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Solving the General Two Water Jugs Problem via an Algorithmic Approach

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***Abstract*—The water jugs problem is a famous problem in artificial intelligence, computer programming, discrete mathematics and psychology. The existing methods of solutions are often non-algorithmic in nature. In this paper, we present an algorithmic approach to solve this problem and describe its implementation in Excel. Illustrative examples are provided.**

***Index Terms*—water jugs problem, arithmetic approach, Diophantine equation, problem-solving, artificial intelligence.**

I. Introduction

THE two water jugs problem is a famous problem in problem-solving [1], geometry [2], recreational mathematics [3], discrete mathematics [4], computer programming [5], cognitive psychology [6, 9 10] and artificial intelligence [11], etc. The problem says:

"You are at the side of a river. You have a 3 liter jug and a 5 liter jug. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure 1 liter of water? "

Several methods can be applied to solve this problem, such as the working backwards approach [1], the geometric approach [2, 3], the digraph approach [4], the search approach (BFS or DFS) [5, 10] and some heuristic methods [6, 8, 9]. However, most of them are not algorithmic in nature and the time and memory taken to solve it could be expensive sometimes. In this paper, we present a simple algorithmic approach to solve this problem, which was introduced by the author in [11, 12]. By using this approach, we can obtain the total amount of water (say V) in the jugs at each pouring step by simple additions or subtractions only, and the actual pouring sequence can be easily determined by referring to the computed values of V. As a side product, we can obtain a particular solution of the associated Diophantine equation of the water jug problem concerned. Due to its simplicity, this approach is suitable for either hand calculation or implementation in computer languages. Unlike the common search methods adopted for solving this problem, no additional memory cost is needed for doing searching and branching such as BFS or DFS, by using this new algorithmic approach. Comparing to the approach described in [13], this new approach has the advantage that we can obtain two possible solutions to solve the water jugs problem instead of one only. Then, we can compare and justify which jug to fill first and end up completing the task with less pouring steps as possible.

The whole paper is organized like this. In the next section, we will introduce the algorithmic approach for solving the general two water jugs problem and describe the mathematical background behind. In the third section, we will illustrate how to apply the new approach by a few examples. In the fourth section, we will discuss briefly how to implement this new approach by means of Excel. Then, we will conclude with some remarks in the final section.

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II. An algorithmic approach

An algorithmic approach to the general two water jugs problem was introduced in [11, 12], which can be applied to solve the problem below:

"Let m, n, d be positive integers. You are at the side of a river. You have a m-liter jug and a n-liter jug, where 0<m<n. The jugs do not have markings to allow measuring smaller quantities. How can you use the jugs to measure d (<n) liters of water? "

The associated Diophantine equation of the problem is given by $mx+ny=d$, whose solvability is described by the theorem below. Reader can refer to [7] for a proof of this important result in number theory.

Theorem 2.1. The Diophantine equation $mx+ny=d$ is solvable *if and only if* gcd(m, n) divides d.

For example, the water jugs problem described in the introduction is solvable since gcd(3, 5) divides 1. However, if the jugs are replaced by a 5-litre jug and a 10-litre jug, then it is unsolvable since gcd(5, 10) does not divide 1. For convenience, let us assume $mx+ny=d$ is solvable in the discussions below. Depending on which jug is chosen to be filled first, there are two possible solutions for solving the two water jugs problems. They are labelled by M_1 and M_2 in the following algorithms:

Algorithm 2.1.
Input: The integers m, n and d, where $0<m<n$ and $d<n$.

Output: An integer sequence corresponding to a feasible solution (called M_1) of the two water jugs problem, by filling the m-litre jug first.
Procedure:

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Step 1. Initialize a dummy variable $k = 0$.
Step 2. If $k \neq d$, then repeat adding m to k and assign the result to k until $k = d$ or $k > n$.
Step 3. If $k > n$, then subtract n from k and assign the result to k.
Step 4. If $k = d$, then stop. Otherwise, repeat the steps from Step 2 to Step 4.

The number of additions (say x_1) and subtractions (say y_1) involved provides a solution to the Diophantine equation $mx+ny=d$, namely $x = x_1$, $y = -y_1$. The actual pouring sequence can be determined by referring to the integer sequence obtained.

Algorithm 2.2.
Input: The integers m, n and d, where $0<m<n$ and $d<n$.

Output: An integer sequence corresponding to a feasible solution (called M_2) of the two water jugs problem, by filling the n-litre jug first.

Procedure:
Step 1. Initialize a dummy variable $k = 0$.
Step 2. If $k \neq d$, then add n to k and assign the result to k.
Step 3. If $k > d$, then repeat subtracting m from k and assign the result to k until $k = d$ or $k < m$.
Step 4. If $k = d$, then stop. Otherwise, repeat the steps from Step 2 to Step 4.

The number of subtractions (say x_2) and additions (say y_2) involved provides a solution to the Diophantine equation $mx+ny=d$, namely $x = -x_2$, $y = y_2$. The actual pouring sequence can be determined by referring to the integer sequence obtained.

III. Illustrative Examples

We now illustrate how to apply the arithmetic approach to solve the two water jugs problem below.

Example 3.1. There are a 3-litre jug and a 5-litre jug. We want to use them to measure 1 liter of water, as described in the introduction. Using the above notations, we have m=3, n=5, d=1 and the associated Diophantine equation is $3x+5y=1$. Since gcd(3, 5) divides 1, so this equation is solvable. By applying Algorithm 2.1, we can obtain the following integer sequence for M_1:

$$\boxed{0} \xrightarrow[+3]{} \boxed{3} \xrightarrow[+3]{} \boxed{6} \xrightarrow[-5]{} \boxed{1}$$

The number of additions and subtractions involved are 2 and 1 respectively, so $x = 2$, $y = -1$ is a solution of the Diophantine equation $3x+5y=1$. Since the integers appeared in the boxes represents the total amount of water in the jugs in successive steps, the corresponding pouring steps of M_1 can be easily obtained below, where (x, y) are used to denote the amounts of water inside the 3-litre jug and the 5-litre jug respectively:

$(0,0) \to (3,0) \to (0,3) \to (3,3) \to (1,5)$

So, the total number of pouring steps involved in M_1 is 4.

Similarly, we can obtain an integer sequence for M_2 by applying Algorithm 2.2, namely:

$$\boxed{0} \xrightarrow[+5]{} \boxed{5} \xrightarrow[-3]{} \boxed{2} \xrightarrow[+5]{} \boxed{7} \xrightarrow[-3]{} \boxed{4} \xrightarrow[-3]{} \boxed{1}$$

The number of additions and subtractions involved are 2 and 3 respectively, so $x = -3$, $y = 2$ is a solution of the Diophantine equation $3x+5y=1$. The corresponding pouring steps for M_2 are as follows:

$(0,0) \to (0,5) \to (3,2) \to (0,2) \to (2,0) \to (2,5) \to (3,4) \to (0, 4) \to (3, 1)$

So, the total number of pouring steps involved in M_2 is 8.

By comparing the number of steps involved in M_1 and M_2, we can see that M_1 provides a more optimal solution to solve this water jug problem.

Example 3.2. There are a 3-litre jug and a 7-litre jug. We want to use them to measure 5-litre of water. So, we have m=3, n=7, d=5 and $3x+7y=5$ is the associated Diophantine equation. Since gcd(3, 7) divides 5, so this equation is solvable. Applying Algorithm 2.1, we can obtain an integer sequence for M_1:

$$\boxed{0} \xrightarrow[+3]{} \boxed{3} \xrightarrow[+3]{} \boxed{6} \xrightarrow[+3]{} \boxed{9} \xrightarrow[-7]{} \boxed{2} \xrightarrow[+3]{} \boxed{5}$$

The number of additions and subtractions involved are 3 and 1 respectively, so $x = 4$, $y = -1$ is a solution of the Diophantine equation $3x+7y=5$. The corresponding pouring steps for M_1 are:

$(0,0) \to (3,0) \to (0,3) \to (3,3) \to (0,6) \to (3,6) \to (2,7) \to (2,0) \to (0, 2) \to (3,2) \to (0,5)$

So, the total number of pouring steps involved in M_1 is 10.

Similarly, we can obtain an integer sequence for M_2 by applying Algorithm 2.2, namely:

$$\boxed{0} \xrightarrow[+7]{} \boxed{7} \xrightarrow[-3]{} \boxed{4} \xrightarrow[-3]{} \boxed{1} \xrightarrow[+7]{} \boxed{8} \xrightarrow[-3]{} \boxed{5}$$

The number of additions and subtractions involved are 2 and 3 respectively, so $x = -3$, $y = 2$ is a solution to the equation $3x+7y=5$. The corresponding pouring steps for M_2 are:

$(0,0) \to (0,7) \to (3,4) \to (0,4) \to (3,1) \to (0,1) \to (1,0) \to (1,7) \to (3, 5)$

So, the total number of pouring steps involved in M_2 is 8.
By comparing the number of steps in M_1 and M_2, we can see that M_2 provides a more optimal solution to solve this water

jug problem. In [13], the single solution obtained by using the Extended Euclidean algorithm to solve this problem is the same as the one obtained by M_1, which is not an optimal solution since we can apply M_2 to solve the problem with 2 less steps.

IV. Implementation of the Algorithm in Excel

For automatic generation of the integer sequences by the proposed new algorithms, we can implement them by using Excel. Fig. 1 shows a simple interactive interface of such an implementation.

	A	B	C	D	E
1				Method 1	Method 2
2	m	3		0	0
3	n	5		3	5
4	d	1		6	2
5				1	7
6				stop	4
7					1
8					stop
9					

Fig 1. An interactive interface of the algorithms implemented in Excel.

Users can input the values of *m*, *n*, *d* in the cells B2, B3 and B4. Then, the integer sequences generated by Method 1 (M_1) and Method 2 (M_2) will be shown readily by sliding the cells D3 and E3 downwards until the word "stop" appears. The input commands for the cells D3 and E3 are like these:

For D3:
"=IF(D2=$B$4,"stop",IF(D2>$B$3,D2-$B$3,D2+$B$2))"

For E3:
"=IF(E2=$B$4,"stop",IF(E2>$B$2,E2-$B$2,E2+$B$3))"

With the generated integer sequences, we can easily determine the actual water pouring steps, which have been illustrated in the examples in section 3. Also, by looking at the integer sequences generated by such an interactive Excel program, we can obtain the special solutions of the associated Diophantine equation of the problem concerned. For instance, in the column of "Method 1", the length of the integer sequence (excluding the zero at the beginning) is 3 and there are 2 integers greater than or equal to 3, which corresponds to a solution of x. By subtracting the length of the integer sequence from the value of x, we can obtain a solution of y. So, $x = 2$ and $y = -1$ is a particular solution of $3x+5y=1$. Similarly, in the column of "Method 2", the length of the integer sequence (excluding the zero at the beginning) is 5 and there are 2 integers greater than or equal to 5, which corresponds to a solution of y. So, $y = 2$ and $x = -3$ is a particular solution of $3x+5y=1$, where the value of x is calculated by subtracting the length of the integer sequence from the value of y.

V. Concluding Remarks

A simple algorithmic approach to solve the general two water jugs problem and a short description of its implementation in Excel is presented in this paper. This approach is suitable for either hand calculations or implementation in the common computer languages. The integer sequences for M_1 or M_2 can be obtained easily by using simple additions and subtractions only. There is no additional memory cost spent on searching and branching like what common search methods do, in order to determine the actual pouring sequence. Due to its novelty and simplicity, this approach is suitable for introduction to undergraduate students or researchers in the areas of artificial intelligence, discrete mathematics, computer sciences, engineering, number theory, problem-solving or cognitive psychology. Further development of this new approach and the implemented Excel program to explore and formulate a criteria on how to obtain an optimal solution of the general two water jugs problem, in the sense that the number of pouring steps involved is least possible, will be a meaningful and challenging research topic for pursue in the near future.

References

[1] G. Polya, *How to Solve It.* NJ: Princeton University Press, 1945.
[2] H.S.M.Coxeter and S.L.Greitzer, *Geometry Revisited*. Washington D.C.: The Mathematical Association of America, 1967.
[3] H. E. Dudeney, *Amusements in Mathematics*. NY: Dover, 1970.
[4] C. J. McDiarmid and J. R. Alfonsin, "Sharing jugs of wine," *Discrete Mathematics*, vol. 125, 1994, pp. 279–287.
[5] B. Harvey, *Computer Science Logo Style: Symbolic Computing* (Vol. I), MA: MIT, 1997.
[6] M. K. Colvin, K. Dunbar and J. Grafman, "The effects of frontal lobe lesions on goal achievement in the water jug task," *Journal of Cognitive Neuroscience*, 13(8), 2001, pp. 1129–1147.
[7] D. M. Burdon, *Elementary Number Theory, 5th edition*, NY: McGraw-Hill, 2002.
[8] S. L. Beilock and M.S. DeCaro, "From poor performance to success under stress: working memory, strategy selection and mathematical problem solving under pressure," *Journal of Experimental Psychology*, 33(6), 2007, pp. 983–998.
[9] H. P. Carder, S. J. Handley and T. J. Perfect, "Counterintuitive and alternative moves choice in the water jug tasks," *Brain and Cognition*, vol. 66, 2008, pp. 11–20.
[10] S. Abu Naser, "Developing visualization tool for the teaching AI searching algorithms," *Information Technology Journal*, 7(2), 2008, pp. 350–355.
[11] Y. K. Man, "Solving the water jugs problem by an integer sequence approach," *International Journal of Mathematical Education in Science & Technology*, 43(1), 2012, pp.109–113.
[12] Y. K. Man, "A non-heuristic approach to the general two water jugs problem," *Journal of Communication and Computer*, 10, 2013, pp.904-908.
[13] R. S. Mary, "An alternative arithmetic approach to the water jugs problem," *Proceedings on National Conference on Computational Intelligence for Engineering Quality Software*, 1, 2014, pp. 10-13.