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A Cell-centered Scheme for Solving Anisotropic Diffusion Equations on Skewed Meshes

Lina Chang

***Abstract*—An algorithm is developed to interpolate node unknowns from cell-centered ones for the construction of diffusion schemes on skewed meshes. And a cell-centered scheme for solving anisotropic problems is then constructed. Its main characteristic lie in that an explicit expression is obtained for the interpolation. And the discontinuity can be dealt with strictly. The effectiveness of the scheme is demonstrated by the numerical experiments.**

***Index Terms*—diffusion equation, cell-centered scheme, anisotropic, discontinuity.**

I. INTRODUCTION

Skewed meshes arise in various fields, e.g., grid generation on physical domains with complex geometry. For simulating problems such as heat transfer, plasma physics, oil reservoir etc., anisotropic diffusion equations need to be solved on these meshes.

In the construction of finite volume schemes, due to the skewness of the grids, as well as the anisotropy, auxiliary unknowns defined at the vertices or edges are often introduced in addition to the cell-centered unknowns. The schemes with nine or even more stencils are then resulted on quadrilateral meshes.

In the early local support operator method (LSOM) [1], auxiliary unknowns defined at the edges are introduced and treated as primary unknowns, so that the flux can be discretized with local stencils. The edge unknowns are also the primary ones in the hybrid finite volume scheme [2] and at the discontinuous interfaces of SUSHI shceme[3]. On the other hand, vertex unknowns are introduced and used as the primary ones in the diamond schemes [4] [5]. The supplementary primary unknowns will increase the computational cost since the diffusion equation needs to be solved implicitly.

So diffusion schemes with cell-centered unknowns only are preferred. And the cell-centered scheme is constructed by interpolating the auxiliary unknowns from cell-centered ones. Usually, a local linear system is formulated and solved, satisfied by auxiliary (edge or vertex) unknowns in Refs.[6]-[9].

To avoid the need of solving such local linear systems, an explicit interpolation algorithm is presented in Ref.[10] for the calculation of node unknowns. In Refs. [11][12], the auxiliary unknowns are defined at a specific point on the edge. The position of the point depends on the mesh geometry and the distribution of the diffusion tensor. Then an interpolation method with two stencils are obtained. By enlarging the stencils, the auxiliary unknowns can be defined at any point on the edge in Ref. [13].

In this paper, the interpolation method given in Ref.[10] is extended to handle anisotropic problems. Its main characteristic are the following:

- It can handle anisotropic diffusion problems on skewed meshes.
- It gives an explicit expression for calculating the auxiliary unknowns from cell-centered ones. Specifically, the auxiliary unknowns are defined as the difference of the values at the adjacent nodes.
- It deals rigorously with discontinuities.
- It results a nine-point scheme on skewed quadrilateral meshes.

The remainder of this paper is organized as follows. First we describe the problems. The method for the interpolation of the auxiliary unknowns is then given in section 3, followed by numerical examples in section 4. Finally, we conclude in section 5.

II. PROBLEM DESCRIPTION

Let Ω be an open bounded subset of R^d with $\partial\Omega$ being its boundary. We consider the following diffusion problem

$$-\nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u) = f \qquad \text{in } \Omega, \tag{1}$$

where

- u is a scalar function. In the case of heat conduction, u denotes the temperature. For flows through porous media, u represents the pressure.
- f is the intensity of sources.
- $\mathcal{D}(\mathbf{x}) = (d_{i,j})$ is a given tensor which is symmetric. Moreover, there exists a constant $c > 0$ such that

$$\mathcal{D}(\mathbf{x})\xi \cdot \xi \geq c|\xi|^2 \text{ for all } \xi \in \mathrm{R^d}. \tag{2}$$

We consider boundary conditions of the Dirichlet type

$$u = \bar{u}, \ on \ \partial\Omega.$$

Suppose that the domain $\Omega \in R^2$ is made up of a number of non-overlapping polygonal cells $\{\Omega_i\}$. The vertices of cell Ω_i are labeled with M_r, M_{r+1}, M_{r+2}, M_{r+3}, $\cdots$ (see Figure 1). C_i is its centroid. $\delta_{r,r+1} = M_rM_{r+1}$ is the common edge between cell Ω_i and its adjacent cell Ω_j. We assume that each cell Ω_i is star-shaped with respect to its centroid C_i, which means that any ray emanating from C_i intersects the boundary of Ω_i at only one point.

By integrating (1) over Ω_i and using the Green formula, we obtain

$$-\int_{\partial\Omega_i} \mathcal{D}\nabla u \cdot \mathbf{n} dl = f_i|\Omega_i|, \tag{3}$$

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L. Chang is with the Key Laboratory of Science and Technology on Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing, China, e-mail: changln@iapcm.ac.cn.

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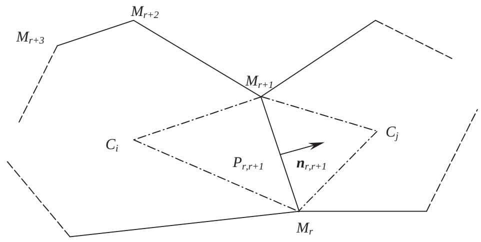

Fig. 1. Stencils for the descretization of the flux

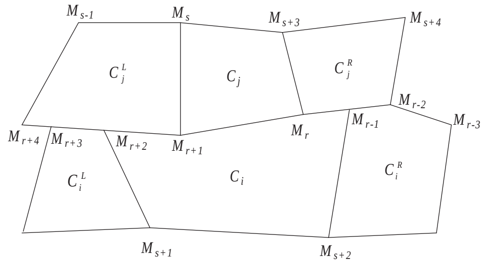

Fig. 2. Schematic skewed mesh

where $\mathbf{n}$ is the unit vector outward normal to the length element dl on $\partial\Omega_i$, and u_i is the mean value of u in Ω_i:

$$u_i = \frac{1}{|\Omega_i|}\int_{\Omega_i} u d\Omega.$$

$\mathcal{D}_i$ and f_i are defined in a similar fashion.

Defining the normal component of the flux density on edge $\sigma \in \partial\Omega_i$ as

$$F_\sigma = -\int_\sigma \mathcal{D}\nabla u \cdot \mathbf{n}_\sigma dl/|\sigma|, \tag{4}$$

we can write

$$-\int_{\partial\Omega_i} \mathcal{D}\nabla u \cdot \mathbf{n} dl = \sum_{\sigma\in\partial\Omega_i} F_\sigma |\sigma|.$$

A control volume scheme is determined when the discretization of F_σ is specified on each edge.

III. Interpolation of the node unknowns

Eq. (4) is rewritten as

$$F_\sigma = -\int_\sigma \nabla u \cdot \mathcal{D}\mathbf{n}_\sigma dl/|\sigma|,$$

since $\mathcal{D}$ is symmetric.

We introduce the following notations

$$\mathbf{n}^{\mathcal{D}_i}_{r,r+1} = \mathcal{D}_i \mathbf{n}_{i,j}, \quad \kappa_i = \mathbf{n}^{\mathcal{D}_i}_{r,r+1} \cdot \mathbf{n}_{r,r+1}, \tag{5}$$

$$\mathbf{n}^{\mathcal{D}_j}_{r,r+1} = \mathcal{D}_j \mathbf{n}_{i,j}, \quad \kappa_j = \mathbf{n}^{\mathcal{D}_j}_{r,r+1} \cdot \mathbf{n}_{r,r+1}, \tag{6}$$

where $\mathcal{D}_i$ and $\mathcal{D}_j$ are the diffusion tensors defined on Ω_i and Ω_j respectively.

Assuming that the center C_i, as well as C_j, is not on the line which contains $M_r M_{r+1}$, we can express the flux density on edge $\delta_{r,r+1}$ as

$$F_{r,r+1} = -\frac{\lambda_i\lambda_j}{\lambda_i+\lambda_j}(u_j - u_i - \frac{D_{r,r+1}}{|\delta_{r,r+1}|}(u_{r+1}-u_r)), \tag{7}$$

where u_{r+1} and u_r are the unknowns defined at the vertex M_{r+1} and M_r, respectively.

$$\lambda_i = \frac{\kappa_i}{h_i}, \qquad \lambda_j = \frac{\kappa_j}{h_j}$$

h_i(resp. h_j) is the distance from C_i (resp. C_j) to $\delta_{r,r+1}$. The details of the derivation of Eq.(7) can be found in Ref.[13].

$$D_{r,r+1} = -\frac{\mathbf{n}^{\mathcal{D}_i}_{r,r+1}\cdot\mathbf{t}_{r,r+1}}{\lambda_i} - \frac{\mathbf{n}^{\mathcal{D}_j}_{r,r+1}\cdot\mathbf{t}_{r,r+1}}{\lambda_j} + (\mathbf{x}_j - \mathbf{x}_i)\cdot\mathbf{t}_{r,r+1}. \tag{8}$$

$\mathbf{t}_{r,r+1}$ is the unit vector tangential to $\delta_{r,r+1}$.

Now, the third item $u_{r+1} - u_r$ remains unknown. In the following, we shall show how to derive the explicit expression for $u_{r+1} - u_r$. The problems with discontinuous coefficient will be considered later.

To obtain the interpolation, we enlarge the stencils so that the adjacent cells are involved for constructing the flux. As an example, we choose the cell Ω_i^L, Ω_i^R, Ω_j^L and Ω_j^R as the stencils for interpolation, as shown in Figure 2.

With Ω_i^L and Ω_j^R, the discretization of $F_{r,r+1}$ can be written as

$$F^{LR}_{r,r+1} = -\tau_{LR}(u_j^R - u_i^L - \frac{D^{LR}_{r,r+1}}{|\delta_{r,r+1}|}(u_{r+1}-u_r)), \tag{9}$$

where u_i^L and u_j^R are the values defined at the centers of Ω_i^L and Ω_j^R, respectively.

$$\tau_{LR} = \frac{\lambda_i^L\lambda_j^R}{\lambda_i^L+\lambda_j^R}, \qquad \lambda_i^L = \frac{\kappa_i^L}{h_i^L}, \qquad \lambda_j^R = \frac{\kappa_j^R}{h_j^R}$$

h_i^L(resp. h_j^R) is the distance from C_i^L (resp. C_j^R) to $\delta_{r,r+1}$.

$$D^{LR}_{r,r+1} = -\frac{\mathbf{n}^{\mathcal{D}_i}_{r,r+1}\cdot\mathbf{t}_{r,r+1}}{\lambda_i^L} - \frac{\mathbf{n}^{\mathcal{D}_j}_{r,r+1}\cdot\mathbf{t}_{r,r+1}}{\lambda_j^R} + (\mathbf{x}_j^R - \mathbf{x}_i^L)\cdot\mathbf{t}_{r,r+1}. \tag{10}$$

Similarly, with Ω_i^R and Ω_j^L, we obtain

$$F^{RL}_{r,r+1} = -\tau_{RL}(u_j^L - u_i^R - \frac{D^{RL}_{r,r+1}}{|\delta_{r,r+1}|}(u_{r+1}-u_r)), \tag{11}$$

where u_i^R and u_j^L are the values defined at the centers of Ω_i^R and Ω_j^L, respectively.

$$\tau_{RL} = \frac{\lambda_i^R\lambda_j^L}{\lambda_i^R+\lambda_j^L}, \qquad \lambda_i^R = \frac{\kappa_i^R}{h_i^R}, \qquad \lambda_j^L = \frac{\kappa_j^L}{h_j^L}$$

h_i^R(resp. h_j^L) is the distance from C_i^R (resp. C_j^L) to $\delta_{r,r+1}$.

$$D^{RL}_{r,r+1} = -\frac{\mathbf{n}^{\mathcal{D}_i}_{r,r+1}\cdot\mathbf{t}_{r,r+1}}{\lambda_i^R} - \frac{\mathbf{n}^{\mathcal{D}_j}_{r,r+1}\cdot\mathbf{t}_{r,r+1}}{\lambda_j^L} + (\mathbf{x}_j^L - \mathbf{x}_i^R)\cdot\mathbf{t}_{r,r+1}. \tag{12}$$

$F^{LR}_{r,r+1}$ and $F^{RL}_{r,r+1}$ are both the discretization of flux across $\delta_{r,r+1}$. So we have

$$F^{LR}_{r,r+1} = F^{RL}_{r,r+1} \tag{13}$$

Combining Eqs.(9), (11) and (13), we finally obtain

$$u_{r+1} - u_r = \frac{\tau_{LR}(u_j^R - u_i^L) - \tau_{RL}(u_j^L - u_i^R)}{\tau_{LR}D^{LR}_{r,r+1} - \tau^{RL}_{r,r+1}D_{RL}}|\delta_{r,r+1}|. \tag{14}$$

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By substituting Eq.(14) into Eq.(7), the discretization of the flux is obtained with cell-centered unknowns only.

Note that Eq.(14) is used for the construction of flux. So the method presented here is different form that in Ref.[13], which aims to reconstruct the unknowns defined at the edges.

For discontinuous problems, the adjacent cells across the discontinuity cannot be chosen as the stencil for interpolation. For example, if the diffusion tensor is discontinuous between Ω_i^L and Ω_i, other adjacent cell will be used as a substitute for Ω_i^L.

IV. Numerical examples

Assume that the exact solutions are known. Let u_i be the value of the exact solution at the centroid of the cell Ω_i, u_i^h the numerical solution, $|\Omega_i|$ the area of Ω_i and $N(\mathcal{J})$ the number of cells. Then we calculate the relative asymptotic errors for the solution using the mean-square norm

$$E_{l_2}^u = \left(\frac{\sum\limits_{i=1}^{N(\mathcal{J})} \left(u_i^h - u_i\right)^2 |\Omega_i|}{\sum\limits_{i=1}^{N(\mathcal{J})} {u_i}^2 |\Omega_i|} \right)^{\frac{1}{2}}. \tag{15}$$

Similarly, the relative asymptotic errors for the flux is defined as

$$E_{l_2}^F = \left(\frac{\sum\limits_{\sigma\in\Gamma} \left(F_\sigma^h - F_\sigma\right)^2 |S_\sigma|}{\sum\limits_{\sigma\in\Gamma} {F_\sigma}^2 |S_\sigma|} \right)^{\frac{1}{2}}, \tag{16}$$

where $\Gamma = \Gamma_I \cup \Gamma_O$, and Γ_I and Γ_O are the sets of interior and boundary edges respectively. S_σ is a representative area for σ. In our numerical test, S_σ is calculated by summing the area of the adjacent two sub-triangles. Note that there is only one adjacent sub-triangle for the boundary.

Calculations are performed on a sequence of grids with different mesh sizes. For two grids $\mathcal{J}_1$ and $\mathcal{J}_2$, we denote the corresponding asymptotic errors by E_1 and E_2, respectively. Then the order of convergence is approximated by

$$q = -2\frac{\log(E_2) - \log(E_1)}{\log(N_2) - \log(N_1)}, \tag{17}$$

where N_1 and N_2 are the number of unknowns in the meshes $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. The order of convergence is also defined with respect to the mesh size

$$\eta = \frac{\log(E_2) - \log(E_1)}{\log(h_2) - \log(h_1)}, \tag{18}$$

where $h_1 = \sup\limits_{\Omega_i\in\mathcal{J}_1} \{\mathrm{diam}\ \Omega_\mathrm{i}\}$ and $h_2 = \sup\limits_{\Omega_i\in\mathcal{J}_2} \{\mathrm{diam}\ \Omega_\mathrm{i}\}$.

A. Mildly anisotropy problem

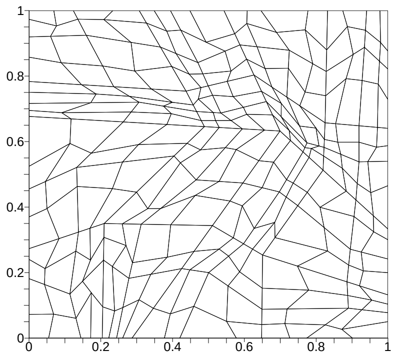
This test originates from Refs. [14]. The diffusion tensor is given by

$$\mathcal{D} = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}.$$

The exact solution is smooth and defined as

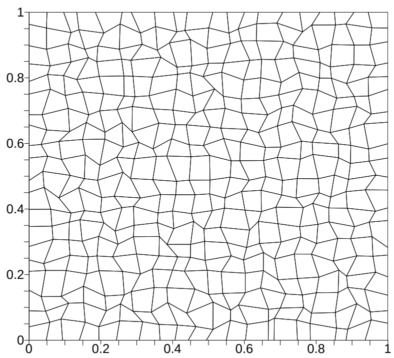
$$\bar{u}(x,y) = \sin((1-x)(1-y)) + (1-x)^3(1-y)^2,$$

Dirichlet boundary condition is consider here. The boundary condition and source term are specified by the exact solution.



256 cells

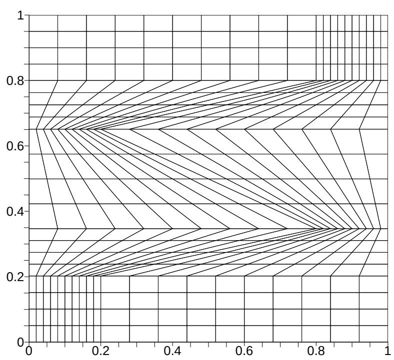
Fig. 3. Sheshtakov meshes



20×20 cells

Fig. 4. Random meshes

We perform the calculation on the Sheshtakov meshes, random meshes, and Kershaw meshes, presented in Figure 3, Figure 4 and Figure 5 respectively. Table I lists the convergence analysis data. It shows that our scheme is close to second order accurate for the solution. And the convergence rate of the flux is more than one.



20×20 cells

Fig. 5. Kershaw meshes

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TABLE I
THE RESULTS FOR MILDLY ANISOTROPY PROBLEM ON VARIOUS SKEWED MESHES

Mesh	N	$E^u_{l_2}$	$q^u_{l_2}$	$E^F_{l_2}$	$q^F_{l_2}$
Sheshtakov	256	$0.375e^{-2}$	–	$0.207e^{-1}$	–
	1024	$0.935e^{-3}$	2.00	$0.872e^{-2}$	1.25
	4096	$0.264e^{-3}$	1.82	$0.306e^{-2}$	1.51
	16384	$0.703e^{-4}$	1.91	$0.107e^{-2}$	1.52
	65536	$0.185e^{-4}$	1.93	$0.382e^{-3}$	1.49
Random	400	$0.134e^{-2}$	–	$0.775e^{-2}$	–
	1600	$0.316e^{-3}$	2.09	$0.359e^{-2}$	1.11
	6400	$0.788e^{-4}$	2.00	$0.173e^{-2}$	1.06
	25600	$0.225e^{-4}$	1.81	$0.833e^{-3}$	1.05
	102400	$0.487e^{-5}$	2.21	$0.410e^{-3}$	1.02
Kershaw	400	$0.124e^{-1}$	–	$0.627e^{-1}$	–
	1600	$0.478e^{-2}$	1.37	$0.247e^{-1}$	1.34
	6400	$0.141e^{-2}$	1.76	$0.880e^{-2}$	1.49
	25600	$0.373e^{-3}$	1.92	$0.302e^{-2}$	1.54
	102400	$0.951e^{-4}$	1.97	$0.104e^{-2}$	1.54

TABLE II
THE RESULTS FOR THE DISCONTINUOUS PROBLEM ON RANDOM MESHES

N	$E^u_{l_2}$	$q^u_{l_2}$	$E^F_{l_2}$	$q^F_{l_2}$
64	$0.268e^{-1}$	–	$0.892e^{-1}$	–
256	$0.632e^{-2}$	2.08	$0.239e^{-1}$	1.90
1024	$0.159e^{-2}$	1.99	$0.197e^{-1}$	0.28
4096	$0.399e^{-3}$	2.00	$0.697e^{-2}$	1.50
16384	$0.971e^{-4}$	2.04	$0.309e^{-2}$	1.17

B. Discontinuous problem

We consider a diffusion problem with the discontinuous piecewise constant tensor function

$$\mathcal{D} = \begin{cases} \mathcal{D}_1, \\ \mathcal{D}_2, \end{cases}$$

where

$$\mathcal{D}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } x \le 0.5,$$

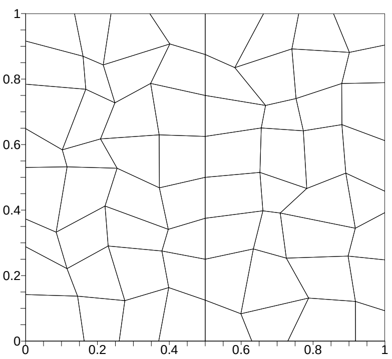
$$\mathcal{D}_2 = \begin{pmatrix} 100 & 0 \\ 0 & 0.01 \end{pmatrix} \text{ if } x > 0.5.$$

The analytical solution is given by

$$\bar{u} = \begin{cases} \cos(\pi x)\sin(\pi x) & \text{if } x \le 0.5, \\ 0.01\cos(\pi x)\sin(\pi x) & \text{if } x > 0.5, \end{cases}$$

and the source term by $f = -\nabla(\mathcal{D}\nabla\bar{u})$. Here the Dirichlet conditions are considered on the boundary. This example comes from Ref. [15].

Calculation are performed on a series of random meshes, displayed in Figure 6. The errors on those meshes are presented in Table II. Our scheme is second order accurate for the solution. For the flux, the convergence rate is not stable. The reason may lie in that the skewness of our random grids deteriorates or alleviate randomly as the grids are refined, which will affect the convergence order more or less.



20×20 cells

Fig. 6. Random meshes with a straight line at the discontinuity

V. CONCLUSIONS

In this paper an interpolation method is developed for solving heterogeneous anisotropic diffusion problems on skewed meshes. Using the continuity condition of the solution and flux, an explicit expression is derived for interpolating the node unknowns from the cell-centered unknowns. Discontinuity can be dealt with strictly by choosing the proper stencils. A scheme with cell-centered unknowns only is resulted. And there is no need for reconstructing the gradient, which makes the scheme concise.

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