A Numerical Scheme for the Black-Scholes Equation with Variable Interest Rate Using Spectral Collocation and Backward-Differences

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Abstract— **The Black-Scholes equation models the fair value of a European call option under certain market assumptions. The terminal condition is derived from the difference between the stock price upon maturity and the option strike price, while the boundary conditions are derived from the put-call parity. In this work, we modify one market assumption, namely that of constant interest rate, and numerically solve a version of the Black-Scholes equation with time-dependent risk-free interest rates. We use a Chebyshev collocation scheme that performs spectral discretization in the stock variable and backward-difference in the time variable. We present numerical evidence of the spectral accuracy of the scheme against the known analytic solution.**

Index Terms—**Black-Scholes equation, differentiation matrices, spectral collocation method, variable interest rate**

I. THE BLACK-SCHOLES MODEL WITH VARIABLE INTEREST

Let $V(S,t)$ denote the fair value of a European call option with an underlying asset of price S at time t. Let r be the risk-free interest rate for all t from 0 to maturity time T, and let $\sqrt{v}$ be the volatility of the stock. In the 1973 paper of Fischer Black and Myron Scholes [1], the authors derived the following equation that governs the time-evolution of the function V:

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} \qquad (1)$$

Theoretically, the stock price S can increase without bound. But in reality and in numerical computations, we often set a maximum value for S, say at $S = S_{max}$. The boundary condition at $S = S_{max}$ with the strike price E is derived by discounting from S_{max} the interest incurred by E for the time $(T\text{-}t)$ at a constant annualized interest rate r. This is a consequence of the put-call parity [1]. On the other hand, at $S=0$, the option value is 0. Thus the boundary conditions are:

$$V(0,t) = 0$$

and

$$V(S_{max}, t) = S_{max} - Ee^{-r(T-t)}.$$

The option is exercised if the stock price exceeds the strike price of the call option upon maturity. The holder of

the option can buy the underlying asset at price E and sell it at market price S. The pay-off is $S\text{-}E$. Thus, the terminal condition is:

$$V(S,T) = \max(S - E, 0)$$

Equation (1) was derived under certain assumptions—that an arbitrage-free market exists, that the underlying stock price follows a geometric Brownian motion with constant drift and volatility, that there are no transaction costs or dividend yields, and that securities are continuous variables, and hence perfectly divisible. Admittedly, these assumptions are not always met in reality.

In the recent years, some of the market assumptions have been relaxed. In particular, the Vasicek model [8] assumes that the interest rate follows a geometric Brownian motion and satisfies the stochastic differential equation

$$dr = \frac{I - r}{C}dt + \varrho dB_t$$

where I and C are constants. This is a mean reverting model where the interest rate approaches the limiting value of I. Under the assumption that ϱ is 0 (this is the standard deviation of the interest rate), the equation for r becomes an ordinary differential equation whose solution is

$$r(t) = r_0\, e^{-t/c} + I(1 - e^{-t/c}) \qquad (2)$$

Given this time-dependent interest rate, the solution to the Black-Scholes PDE is

$$V(S,t) = S\Phi(z) - E\exp\left(-\int_0^{T-t} r(\tau)d\tau\right)\Phi\left(z\sqrt{v(T-t)}\right)$$

where

$$z = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{v}{2}\right)(T - t)}{\sqrt{v(T - t)}}.$$

Here the function Φ is the cumulative standard normal distribution.

In this work, we solve the PDE in (1) with the variable interest rate given in (2). We use Chebyshev spectral collocation in the stock variable and backward-differences in the time variable.

The work is not without precedent. In [5], the authors present a spectral collocation scheme in the stock variable and a time-discretization that yields an differential algrebraic equation (DAE) which they solved using fourth-order Runge-Kutta method. Our scheme is conceptually

simpler in that only first order backward-difference is used in discretizing the time variable.

II. METHOD

A. Spectral Collocation

We first work on the stock variable S, discretizing the stock values on the interval $[0, S_{max}]$ using Chebyshev grid points given by the equation:

$$S_k = \frac{S_{max}}{2}\widehat{S_k} + \frac{S_{max}}{2}, where$$
$$\widehat{S_k} = \cos\left(k\pi/N\right), k = 1, 2, \dots, N.$$

Below is an illustration of the distribution of the Chebyshev points with *N=20*.

Fig 1   Discrete Chebyshev points with N=20

Notice that the points are not equally spaced but are instead more clustered towards the ends. Discretizing the stock variable in this manner makes the interpolating polynomial more accurate and less prone to the Runge phenomenon [7].

We then interpolate V at each point S_k using a global interpolating polynomial p of Lagrange form. From the minimax property of the Chebyshev polynomial, we are assured of minimal error in interpolation.

In the Black-Scholes PDE, V is given as a function of its partial derivatives, and so we approximate $\frac{\partial V}{\partial S}$ by differentiating the interpolating polynomial. Note that the minimax property applies to the interpolating polynomial and not its derivative. Nevertheless, we can still be confident of a small error in our interpolation of the derivative. [2, 3, 4]

Furthermore, since differentiation is a linear operation, we can represent it by matrix multiplication. In this case, we use the Chebyshev differentiation matrix D so that $D * V \cong \frac{\partial V}{\partial S}$. Weidman and Reddy [9] provide a Matlab code that computes D for a given N.

Hence, we are now able to simplify the Black-Scholes PDE

$$\frac{\partial V}{\partial t} = rV - rS\frac{\partial V}{\partial S} - \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}$$

as follows:
$$\frac{\partial V}{\partial t} = rV - rS(DV) - \frac{1}{2}vS^2(D^2V).$$

B. Backward-differences

For the time variable, we discretize using a first order backward-difference. The finite-difference scheme allows us to solve differential equations by approximating the derivative of a function on a discrete grid. In this case, we partition the given time frame *[0, T]* into $\frac{T}{\Delta t} + 1$ equally spaced intervals of length Δt.

The backward-difference approximation is obtained from the definition of a derivative:

$$\frac{dV}{dt} = \lim_{\Delta t \to 0} \frac{V(t) - V(t - \Delta t)}{\Delta t}$$

We take Δt to be a very small number and get:

$$\frac{\partial V}{\partial t} \cong \frac{(V^n - V^{n-1})}{\Delta t},$$

where V^n is the value of the call option at the nth time step.

Note that V^n is known for the final time step $\frac{T}{\Delta t} + 1$, which allows us to use this formula recursively, marching backward in time to obtain the price of the call option today *(t=0)*, which is what we want.

Substituting back into the Black-Scholes PDE and with spectral differentiation on *S*, we get:

$$\frac{(V^n - V^{n-1})}{\Delta t} = rV - rS(DV) - \frac{1}{2}vS^2(D^2V).$$

Solving for V^{n-1}, we have:

$$V^{n-1} = V^n - \Delta t\left(rV - rS(DV) - \frac{1}{2}vS^2(D^2V)\right),$$

which can be simplified to

$$V^{n-1} = \left((1 - r\Delta t)I + L\Delta t\right)V^n,$$

where I is the *N+1 by N+1* identity matrix, and $L = \frac{1}{2}vS^2D^2 + rSD$. L is called the spectral operator.

To account for the final and boundary conditions of the Black-Scholes PDE, it is necessary to adjust the spectral operator L for each recursion $V^{n-1} = \left((1 - r\Delta t)I + L\Delta t\right)V^n$. This can be done simply by replacing the necessary rows and columns with the corresponding rows and columns of the identity matrix.

III. TIME-DEPENDENT INTEREST RATES

In the above discussion, r has been kept constant, as this is one of the assumptions of the Black-Scholes model. However, we now choose r to be dependent on time, as is more representative of realistic conditions. We find that the method can be applied in much the same way.

With variable interest rates, the Black-Scholes equation may be written as:

$$\frac{\partial V}{\partial t} = r(t)V - r(t)S\frac{\partial V}{\partial S} - \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}.$$

From (2), suppose $r(t) = r_0\,e^{-t/c} + I\left(1 - e^{-t/c}\right)$. As mentioned previously, (2) gives the analytic solution to the time-dependent interest rate modeled by Vasicek [8]. The graph of the function is shown below. Inputting parameters r_0, *I*, and c into the equation, we use the same equally-spaced time grid to discretize the function r and obtain the required interest rate value for each step in our backward recursion.
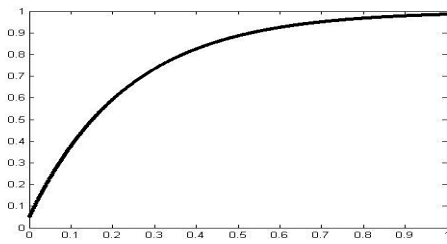
Fig 2 A graph of the Vasicek interest rate model with parameters $r_0 = 0.05, a = 4, b = 1$;

IV. RESULTS

We implemented the above scheme in Matlab and ran the program several times with varying input parameters and compared them against the known analytic solution. The graph below shows one such run of the program with parameters E=2.2, M=34000, N=30, S_{max}=3, T=0.5, a=0.05, b=0.5, r=0.5, v=$\sqrt{0.15}$. The following result is typical of what we have observed in general.
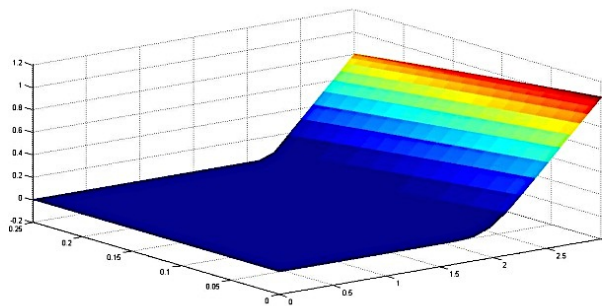

Fig 3 The numerical solution to the Black-Scholes equation with variable interest rates and parameters as stated above

Notice that at $t=T$, the function is shaped like a hockey-stick, with a sharp corner at the strike price E. This tells us that the strike price is the indifference point in exercising the call option; that is, at prices greater than the strike price, the pay-off to the call option is positive, while at prices less than the strike price, they pay-off is negative. Further, the sharp corner smoothens out as we step backward in time, which is expected of parabolic PDEs. At the initial time, the function is smooth.

Using these particular parameters, when compared with the analytic solution, the norm of error is 0.1625. This is a grid matrix with 6801 time steps by 31 stock points.
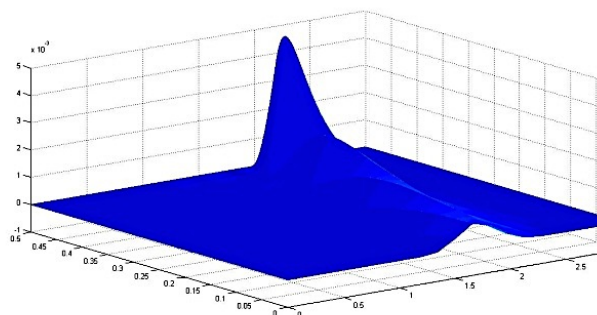

Fig 4 Mesh of the error between the analytic and numerical solution in the order of 10^{-3}

The error norm is graphed above. Pointwise, the maximum error is in the order of 10^{-3}. The error spikes at the point where the first derivative of the call option value is discontinuous, namely, at the strike price. This eventually smoothens out as the solution is computed backward in time. Again, this is typical because of the parabolic nature of the Black-Scholes equation.

V. CONCLUSION

We have developed a numerical method for solving the Black-Scholes equation with time-dependent interest rates using spectral collocation in stock and backward-differences in time. With multiple runs of the program, we have given numerical evidence that the scheme works based on the error of the numerical solution against the known analytic solution.

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