitrary normal form formulas. This concept is closely related to a set theoretic approach to hypergraph duality which is discussed in advance.**

Keywords: propositional-logic, normal-form-formula, hypergraph, duality

1 Introduction

There is a well known notion of duality in propositional logic (cf. e.g. [3, 14]). Given a well formulated expression F over a set V of atoms such that it contains only the Boolean operators $\wedge, \vee, \neg$. Then the dual expression F^d essentially is obtained by exchanging $\wedge$ and $\vee$ and by replacing each atom x by its negation $\neg x$. For a precise inductive definition of this concept of duality cf. [14]. Specifically if F is a formula in conjunctive normal form (CNF), then its dual F^d is a tautologically equivalent formula in disjunctive normal form (DNF). A similar concept exists for Boolean functions (also called truth functions [10]). Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, then its dual f^d is defined pointwise by $f^d(x) = \overline{f}(\overline{x}), x \in \{0,1\}^n$. An often studied problem in this context is DUALIZATION of Boolean functions (or, closely related, hypergraphs, see below) [4, 5, 7, 8, 9]. Here one is given the prime CNF of a (monotone or positive) Boolean function f and is asked for the prime CNF of its dual f^d. As mentioned, there is a closely related duality concept for hypergraphs resting on its transversal hypergraph. By definition a transversal (or cover) of a hypergraph $H = (V, E)$ is a subset of the vertex set $t \subseteq V$ such that $\forall e \in E : t \cap e \neq \emptyset$. A *minimal* transversal properly contains no other transversal of H. Given H then the transversal or *dual* hypergraph H^d has the same vertex set and the set of all minimal transversals as edge set. The problem of computing the dual of a hypergraph is equivalent to the problem of computing the (monotone) dual of a (monotone) Boolean function [7, 8]. In this paper we propose a duality concept for hypergraphs called *diagonalized duality*. Essentially it exchanges the roles of vertices and hyperedges (thus modifying the classical duality notion [2] exchanging rather the index sets of the vertex and the hyperedge sets). In a second step this notion of diagonalized duality is also transfered to propositional logic. More precisely, it will be defined for (arbitrary) normal form propositional formulas resulting in exchanging the roles of appropriately defined notions of *clauses* and *variables*, resp. *literals*.

2 Diagonalized Duality of Hypergraphs

Compared to the duality principle based on the transversal hypergraph there is a more classical notion of hypergraph duality. This essentially exchanges the roles of its vertex and (hyper)edge sets. Let I be a (finite) index set and let $E := (e_i)_{i \in I}$ be a collection of sets with $V := \bigcup_{i \in I} e_i$. Then $H = (V, E)$ defines a hypergraph with vertex set V and collection of (hyper)edges E. Since the edges are distinguished by its indices it is possible that $e_i = e_j, i \neq j$. In this case we also call H a *hypergraph over* I which is indicated by $H = (V, E : I)$. The following notion of duality is well settled (cf. e.g. [2]):

Definition 1 *Let $H = (V, E : I)$ be a hypergraph over fixed index set I. The* (set theoretical) dual *hypergraph H^* of H is defined as follows: For each $x \in V$ let $x^* := \{i \in I : x \in e_i\}$ be the* dual (hyper)edge *corresponding to x. Then set $H^* = (V^*, E^*)$, where $V^* := \bigcup_{x \in V} x^*$ and $E^* := \{x^* : x \in V\}$.*

This duality actually exchanges the roles of the index and the vertex sets since we have $V^* = I$ and $E^* = (x^*)_{x \in V}$. Hence H^* is a hypergraph over V, namely $H^* = (I, E^* : V)$. From this expression immediately follows syntactically $H^{**} := (H^*)^* = (V, E^{**} : I)$. An inspection of the incidence matrices of H and H^{**} then tells us that the duality is consistent in the following sense:

Lemma 1 $H^{**} = H$.□

In the sequel we are interested in a set theoretic duality "living" on the vertex and edge sets itself rather than on the index sets. Let us consider a restricted class of hypergraphs, namely those in which each edge occurs only

*Mathematics Group, Department 4, HTW Berlin, D-10313 Berlin, Germany, Email: porschen@htw-berlin.de

ISBN: 978-988-19253-2-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

once. Then instead of speaking about a collection of edges whose members are refered to by an index set, the edges now constitute a (finite) *set*. Thus from now on, a hypergraph is a pair $H = (V, E)$ where $V = V(H)$ is a finite set, the *vertex set* and $E = E(H)$ is a family of subsets of V the *(hyper)edge set* such that for each $x \in V$ there is an edge containing it. An edge with $|e| = 1$ is called a *loop*, and if we have $|e| \geq 2$ for all edges of a hypergraph then it is called *loopless*. For a vertex x of H, let $E_x = \{e \in E : x \in e\}$ be the set of all edges containing x. Then $\omega_H(x) := |E_x|$ denotes the *degree* of the vertex x in H, we simply write $\omega(x)$ when there is no danger of confusion.

Let $\mathcal{H}$ denote the set of all (finite, not necessarily loopless) hypergraphs $H = (V, E)$ with $E \subseteq 2^V \setminus \{\emptyset\}$ such that $V = \bigcup_{e \in E} e$, where 2^V denotes the power set of V. Recall that a *hypergraph isomorphy* $H_1 \cong H_2$ is a bijection $V(H_1) \to V(H_2)$ which in both directions preserves the structure of hyperedges.

Remark 1 *It is not hard to verify that $H_1 \cong H_2$ implies $H_1^* \cong H_2^*$. Moreover, the reverse implication can be concluded then directly via Lemma 1. So, we always have $H_1 \cong H_2$ if and only if $H_1^* \cong H_2^*$.*

Recall that a hypergraph is called *Sperner* if no hyperedge is contained in another hyperedge [2]. A hypergraph $H = (V, E) \in \mathcal{H}$ is called *linear* [2] if for all $e, e' \in E$ it holds that $|e \cap e'| \leq 1$ whenever $e \neq e'$. Let $\mathcal{H}_{\text{lin}} \subseteq \mathcal{H}$ denote the class of all linear hypergraphs.

We define $\mathcal{H}(n) \subseteq \mathcal{H}$ as the collection of those hypergraphs having a vertex set of cardinality $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, and set $[n] := \{1, 2, \ldots, n\}$. Let $H = (V, E) \in \mathcal{H}(n)$ be fixed. For each $x \in V$ considering the set $E_x := \{e \in E : x \in e\}$ yields an equivalence relation on V, namley $x \sim_H y :\Leftrightarrow E_x = E_y$. Let $[x]_H$ denote the equivalence class containing x and let $\tilde{V}_H := V/\sim_H$ denote the corresponding quotient space. Define $\mathcal{H}_k(n) = \{H = (V, E) \in \mathcal{H}(n) : |\tilde{V}_H| = k\}$, for positive integer $k \leq n$, i.e., the collection of all hypergraphs admitting a quotient space as above of cardinality k. For example, as is not hard to see the class $\mathcal{H}_1(n)$ consists of all hypergraphs of the form $H = (V, \{e = V\})$, i.e., those hypergraphs consisting of only one edge containing all n vertices, for appropriate $n \in \mathbb{N}$. Clearly, for each $H \in \mathcal{H}(n)$ there is a unique $k \in [n]$ such that $H \in \mathcal{H}_k(n)$ and as disjoint union we have $\mathcal{H}(n) = \bigcup_{k \in [n]} \mathcal{H}_k(n)$.

Definition 2 *For fixed number n of vertices, we call $\mathcal{H}_n(n)$ the* n-diagonal class (of hypergraphs)*, and the collection $\mathcal{H}_{\text{diag}} := \bigcup_{n \in \mathbb{N}} \mathcal{H}_n(n) \subseteq \mathcal{H}$ is called the* diagonal *class.*

This class obviously contains all hypergraphs H consisting of loops only, i.e., $E(H) = \{\{x\} : x \in V(H)\}$. For the set theoretical dual of a member in $\mathcal{H}$ the same holds true.

Lemma 2 *If $H \in \mathcal{H}$ then $\mathcal{H}^* \in \mathcal{H}_{\text{diag}}$.*

Proof. Let $H = (V, E) \in \mathcal{H}$ with $m := |E|$, then by definition $H^* = ([m], E^* : V)$. Since for every $x^* \in E^*$ we have $x^* = \{i \in [m] : x \in e_i\}$, and as basis for the equivalence relation on $V^* = [m]$ we consider the sets $E_i^* := \{x^* \in E^* : i \in x^*\}$, for $i \in [m]$, one immediately verifies that $x \in e_i \Leftrightarrow x^* \in E_i^*$. Moreover, since $H \in \mathcal{H}$ there are no $i, j \in [m]$ such that $i \neq j$ and $e_i = e_j$. This in turn means that the same holds true for the sets E_i^*, $i \in [m]$, implying $H^* \in \mathcal{H}_{\text{diag}}$. □

Let $\mathcal{H}_{\geq 2}$ be the collection of all hypergraphs such that each vertex has degree at least two, similar for the linear class, then we have the following observations.

Theorem 1 *$\mathcal{H}_{\text{lin},\geq 2} \subseteq \mathcal{H}_{\text{diag}}$, but $\mathcal{H}_{\text{lin},\geq 2} \neq \mathcal{H}_{\text{diag}}$, so $\mathcal{H}_{\text{lin},\geq 2}$ is a proper subclass. Moreover if $H \in \mathcal{H}_{\text{diag}}$ then $G \in \mathcal{H}_{\text{diag}}$ for all $G \in \mathcal{H}$ with $G \cong H$.*

Proof. Let $H = (V, E) \in \mathcal{H}_{\text{lin},\geq 2}$ such that $|V| =: n$, for positive integer n, and suppose $|\tilde{V}_H| = k < n$. Hence there are $x, y \in V$ such that $E_x = E_y$ and $x \neq y$. Since $\omega(v) \geq 2$ for every vertex v of H it is guaranteed that $|E_x| = |E_y| \geq 2$. Thus we have $e, e' \in E$, $e \neq e'$, such that $x, y \in e \cap e'$ contradicting the linearity of H. For proving the second claim regarding properness of the inclusion, consider $H = (V, E) \in \mathcal{H}_{\geq 2}$ with $V = \{a, b, x, y\}$ and $E = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{a, x, y\}$, $e_2 = \{b, x, y\}$, $e_3 = \{a, x\}$, $e_4 = \{a, b\}$; thus H is not linear. But $H \in \mathcal{H}_{\text{diag}}$ holds, as can be verified easily. Finally, let $G, H \in \mathcal{H}$ be in the same isomorphism class and $H \in \mathcal{H}_{\text{diag}}$. Since $x \in e \in E(H) \Leftrightarrow \varphi(x) \in \varphi(e) \in E(G)$, for an appropriate isomorphism $\varphi : V(H) \to V(G)$, we see that $E(H)_x = E(H)_y$ if and only if $E(G)_{\varphi(x)} = E(G)_{\varphi(y)}$ for arbitrary $x, y \in V(H)$. Thus the last statement is verified where we set $\varphi(e) := \{\varphi(x) : x \in e\}$. □

On the other hand notice that $H \in \mathcal{H}_{\text{diag}}$ obviously needs not to possess the Sperner property, i.e., $e \subseteq e' \Rightarrow e = e'$, as the example in the preceding proof indicates. A basic notion for our new duality concept is the following.

Definition 3 *For $H = (V, E) \in \mathcal{H}$, the (unique) hypergraph $\tilde{H} := (\tilde{V}_H, \tilde{E}_H)$ is called the* diagonalization *of H, where $\tilde{E}_H := \{\tilde{e} : e \in E\}$ and $\tilde{e} := \{[x]_H : x \in e\}$.*

For providing an example we start with $H = (V, E)$ with $V = \{a, b, c, d\}$ and $E = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{a, b, c\}$, $e_2 = \{a, b\}$, $e_3 = \{c, d\}$, $e_4 = \{d\}$. Following the construction rules as stated above we obtain $\tilde{H} = (\tilde{V}_H, \tilde{E}_H)$ with $\tilde{V}_H = \{[a]_H, [c]_H, [d]_H\}$. Moreover, $\tilde{E}_H = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, where $\tilde{e}_1 = \{[a]_H, [c]_H\}$,

ISBN: 978-988-19253-2-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\tilde{e}_2 = \{[a]_H\}$, $\tilde{e}_3 = \{[c]_H, [d]_H\}$, $\tilde{e}_4 = \{[d]_H\}$. Further observe that despite of $H \notin \mathcal{H}_{\text{diag}}$, namely $H \in \mathcal{H}_3(4)$, we have $\tilde{H} \in \mathcal{H}_3(3)$ thus $\tilde{H} \in \mathcal{H}_{\text{diag}}$ which is a fact in general.

Lemma 3 *For $H \in \mathcal{H}$, we have $|\tilde{E}_H| = |E|$, moreover*
(i) $\tilde{H} \in \mathcal{H}_{\text{diag}}$,
(ii) $H \in \mathcal{H}_{\text{diag}}$ if and only if $H \cong \tilde{H}$,
(iii) if $H \in \mathcal{H}_{\text{lin},\geq 2}$ then $\tilde{H} \in \mathcal{H}_{\text{lin},\geq 2}$, and
(iv) $H \cong G$ implies $\tilde{H} \cong \tilde{G}$.

Proof. Let $H = (V, E) \in \mathcal{H}$ be arbitrarily chosen. For the first assertion, suppose there are $e, e' \in E$ such that $e \neq e'$ but $\tilde{e} = \tilde{e}'$. Consider a vertex $c \in e$ with $c \notin e'$ implying $e' \notin E_c$. Since for $x \in e'$ we have $e' \in E_x$ it follows that $c \not\sim_H x$ for all $x \in e'$. Therefore $[c]_H \notin \tilde{e}'$ yielding a contradiction and implying $e \subseteq e'$. The reverse implication can be shown analogously, thus each e induces a unique $\tilde{e}$ (where often holds $|\tilde{e}| \leq |e|$). So, we always have $|\tilde{E}_H| = |E|$ which now is an immediate consequence of the fact that E, as a set, contains distinct edges only. For verifying assertion (i), let $n := |V|$ then there is a fixed $k := |\tilde{V}_H| \in [n]$ such that $H \in \mathcal{H}_k(n)$ and by construction we obtain $\tilde{H} \in \mathcal{H}_i(k)$ for $i \in [k]$. To verify $i = k$ it suffices to prove that $[x]_H \sim_{\tilde{H}} [y]_H$ if and only if $x \sim_H y$ establishing that the class structure is respected. Indeed, let $x, y \in V$ with $x \neq y$ such that $[x]_H \sim_{\tilde{H}} [y]_H$. Because every $e \in E_x$ induces a unique $\tilde{e} \in E_{[x]_H}$ it follows that $E_{[x]_H} = E_{[y]_H}$ implies $E_x = E_y$ thus $x \sim_H y$, the converse implication can be treated analogously. Hence $\tilde{H}$ indeed is a member of the diagonal class. Next consider claim (ii) and assume $H \in \mathcal{H}_{\text{diag}}$ meaning $H \in \mathcal{H}_n(n)$. From the proof above it follows that $\tilde{H} \in \mathcal{H}_n(n)$ also. Clearly we have $|\tilde{V}_H| = |V| = n$ and $x \mapsto [x]_H = \{x\}$ provides a bijection of the vertex sets of H and $\tilde{H}$. Therefore every e is in bijection to $\tilde{e}$ yielding $H \cong \tilde{H}$. Conversely assume $H \cong \tilde{H}$ for any $H \in \mathcal{H}$, then by part (i) and the second statement of Theorem 1 we obtain $H \in \mathcal{H}_{\text{diag}}$. Let $H \in \mathcal{H}_{\text{lin},\geq 2}$ then $H \in \mathcal{H}_{\text{diag}}$ according to the first statement of Theorem 1 therefore $H \cong \tilde{H}$ due to part (ii) above. Hence, we must have $\tilde{H} \in \mathcal{H}_{\text{lin},\geq 2}$ which proves (iii). Finally addressing (iv) let $H = (V, E), G = (W, F) \in \mathcal{H}$ and $\varphi : H \to G$ be an isomorphism. We claim that the induced mapping $\tilde{\varphi} : \tilde{V}_H \to \tilde{W}_G$ with $\tilde{\varphi}([x]_H) := [\varphi(x)]_G$, for every $x \in V$, yields an isomorphism between $\tilde{H}$ and $\tilde{G}$. In fact, from $e \in E \Leftrightarrow \varphi(e) \in F$ we obtain $x \sim_H y \Leftrightarrow E_x = E_y \Leftrightarrow F_{\varphi(x)} = F_{\varphi(y)} \Leftrightarrow \varphi(x) \sim_G \varphi(y)$. Therefore $[x]_H \in \tilde{e} \Leftrightarrow [\varphi(x)]_G \in \widetilde{\varphi(e)}$ according to $\sim_G$. Hence, we obtain $\widetilde{\varphi(e)} = \tilde{\varphi}(\tilde{e})$, and we derive $\tilde{e} \in \tilde{E}_H \Leftrightarrow e \in E \Leftrightarrow \varphi(e) \in F \Leftrightarrow \widetilde{\varphi(e)} = \tilde{\varphi}(\tilde{e}) \in \tilde{F}_G$. □

Notice that the reverse implications of statements (iii), (iv) above do not hold true in general. The property stated in part (i) of the preceding lemma directly leads to the following notion.

Definition 4 *Let $H = (V, E) \in \mathcal{H}$ be a hypergraph. Then $H^\# = (V^\#, E^\#) := (\tilde{E}_H, \{E_{[x]_H}\} : \tilde{V}_H)$ is called the* diagonalized dual (hypergraph) *of H. Here $E_{[x]_H} = \{\tilde{e} \in \tilde{E}_H : [x]_H \in \tilde{e}\}$.*

If $H \in \mathcal{H}$ is Sperner then $H^\#$ in general is not. For instance consider the hypergraph H given by $V = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2\}$ where $e_1 = \{x_1, x_2\}, e_2 = \{x_2, x_3\}$. We see that $H \in \mathcal{H}_3(3)$ and for the edges of $H^\#$ we have $E_{[x_1]_H} = \{\tilde{e}_1\}$, $E_{[x_2]_H} = \{\tilde{e}_1, \tilde{e}_2\}$, $E_{[x_3]_H} = \{\tilde{e}_2\}$ obviously not possessing the Sperner property over the vertex set $\{\tilde{e}_1, \tilde{e}_2\}$. It is left to the reader to verify that in this case $H^\# \in \mathcal{H}_{\text{diag}}$ holds, more precisely $H^\# \in \mathcal{H}_2(2)$. Below it is shown that this is a fact in general (cf. Theorem 2).

Lemma 4 *For $H \in \mathcal{H}$ we have: (i) $H^\# \in \mathcal{H}$, and (ii) $H^\# \cong \tilde{H}^*$.*

Proof. Let $H = (V, E) \in \mathcal{H}$ be fixed. Suppose that there are $[x]_H \neq [y]_H$ such that $E_{[x]_H} = E_{[y]_H}$ then $E_x = E_y$ then $x \sim_H y$ then $[x]_H = [y]_H$. Hence $H^\# \in \mathcal{H}$ establishing (i). To prove assertion (ii) set $|E| =: m$, then H can be regarded as a hypergraph over index set $[m]$, namely $H = (V, E : [m])$. By the definition of diagonalization and because of $|\tilde{E}_H| = |E|$ according to Lemma 3 we have $\tilde{H} = (\tilde{V}_H, \tilde{E}_H : [m])$. Now by Definition 1 it follows that $\tilde{H}^* = ([m], \tilde{E}_H^* : \tilde{V}_H)$ where $\tilde{E}_H^* = \{[x]_H^* : [x]_H \in \tilde{V}_H\}$ and $[x]_H^* := \{i \in [m] : [x]_H \in \tilde{e}_i\}$. On the other hand we have $H^\# = (\tilde{E}_H, \{E_{[x]_H}\} : \tilde{V}_H)$ as a hypergraph over the index set $\tilde{V}_H$. Identifying $E_{[x]_H} := \{\tilde{e} \in \tilde{E}_H : [x]_H \in \tilde{e}\}$ and $[x]_H^*$ the claimed hypergraph isomorphy can be derived, which completes the proof of (ii). □

Theorem 2 *For $H \in \mathcal{H}$ we have: (i) $H^\# \in \mathcal{H}_{\text{diag}}$, and (ii) if $H \in \mathcal{H}_{\text{lin},\geq 2}$ then $H^\# \in \mathcal{H}_{\text{lin},\geq 2}$.*

Proof. Let $H = (V, E) \in \mathcal{H}$. For proving assertion (i), we notice that $\tilde{H}^* \in \mathcal{H}_{\text{diag}}$ according to Lemma 2, because $\tilde{H} \in \mathcal{H}$. Therefore $H^\# \in \mathcal{H}_{\text{diag}}$ in view of Lemma 4, (ii), and the second statement in Theorem 1. In order to verify part (ii), recall that in view of Lemma 3 we have $|\tilde{E}_H| = |E|$ meaning that every $e \in E$ induces a unique $\tilde{e} \in \tilde{E}_H$ and vice versa. Now assume that $H \in \mathcal{H}_{\text{lin},\geq 2}$, but $H^\# = (V^\#, E^\#) \notin \mathcal{H}_{\text{lin},\geq 2}$. Recall from Definition 4 that $V^\# = \tilde{E}_H$ and $E^\# = \{E_{[x]_H} : [x]_H \in \tilde{V}_H)\}$. So, there are $x, y \in V$ with $[x]_H \neq [y]_H$, and moreover there are $e, e' \in E$ with $e \neq e'$ such that the corresponding $\tilde{e}, \tilde{e'} \in E_{[x]_H} \cap E_{[y]_H}$. In turn that means $[x]_H, [y]_H \in \tilde{e} \cap \tilde{e'}$ and therefore $x, y \in e \cap e'$ yielding a contradiction to the linearity of H.□

For hypergraphs of the diagonal class we recover the usual behavior of dualization in the sense of isomorphy as stated next.

ISBN: 978-988-19253-2-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Theorem 3 *For $H \in \mathcal{H}_{\rm diag}$ we have: (i) $H^{\#} \cong H^*$, and (ii) $H^{\#\#} := (H^{\#})^{\#} \cong H$.*

PROOF. For $H \in \mathcal{H}_{\rm diag}$ we have $H \cong \tilde{H}$ according to Lemma 3, (ii). Using Lemma 4, (ii), we conclude that $H^{\#} \cong \tilde{H}^* \cong H^*$ holds according to Remark 1, hence assertion (i) is true. For proving (ii), observe that with Theorem 2, (i), we have $H^{\#} \in \mathcal{H}_{\rm diag}$. Hence we can apply (i) to obtain $(H^{\#})^{\#} \cong (H^{\#})^*$. Applying (i) again, this time to H, yields $H^{\#\#} \cong H^{**} = H$ where Lemma 1 has been used. □

Theorem 4 *Let $H \in \mathcal{H}$ then $H^{\#\#} \cong \tilde{H}$.*

PROOF. Let $H \in \mathcal{H}$ be arbitrarily fixed and set $G := H^{\#} \in \mathcal{H}_m(m)$ for some unique $m \in \mathbb{N}$ in view of Theorem 2, part (i). Due to Theorem 3, (i), we see $G^{\#} \cong G^*$. Relying on relation $H^{\#} \cong \tilde{H}^*$ according to Lemma 4, part (ii), we finally obtain $H^{\#\#} = G^{\#} \cong G^* = (H^{\#})^* \cong (\tilde{H}^*)^* = \tilde{H}$. Here for the last equality Lemma 1 has been used. □

Clearly, by construction the n-diagonal classes $\mathcal{H}_n(n)$ (for $n \in \mathbb{N}$) have in $\mathcal{H}$ the distinguished property that iterating twice the diagonalized dualization on one of its members recovers exactly this member at least in the sense of hypergraph isomorphy. Moreover, given a hypergraph $H \in \mathcal{H}$ the problem of computing the diagonalized dual lies in the complexity class P, because it essentially requires a computation of the diagonalization $\tilde{H}$ of H which obviously can be executed in polynomial time.

3 Diagonalized Duality in Logic

Based on the concept of hypergraph duality introduced in the last section, in the sequel we define an analogous duality for propositional normal form formulas (NFF's). After having translated the notions to monotone formulas (which can be done immediately) the concept also is generalized to arbitrary NFF's. First the basic notation is fixed. Let $V = \{x_1, \ldots, x_n\}$ ($n \in \mathbb{N}$) be a finite set of atoms also called *(propositional) variables* and let $\overline{x} := \neg x$ denote the negation of atom x. For $X \subseteq V$ we set $\overline{X} := \{\overline{x} : x \in X\}$ and $L = V \cup \overline{V}$ for the set of *literals* over V. The *positive literal over* x is x itself and $\overline{x}$ is the *negative literal* over x. Let $\mathrm{OP} := \{\wedge, \vee, \rightarrow, \leftrightarrow, \oplus\}$ be the set of Boolean operators corresponding to AND, OR, IMPLICATION, EQUIVALENCE, XOR. Let $\odot \in \mathrm{OP}$ be a fixed operator, then a finite $\odot$-junction $c = l_1(c) \odot \cdots \odot l_{|c|}(c)$ is called a $\odot$-*clause* (for short *clause* if no ambiguity can occur) if $l_1(c) \neq l_j(c), i \neq j$. Hence such a clause can be regarded as a $|c|$-cardinality set of its literals if the juncting operator is fixed. Let $\mathrm{CL}_\odot$ denote the set of the $\odot$-clauses. Clearly, from the point of view of sets the objects $\mathrm{CL}_\odot, \odot \in \mathrm{OP}$, are essentially the same. Hence, when we simply write CL, and speak of *clauses* when a specific junction is not of importance. Given a clause c we denote with $L(c)$ the set of its literals and by $V(c)$ the set of variables over which its literals are defined. In the sense of a set interpretation of c we actually have that c is identified with $L(c)$. Let two different operators $\odot, \otimes \in \mathrm{OP}$ be fixed. A finite $\otimes$-junction $C = c_1(C) \otimes \cdots \otimes c_{|C|}(C)$ of different clauses $c_i(C) \in \mathrm{CL}_\odot$ is called a $\otimes, \odot$-*normal form formula*. Let $\mathcal{C} := \mathcal{C}_{\otimes,\odot}$ denote the set of these formulas. Notice that an NFF can be identified with its set of clauses $\mathrm{CL}(C) \equiv C$ whenever the underlying Boolean operators are fixed. For the set of literals in C we have $L(C) = \bigcup_{c \in C} L(c)$ and similarly for the set of variables we have $V(C) = \bigcup_{c \in C} V(c)$. A clause c is called *positive monotone* if $L(c) = V(c)$ and *negative monotone* if $L(c) = \overline{V(c)}$. An NFF C is called *positive (negative) monotone* if it contains only positive (negative) monotone clauses. Let $\mathcal{C}^{+(-)}$ denote the set of all positive (negative) monotone formulas. Moreover, let us transfer the notion of linearity to arbitrary NFF formulas (cf. [18]). An NFF C is called *linear* if for all $c_1, c_2 \in C$ with $c_1 \neq c_2$, we have $|V(c_1) \cap V(c_2)| \leq 1$. Let $\mathcal{C}_{\rm lin} \subseteq \mathcal{C}_{\otimes,\odot}$ denote the class of all linear formulas. In the remainder of this section, we restrict the consideration to positive monotone formulas, simply called monotone, since negative monotone formulas can be treated analogously. Observe that a positive monotone formula C contains no negative literals. Hence, it corresponds to a hypergraph $H_C \in \mathcal{H}$ whose vertex set is identified with $V(C)$ and whose edge set is identified with $\mathrm{CL}(C)$. Conversely, given a hypergraph $H = (V, E) \in \mathcal{H}$, we obtain the corresponding NFF C_H with variable set $V(C_H) = V$ and clause set $\mathrm{CL}(C_H) = E$. Then interpret $C_H \in \mathcal{C}_{\otimes,\odot}$ for any fixed pair of distinct Boolean operators in OP. Given two formulas $C, C' \in \mathcal{C}^+$ we write $C \cong C'$ if and only if $H_C \cong H_{C'}$ in the sense of hypergraph isomorphy. For $\otimes, \odot \in \mathrm{OP}$, let $C \in \mathcal{C}^+_{\otimes,\odot}$ be a monotone NFF. The *monotone diagonalized dual* $C^{\#}$ of C is the monotone NFF in $\mathcal{C}^+_{\odot,\otimes}$ defined by $C^{\#} := C_{H_C^{\#}}$. An NFF C is called a *monotone diagonal formula* if H_C is a member of the diagonal class of hypergraphs. Let the set of monotone diagonal $(\otimes, \odot)$-NFF's be denoted as $\mathcal{D}^+_{\odot,\otimes} \subset \mathcal{C}^+_{\odot,\otimes}$.

Theorem 5 *Let $C \in \mathcal{C}^+_{\otimes,\odot}$ be a monotone formula, then $C^{\#} \in \mathcal{D}^+_{\odot,\otimes}$ is a diagonal monotone formula. Moreover, if $C \in \mathcal{D}^+_{\odot,\otimes}$ then $C^{\#\#} := (C^{\#})^{\#} \cong C$.*

PROOF. The proof of the first claim can be reduced to showing that $H_{C^{\#}} = H_C^{\#}$ is a member of a diagonal class of hypergraphs. But this in turn is guaranteed in any case according to Theorem 2, (i). The assertion then follows by exchanging the involved Boolean operators. The second claim follows likewise according to Theorem 3, (ii). □

For example if C is a diagonal monotone formula in CNF then $C^{\#}$ is a diagonal monotone formula in DNF such

ISBN: 978-988-19253-2-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

that the clauses of C are its variables and vice versa. Given an NFF C the problem of computing the monotone diagonalized dual is in P because this holds for computing the diagonal dual of the corresponding hypergraph. For the twicely iterated monotone diagonalized dual in the general case we have:

Theorem 6 *If $C \in \mathcal{C}^+_{\otimes,\odot}$ then $C^{\#\#} \cong \tilde{C} := C_{\tilde{H}_C} \in \mathcal{C}^+_{\otimes,\odot}$. Where $\tilde{H}_C$ is the diagonalization of H_C according to Definition 3.*

PROOF. Clearly the operators in C are exchanged twicely, hence, if C is a monotone $(\otimes,\odot)$-NFF, so is $C^{\#\#}$. The rest of the proof immediately follows from Theorem 4 which states for the corresponding hypergraphs that $H_{C^{\#\#}} = H_C^{\#\#} \cong \tilde{H}_C$. Hence $C^{\#\#} \cong C_{\tilde{H}_C} = \tilde{C}$. □

In analogy to $\mathcal{H}_{\text{lin},\geq 2}$ let $\mathcal{C}^+_{\text{lin},\geq 2} \subseteq \mathcal{C}^+_{\odot,\otimes}$ denote the collection of those monotone linear formulas C such that each variable occurs in at least two distinct clauses, then as a direct consequence of Theorem 1 we obtain the following

Corollary 1 *We have $\mathcal{C}^+_{\text{lin},\geq 2} \subseteq \mathcal{D}^+_{\odot,\otimes}$ as a proper subclass.* □

4 Generalization to arbitrary NFF's

Clearly, it is possible also to assign a hypergraph to an arbitrary, i.e., not necessarily monotone normal form formula. Let $C \in \mathcal{C}_{\otimes,\odot}$ be given (for any fixed pair of Boolen operators $\otimes,\odot \in \text{OP}$). Then setting $(L(C), \text{CL}(C))$ defines a hypergraph $H_C \in \mathcal{H}$. Notice that a monotone formula C actually appears as a special case because then $L(C) = V(C)$.

Definition 5 *For $\otimes,\odot \in \text{OP}$, let $C \in \mathcal{C}_{\otimes,\odot}$ be an NFF. The* diagonalized dual *$C^\#$ of C is the NFF in $\mathcal{C}_{\odot,\otimes}$ defined as $C^\# := C_{H_C^\#}$. An NFF C is called a* diagonal formula *if H_C is a member of the diagonal class of hypergraphs. We denote the set of diagonal $(\otimes,\odot)$-NFF's by $\mathcal{D}_{\odot,\otimes} \subset \mathcal{C}_{\odot,\otimes}$.*

Theorem 7 *Let $C \in \mathcal{C}_{\otimes,\odot}$ be an NFF, then $C^\# \in \mathcal{D}_{\odot,\otimes}$ is a diagonal NFF. Moreover, if $C \in \mathcal{D}_{\odot,\otimes}$ then $C^{\#\#} := (C^\#)^\# \cong C$.*□

Given an NFF C the problem of computing the diagonalized dual is in P because this holds for computing the diagonalized dual of the corresponding hypergraph. For the twicely iterated diagonalized dual we have:

Theorem 8 *If $C \in \mathcal{C}_{\otimes,\odot}$ then $C^{\#\#} \cong \tilde{C} := C_{\tilde{H}_C} \in \mathcal{C}_{\otimes,\odot}$. Where $\tilde{H}_C$ is the diagonalization of H_C according to Definition 3.*□

Finally consider the class of *1-intersecting* normal form formulas. Such a formula C has the defining property that $\forall c_i, c_j \in \text{CL}(C)$ with $c_i \neq c_j$ it holds that $|L(c_i) \cap L(c_j)| \leq 1$. Since one has to distinguish between a variable and its negation as distinct objects clearly a 1-intersecting formula corresponds to a linear hypergraph $H_C = (L(C), \text{CL}(C)) \in \mathcal{H}_{\text{lin}}$. Recall that $\mathcal{C}_{\geq 2} \subset \mathcal{C}_{\odot,\otimes}$ denotes the collection of normal form formulas C such that each literal occurs at least twice in C. As in the case of hypergraphs we have that the class of diagonal NFF's essentially contains the class of 1-intersecting NFF's.

Theorem 9 *$\mathcal{D}_{\odot,\otimes}$ contains the subset of $\mathcal{C}_{\geq 2}$ consisting of 1-intersecting formulas as a proper subclass.*

PROOF. A 1-intersecting formula $C \in \mathcal{C}_{\geq 2}$ corresponds to $H_C \in \mathcal{H}_{\text{lin},\geq 2}$. According to Theorem 1 the latter is a proper subclass of $\mathcal{H}_{\text{diag}}$ implying the assertion of the theorem. □

Observe that the last last result indeed generalizes Corollary 1 because 1-intersecting formulas, in general, are not linear. For example the clauses $c_1 := \{\bar{x}, y, z\}$ and $c_2 := \{x, y, \bar{z}\}$ satisfy $|L(c_1) \cap L(c_2)| = 1$ but have identical variable sets. Reversely, every linear (not necessarily monotone) formula obviously is 1-intersecting.

5 Conclusions and open Problems

We introduced a new notion of diagonalized duality for hypergraphs and transfered it more generally to normal form propositional formulas resting on their representability as appropriately defined clause sets. The classes of diagonal formulas resp. hypergraphs are distinguished w.r.t. the diagonalized dual because they are recovered exactly by computing the dual of the dual. On the other hand it is clear that instances of this type are most complex in the sense that they yield a maximal cardinality in the vertex set of its diagonalization. It is an interesting question to be addressed by future work, whether the computation of the transversal hypergraphs can be executed efficiently for the n-diagonal classes of hypergraphs. It might be hoped also that some questions of satisfiability theory can be attacked from this point of view. The propositional satisfiability problem (SAT) for CNF formulas is an essential combinatorial problem, namely one of the first problems that have been proven to be NP-complete [6]. More precisely, it is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover SAT plays a fundamental role in the theory of designing exact algorithms, and it has a wide range of applications because many problems can be encoded as a SAT problem via reduction [13, 12] due to the rich expressiveness of the CNF language. The applicational area is pushed by the fact that meanwhile several powerful solvers for SAT have been developed (cf. e.g. [15, 21] and references therein).

ISBN: 978-988-19253-2-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Also from a theoretical point of view one is interested in classes for which SAT can be solved more efficiently, namely in polynomial time, or even in linear time [1, 16]. The same holds for prominent variants of SAT, namely exact satisfiability (XSAT) or Not-All-Equal satisfiability (NAE SAT). Specifically consider NAE SAT for monotone CNF instances which is NP-hard [11, 19]. Here a truth assignment is searched setting in every clause one literal to true and one to false. NAE SAT is equivalent to hypergraph 2-colourability where a 2-colouring of the vertex set is searched such that no hyperedge appears to be monochromatic. The last problem could be studied for restricted classes $\mathcal{H}_k(n)$ for which it might be solvable more efficiently. For a study on the complexity of these problems on linear propositional formula classes we refer to [17, 18, 20].

References

[1] Aspvall, B., Plass, M.R., Tarjan, R.E., "A linear-time algorithm for testing the truth of certain quantified Boolean formulas," *Inform. Process. Lett.* pp. 121-123, 8/1979.

[2] Berge, C., *Hypergraphs*, North-Holland, Amsterdam, 1989.

[3] Enderton, H.B., *A mathematical Introduction to Logic*, Academic Press, Massachusetts, 2001.

[4] Bioch, J.,C., Ibaraki, K., "Complexity of Identification and Dualization of Positive Boolean Functions," *Information and Computation*, pp. 50-63, 123/1995.

[5] Boros, E., Gurvich, V., Khachiyan, L., Makino, K., "Dual-Bounded Generating Problems: Partial and Multiple Transversals of a Hypergraph," *SIAM J. Comput.*, pp. 2036-2050, 30/2001.

[6] Cook, S.A., "The Complexity of Theorem Proving Procedures," *3rd ACM Symposium on Theory of Computing*, pp. 151-158, 1971.

[7] Eiter, T., Gottlob, G.," Identifying the Minimal Transversals of a Hypergraph and Related Problems," SIAM J. Comput., pp. 1278-1304, 24(6)/1995.

[8] Eiter, T., Gottlob, G., "Hypergraph Transversal Computation and Related Problems in Logic and AI," *Lecture Notes in Artificial Intelligence*, pp. 549-564, 2424/2002.

[9] Eiter, T., Gottlob, G., Makino, K., "New Results on Monotone Dualization and Generating Hypergraph Transversals," *SIAM J. Comput.*, pp. 514-537, 32/2003.

[10] Flum, J., Ebbinghaus, H.D., Thomas, W., *Einführung in die mathematische Logik*, Spektrum-Verlag, Heidelberg, 1998.

[11] Garey M.R., Johnson, D.S., *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman and Company, San Francisco, 1979.

[12] Gu, J., Purdom, P.W., Franco, J., Wah, B.W., "Algorithms for the Satisfiability (SAT) Problem: A Survey," in: D. Du, J. Gu, P. M. Pardalos (Eds.), *Satisfiability Problem: Theory and Applications*, DIMACS Workshop, March 11-13, 1996, DIMACS Series, V35, pp. 19-151, American Mathematical Society, Providence, Rhode Island, 1997.

[13] Karp, R.M., "Reducibility Among Combinatorial Problems," in: *Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y.*, New York: Plenum, pp. 85-103, 1972.

[14] Kleine Büning, H., Lettman, T., *Aussagenlogik: Deduktion und Algorithmen*, B. G. Teubner, Stuttgart, 1994.

[15] Le Berre, D., Simon, L., "The Essentials of the SAT 2003 Competition," *6th International Conference on Theory and Applications of Satisfiability Testing (SAT'03)*, LNCS, pp. 172-187, 2919/2004.

[16] Minoux, M., "LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation," *Inform. Process. Lett.*, pp. 1-12, 29/1988.

[17] Porschen S., Schmidt, T., Speckenmeyer, E., Wotzlaw, A., "XSAT and NAE-SAT of linear CNF classes," *Discrete Appl. Math.*, pp. 1-14, 167/2014.

[18] Porschen, S., Speckenmeyer, E., Zhao, X., "Linear CNF formulas and satisfiability," *Discrete Appl. Math.*, pp. 1046-1068, 157/2009.

[19] Schaefer, T.J., "The complexity of satisfiability problems," *Conference Record of the Tenth Annual ACM Symposium on Theory of Computing*, San Diego, California, pp. 216-226, 1978.

[20] Schmidt, T., *Computational complexity of SAT, XSAT and NAE-SAT for linear and mixed Horn CNF formulas*, dissertation, Univ. Köln, 2010.

[21] Speckenmeyer, E., Min Li, C., Manquinho V., Tacchella, A., (Eds.), *Special Issue on the 2007 Competitions, Journal on Satisfiability, Boolean Modeling and Computation*, 4 (2008).

ISBN: 978-988-19253-2-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)