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Numerical Analysis for Some Parallel Algorithms with Large Step Length Spatial Discretizations for 2-dimensional Heat Equation

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***Abstract*—Some simple and efficient domain decomposition algorithms for 2-dimensional heat equation are studied. The values at the interface points are calculated by using implicit schemes derived by discretizing the spatial partial derivatives in two directions with large step lengths alternatively. On sub-domain calculation, three algorithms including traditional fully implicit scheme, Crank-Nicolson scheme and explicit-implicit scheme are considered. By using a constructive reasoning procedure with the help of discrete maximum principle, the stability and accuracy properties of the new algorithms are analyzed. Theoretical analysis and numerical experiments show that by applying these algorithms, the calculation work is simplified, and the stability and convergence conditions are loosen to a quite great extent. The algorithms have perfect accuracy and high parallel efficiency and scalability.**

***Index Terms*—heat equation, domain decomposition, interface evaluation, large step spatial discretizations, numerical analysis**

I. Introduction

Studies on parallel computational method for diffusion problems have important significants in fast and accurate numerical simulations for many practice problem [1]. On the level of basic studies, parallel numerical solutions for PDEs have two meanings. One is parallel discrete schemes, i.e., to construct discrete schemes with intrinsic parallelism and good stability; the other is parallel algebraic methods, i.e., to solve the algebraic systems derived by discrete schemes in parallel ways. Domain decomposition technique plays an important role in designing parallel discrete schemes. It can raise computation efficiency by dividing a large problem into several small sub-problems and hence decreasing the scale of the calculation. Evaluation of unknowns at the interfacial points is the key in the study of domain decomposition algorithms, its design can directly affect the accuracy, stability and efficiency of the parallel algorithm. There are some researches on parallel discrete schemes for diffusion problem [2]-[14]. In [2], a domain decomposition scheme is proposed for heat equation in one-dimensional geometry with large step length discrete approach for spatial derivative on interface and is extended to two-dimensional geometry with strip decomposition. In this paper, we study the parallel schemes for two-dimensional problem with large step length discretization in both spatial directions for block decomposition and present their stability and convergence properties. Three methods are applied to calculate the values at the interior points. The algorithms are easy to implement and need little communication. With strict theoretical analysis, it shows the schemes have much looser stability and perfect accuracy. Numerical tests verify the theoretical results and show the high efficiency of the algorithms.

Consider the two-dimensional heat equation as follows

$$\begin{aligned} u_t = u_{xx} + u_{yy}, &\qquad (x,y)\in\Omega, t\in(0,T],\\ u(x,y,t)=0, &\qquad (x,y)\in\partial\Omega, t\in(0,T],\\ u(x,y,0)=u_0(x,y), &\qquad (x,y)\in\Omega, \end{aligned} \tag{1}$$

where $\Omega = (0,L_1)\times(0,L_2)$, L_1, L_2 and T are positive constants; u_0 is a known function.

In this paper, some preparation work is given in Section II. Several simplified domain decomposition schemes for heat equation (1) are given in Section III. In Section IV, their stability and convergence properties are proved. Then numerical tests are presented in Section V. Finally some conclusions are given in section VI.

II. Preparation Work

Before give approximations for (1), we do some preparation.

Divide interval $[0,T]$ and $[0,L_1]$, $[0,L_2]$ into N and J_1, J_2 equal small intervals respectively, denote $\tau = \frac{T}{N}$, $t^n = n\tau$, and $h_1 = \frac{L_1}{J_1}$, $h_2 = \frac{L_2}{J_2}$, $h = \max\{h_1, h_2\}$, $x_i = ih_1$, $y_j = jh_2$. Let q_1 and q_2 be positive integers, $Q = \min\{q_1, q_2\}$. Denote $H_1 = q_1h_1$ and $H_2 = q_2h_2$ as large spatial step lengths, $H = \max\{H_1, H_2\}$. For a function $\phi(x,y,t)$ defined at mesh points (x_i, y_j, t^n), let $\phi^n_{ij} = \phi(x_i, y_j, t^n)$. Define the difference operators

$$\begin{aligned} \partial_{t,\tau}\phi(x,y,t) &= \tfrac{f(x,y,t)-f(x,y,t-\tau)}{\tau},\\ \partial_{x,s}\phi(x,y,t) &= \tfrac{f(x+s,y,t)-f(x,y,t)}{s},\\ \partial_{y,s}\phi(x,y,t) &= \tfrac{f(x,y+s,t)-f(x,y,t)}{s},\\ \partial^2_{x,s}\phi(x,y,t) &= \tfrac{f(x-s,y,t)-2f(x,y,t)+f(x+s,y,t)}{s^2},\\ \partial^2_{y,s}\phi(x,y,t) &= \tfrac{f(x,y-s,t)-2f(x,y,t)+f(x,y+s,t)}{s^2}. \end{aligned}$$

Thus $\partial_{t,\tau}\phi^{n+1} = \frac{1}{\tau}(\phi^{n+1}-\phi^n)$ for $n\ge 0$, $\partial_{x,h_1}\phi_{ij} = \frac{1}{h_1}(\phi_{i+1,j}-\phi_{ij})$, $\partial_{y,h_2}\phi_{ij} = \frac{1}{h_2}(\phi_{i,j+1}-\phi_{ij})$, $\partial^2_{x,h_1}\phi_{ij} = \frac{1}{h_1^2}(\phi_{i+1,j}-2\phi_{ij}+\phi_{i-1,j})$, $\partial^2_{y,h_2}\phi_{ij} = \frac{1}{h_2^2}(\phi_{i,j+1}-2\phi_{ij}+\phi_{i,j-1})$, $\partial_{x,H_1}\phi_{ij} = \frac{1}{H_1}(\phi_{i+q_1,j}-\phi_{ij})$, $\partial_{y,H_2}\phi_{ij} = \frac{1}{H_2}(\phi_{i,j+q_2}-\phi_{ij})$, $\partial^2_{x,H_1}\phi_{ij} = \frac{1}{H_1^2}(\phi_{i+q_1,j}-2\phi_{ij}+\phi_{i-q_1,j})$, $\partial^2_{y,H_2}\phi_{ij} = \frac{1}{H_2^2}(\phi_{i,j+q_2}-2\phi_{ij}+\phi_{i,j-q_2})$.

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Denote the mesh ratios as $r_1 = \frac{\tau}{h_1^2}$, $r_2 = \frac{\tau}{h_2^2}$. Let $R_1 = \frac{\tau}{H_1^2}$, $R_2 = \frac{\tau}{H_2^2}$ be the "large" mesh ratios.

For $n \geq 0$, by using implicit schemes with explicit large step length discrete simulation for spatial derivatives in x and y directions and implicit normal step discretization in another direction alternatively,

$$\partial_{t,\tau}\phi_{ij}^{n+1} = \partial_{x,H_1}^2\phi_{ij}^n + \partial_{y,h_2}^2\phi_{ij}^{n+1},$$
$$\partial_{t,\tau}\phi_{ij}^{n+1} = \partial_{x,h_1}^2\phi_{ij}^{n+1} + \partial_{y,H_2}^2\phi_{ij}^n,$$

we deduce

$$\begin{aligned}
P_1\phi_{ij}^{n+1} &=: -r_2\phi_{i,j+1}^{n+1} + (1+2r_2)\phi_{ij}^{n+1} - r_2\phi_{i,j-1}^{n+1} \\
&= R_1\phi_{i+q_1,j}^n + (1-2R_1)\phi_{ij}^n + R_1\phi_{i-q_1,j}^n \\
&=: Q_1\phi_{ij}^n, \\
P_2\phi_{ij}^{n+1} &=: -r_1\phi_{i+1,j}^{n+1} + (1+2r_1)\phi_{ij}^{n+1} - r_1\phi_{i-1,j}^{n+1} \\
&= R_2\phi_{i,j+q_2}^n + (1-2R_2)\phi_{ij}^n + R_2\phi_{i,j-q_2}^n \\
&=: Q_2\phi_{ij}^n.
\end{aligned}$$

Now define

$$\begin{aligned}
L_1\phi_{ij}^{n+1} &=: P_1\phi_{ij}^{n+1} - Q_2\phi_{ij}^n, \\
L_2\phi_{ij}^{n+1} &=: P_2\phi_{ij}^{n+1} - Q_2\phi_{ij}^n.
\end{aligned} \quad (2)$$

Let

$$\begin{aligned}
S_1\phi_{ij}^{n+1} &= (1-\tau\partial_{x,h_1}^2 - \tau\partial_{y,h_2}^2)\phi_{ij}^{n+1} - \phi_{ij}^n, \\
S_2\phi_{ij}^{n+1} &= (I - \frac{\tau}{2}\partial_{x,h_1}^2 - \frac{\tau}{2}\partial_{y,h_2}^2)\phi_{ij}^{n+1} \\
&\quad -(I + \frac{\tau}{2}\partial_{x,h_1}^2 + \frac{\tau}{2}\partial_{y,h_2}^2)\phi_{ij}^n, \\
S_3\phi_{ij}^{2m+2} &= (I - \tau\partial_{x,h_1}^2 - \tau\partial_{y,h_2}^2)\phi_{ij}^{2m+2} - \phi_{ij}^{2m+1} \\
&= (I - \tau\partial_{x,h_1}^2 - \tau\partial_{y,h_2}^2)\phi_{ij}^{2m+2} \\
&\quad -(I + \tau\partial_{x,h_1}^2 + \tau\partial_{y,h_2}^2)\phi_{ij}^{2m}.
\end{aligned} \quad (3)$$

We can see $S_1\phi_{ij}^{n+1} = 0$, $S_2\phi_{ij}^{n+1} = 0$ and $S_3\phi_{ij}^{2m+2}$ respectively stand for the traditional fully implicit scheme, Crank-Nicolson scheme and explicit-implicit scheme. For $M = 1, 2, 3$, substitute $\phi_{pq}^n = v^n exp[i(\alpha x_p + \beta y_q)]$ to the relation $S_M\phi_{pq}^n = 0$, we get the increment factor

$$\begin{aligned}
G_1 &= \frac{v^{n+1}}{v^n} = \frac{1}{1 + 4r_1 sin^2\frac{\alpha h_1}{2} + 4r_2 sin^2\frac{\beta h_2}{2}}, \\
G_2 &= \frac{v^{n+1}}{v^n} = \frac{1 - 2r_1 sin^2\frac{\alpha h_1}{2} - 2r_2 sin^2\frac{\beta h_2}{2}}{1 + 2r_1 sin^2\frac{\alpha h_1}{2} + 2r_2 sin^2\frac{\beta h_2}{2}}, \\
G_3 &= \frac{v^{2m+2}}{v^{2m}} = \frac{1 - 4r_1 sin^2\frac{\alpha h_1}{2} - 4r_2 sin^2\frac{\beta h_2}{2}}{1 + 4r_1 sin^2\frac{\alpha h_1}{2} + 4r_2 sin^2\frac{\beta h_2}{2}},
\end{aligned}$$

where the last relation measures increment ratio by each two even time steps. Obviously, $|G_M| \leq 1$ holds for $\forall r_1, r_2 > 0$, thus $S_M\phi_{ij}^n = 0$ is absolutely stable.

Let u_{ij}^n be the real solution of problem (1), then from Taylor's expansion, we have the truncation error

$$\begin{aligned}
L_1u_{ij}^{n+1} &= O(\tau(\tau + h_2^2 + H_1^2)), \\
L_2u_{ij}^{n+1} &= O(\tau(\tau + h_1^2 + H_2^2)), \\
S_1u_{ij}^{n+1} &= O(\tau(\tau + h_1^2 + h_2^2)), \\
S_2u_{ij}^{n+1} &= O(\tau(\tau^2 + h_1^2 + h_2^2)), \\
S_3u_{ij}^{2m+2} &= O(\tau(\tau^2 + h_1^2 + h_2^2)).
\end{aligned} \quad (4)$$

III. Simplified Domain Decomposition Algorithms

For simplicity, we first consider the case of decomposing the domain into four sub-domains with interfacial line $x = x_k$ and $y = y_l$. Here and below, we will refer to points (x_i, y_j, t^n) as boundary points (BPs) if $i = 0$ or J_1, or if $j = 0$ or J_2, or if $n = 0$; similarly, we call them interface points (IFPs) if $i = k$ and $n > 0$, or if $j = l$ and $n > 0$; otherwise we call them interior points (IPs).

The numerical approximation U_{ij}^n to u_{ij}^n is defined by

$$U_{ij}^n = u_{ij}^n, \quad BPs; \quad (5)$$
$$L_1U_{kj}^n = 0, \quad IFPs\ (x_k, y_j, t^n),\ j \neq l; \quad (6)$$
$$L_2U_{il}^n = 0, \quad IFPs\ (x_i, y_l, t^n),\ i \neq k; \quad (7)$$
$$L_1U_{kl}^n = 0, \ or \ L_2U_{kl}^n = 0; \quad (8)$$
$$S_MU_{ij}^n = 0, \quad IPs\ (n = 2m\ for\ M = 3); \quad (9)$$

where $M = 1, 2, 3$.

The algorithms are carried out as follows.

Step 1 First, U_{ij}^n at boundary points are defined.

Step 2 Values at interface points are calculated. Two choices are provided to realize this object.

Case 1: For $L_1U_{kl}^n = 0$ in (8), first, equation system $L_1U_{kj}^n = 0$, $j = 1, 2, \cdots, J_2 - 1$ are solved with U_{k0}^n, $U_{kJ_2}^n$ and U_{ij}^{n-1} $(i = 1, 2, \cdots, J_1 - 1; j = 1, 2, \cdots, J_2 - 1)$ already known. Next, two systems of equations are considered respectively. They are $L_2U_{il}^n = 0$, $i = 1, 2, \cdots, k-1$, with boundary conditions U_{0l}^n and U_{kl}^n known, and $L_2U_{il}^n = 0$, $i = k+1, k+2, \cdots, J_1 - 1$, with boundary conditions U_{kl}^n and $U_{J_1l}^n$ known.

Case 2: For $L_2U_{kl}^n = 0$ in (8), first, resolve $L_2U_{il}^n = 0$, $i = 1, 2, \cdots, J_1 - 1$ with U_{0l}^n, $U_{J_1l}^n$ and U_{ij}^{n-1} $(i = 1, 2, \cdots, J_1 - 1; j = 1, 2, \cdots, J_2 - 1)$ known. Second, consider $L_1U_{kj}^n = 0$, $j = 1, 2, \cdots, l-1$, with boundary conditions U_{k0}^n and U_{kl}^n known, and $L_1U_{kj}^n = 0$, $j = l+1, l+2, \cdots, J_2 - 1$, with boundary conditions U_{kl}^n and $U_{kJ_2}^n$ known.

Hence all the interface values at the n-th time level are derived.

To keep balance on x and y directions, case 1 and case 2 can be chosen alternately. Note that each calculation of interface unknowns is to solve a tridiagonal matrix system and can be realized by sweeping method. The algorithm is simple and economic.

Step 3 Finally, $S_MU_{ij}^n = 0$ are applied to get the interior values in each of the four sub-domains.

On each sub-domain, $S_MU_{ij}^n = 0$ can be realized by the following procedures.

(1) Implicit Scheme

$$\begin{aligned}
&-r_1(U_{i+1,j}^n + U_{i-1,j}^n) + (1 + 2r_1 + 2r_2)U_{ij}^n \\
&-r_2(U_{i,j+1}^n + U_{i,j-1}^n) \\
=\ & U_{ij}^{n-1}.
\end{aligned} \quad (10)$$

(2) CN (Crank-Nicolson) Scheme

$$\begin{aligned}
&-\frac{r_1}{2}(U_{i+1,j}^n + U_{i-1,j}^n) + (1 + r_1 + r_2)U_{ij}^n \\
&-\frac{r_2}{2}(U_{i,j+1}^n + U_{i,j-1}^n) \\
=\ & \frac{r_1}{2}(U_{i+1,j}^{n-1} + U_{i-1,j}^{n-1}) + (1 - r_1 - r_2)U_{ij}^{n-1} \\
&+\frac{r_2}{2}(U_{i,j+1}^{n-1} + U_{i,j-1}^{n-1}).
\end{aligned} \quad (11)$$

(3) EI (Explicit Implicit) Scheme

$$\begin{aligned}
& U_{ij}^{2m-1} \\
= \;& r_1(U_{i+1,j}^{2m-2}+U_{i-1,j}^{2m-2})+(1-2r_1-2r_2)U_{ij}^{2m-2} \\
& +r_2(U_{i,j+1}^{2m-2}+U_{i,j-1}^{2m-2}), \\
& -r_1(U_{i+1,j}^{2m}+U_{i-1,j}^{2m})+(1+2r_1+2r_2)U_{ij}^{2m} \\
& -r_2(U_{i,j+1}^{2m}+U_{i,j-1}^{2m}) \\
= \;& U_{ij}^{2m-1}. \qquad (12)
\end{aligned}$$

EI Scheme (12) means a great save of calculation work.

IV. Theoretical analysis on stability and accuracy properties

Denote Condition (A_1): $1-2R_1 \geq 0$ and $1-2R_2 \geq 0$; Condition (A_2): $1-r_1-r_2 \geq 0$, $1-2R_1 \geq 0$ and $1-2R_2 \geq 0$; Condition (A_3): $1-2r_1-2r_2 \geq 0$.

Algorithms (5)-(9) have the following stability and accuracy properties.

Theorem 1 For the numerical solution U_{ij}^n of Algorithms (5)-(9) and the real solution u_{ij}^n of (1), there is

$$\begin{aligned}
& for\ \ M=1,\ \ under\ \ Condition\ \ (A_1), \\
& \max|u_{ij}^n-U_{ij}^n| = O(\tau+h^2+H\tau+H^3). \\
& for\ \ M=2,\ \ under\ \ Condition\ \ (A_2), \\
& \max|u_{ij}^n-U_{ij}^n| = O(\tau^2+h^2+H\tau+H^3). \\
& for\ \ M=3,\ \ under\ \ Condition\ \ (A_3), \\
& \max|u_{ij}^{2m}-U_{ij}^{2m}| = O(\tau^2+h^2+H\tau+H^3).
\end{aligned}$$

Here C is a positive constant independent of τ and h.

The proof of Theorem 1 is somehow complicated, so we'll show it step by step. First we give a useful property.

Lemma 1 (Discrete maximum principle) For $M = 1,2,3$, under Condition (A_M), if there are

$$\begin{aligned}
z_{ij}^n &\leq 0, && BPs; && (13)\\
L_1 z_{kj}^n &\leq 0, && IFPs\ (x_k,y_j,t^n),\ j\neq l; && (14)\\
L_2 z_{il}^n &\leq 0, && IFPs\ (x_i,y_l,t^n),\ i\neq k; && (15)\\
L_1 z_{kl}^n &\leq 0,\ \ or && L_2 z_{kl}^n \leq 0; && (16)\\
S_M z_{ij}^n &\leq 0, && IPs\ (n=2m\ for\ M=3); && (17)
\end{aligned}$$

then

$$z_{ij}^n \leq 0, \quad \forall i,j,n\ (n=2m\ for\ M=3). \qquad (18)$$

Proof of Lemma 1:

Note that (18) holds for $n=0$ by (13). Now for $M=1$ and 2, suppose the conclusion holds up to some level $n-1$ (where $n\geq 1$); i.e.,

$$z_{ij}^{n-1} \leq 0, \quad \forall i,j; \qquad (19)$$

thus for $L_1 z_{kl}^n \leq 0$ in (16), from (14) and (13), the interface value

$$z_{kj}^n \leq 0,\ \ j=1,2,\cdots,J_2-1,$$

specially, $z_{kl}^n \leq 0$, hence with (13), (19) and (15), there is

$$z_{il}^n \leq 0,\ \ i=1,2,\cdots,J_1-1,\ \ i\neq k,$$

each of z_{kj} and z_{il} is bounded by an average of values of z_{ij}^{n-1}, and the weights in the average are nonnegative with the constraint (A_M). For $L_2 z_{kl}^n \leq 0$ in (16), a similar procedure shows the interface value

$$\begin{aligned}
z_{il}^n &\leq 0,\ \ i=1,2,\cdots,J_1-1,\\
z_{kj}^n &\leq 0,\ \ j=1,2,\cdots,J_2-1,\ \ j\neq l.
\end{aligned}$$

Then notice (13), we can directly know by the maximum principle that

$$z_{ij}^n \leq 0, \quad at\ interior\ points. \qquad (20)$$

For $M=3$, suppose the conclusion holds up to level $2m-2$ (where $m\geq 1$), then by a similar reasoning procedure, and notice (12), we have

$$\begin{aligned}
& -r_1(z_{i+1,j}^{2m}+z_{i-1,j}^{2m})+(1+2r_1+2r_2)z_{ij}^{2m} \\
& -r_2(z_{i,j+1}^{2m}+z_{i,j-1}^{2m}) \\
= \;& r_1(z_{i+1,j}^{2m-2}+z_{i-1,j}^{2m-2})+(1-2r_1-2r_2)z_{ij}^{2m-2} \\
& +r_2(z_{i,j+1}^{2m-2}+z_{i,j-1}^{2m-2}) \leq 0,
\end{aligned}$$

as before, we see (13) stands for $n=2m$, which completes the proof of Lemma 1. ¶

Now we use a constructive reasoning procedure to prove the accuracy property of our domain decomposition discrete scheme.

Proof of Theorem 1:

Denote $e_{ij}^n = u_{ij}^n - U_{ij}^n$. First, we know from (4) and (5)-(9) that

$$\begin{aligned}
e_{ij}^n &= 0, && BPs;\\
L_1 e_{kj}^n &= K_{kj}\tau(\tau+h_2^2+H_1^2), && IFPs\,(x_k,y_j,t^n),\ j\neq l;\\
L_2 e_{il}^n &= K_{il}\tau(\tau+h_1^2+H_2^2), && IFPs\,(x_i,y_l,t^n),\ i\neq k;\\
L_1 e_{kl}^n &= K_{kl}\tau(\tau+h_2^2+H_1^2), && or\\
L_2 e_{kl}^n &= K_{kl}\tau(\tau+h_1^2+H_2^2); &&\\
S_1 e_{ij}^n &= K_{ij}\tau(\tau+h_1^2+h_2^2), && IPs,\\
S_M e_{ij}^n &= K_{ij}\tau(\tau^2+h_1^2+h_2^2), && IPs,\ M=2,3; \qquad (21)
\end{aligned}$$

where $|K_{ij}^n| \leq C$, for $i=1,2,\cdots,J_1-1$ and $j=1,2,\cdots,J_2-1$.

Then we introduce some additive functions to deal with the normal step length discretizations. Denote

$$\begin{aligned}
w_i &= \frac{1}{2}x_i(1-x_i),\ \ i=0,1,\cdots,J_1;\\
\kappa_j &= \frac{1}{2}y_j(1-y_j),\ \ y=0,1,\cdots,J_2; \qquad (22)
\end{aligned}$$

then we have

$$\begin{aligned}
& L_1 w_i = L_2 w_i = \tau,\ \ i=1,2,\cdots,J_1-1;\\
& L_1\kappa_j = L_2\kappa_j = \tau,\ \ j=1,2,\cdots,J_2-1;\\
& S_1 w_i = S_2 w_i = \tau,\ \ S_3 w_i = 2\tau,\ \ i=1,2,\cdots,J_1-1;\\
& S_1\kappa_j = S_2\kappa_j = \tau,\ \ S_3\kappa_j = 2\tau,\ \ j=1,2,\cdots,J_2-1;\\
& 0 \leq w_i \leq \frac{1}{4},\ \ i=0,1,\cdots,J_1;\\
& 0 \leq \kappa_j \leq \frac{1}{4},\ \ j=0,1,\cdots,J_2.
\end{aligned}$$

Now let

$$\theta_{ij} = w_i + \kappa_j, \qquad (23)$$

then we have

$$\begin{aligned}
&L_1\theta_{ij} = L_2\theta_{ij} = 2\tau,\\
&S_1\theta_{ij} = S_2\theta_{ij} = 2\tau,\ \ S_3\theta_{ij} = 4\tau,\\
&\qquad i = 1, 2, \cdots, J_1 - 1;\ j = 1, 2, \cdots, J_2 - 1.\\
&0 \le \theta_{ij} \le \frac{1}{2},\ \ i = 0, 1, \cdots, J_1;\ j = 0, 1, \cdots, J_2.
\end{aligned}$$

Next we introduce some additional functions to treat the large spatial step length terms. Define

$$\delta_i = \ \left\{ \begin{array}{ll} H_1(1 - x_k)x_i, & i \le k.\\ H_1x_k(1 - x_i), & i \ge k. \end{array}\right.$$

$$\sigma_j = \ \left\{ \begin{array}{ll} H_2(1 - y_l)y_j, & j \le l.\\ H_2y_l(1 - y_j), & j \ge l. \end{array}\right. \tag{24}$$

By a thorough calculation, we have

$$\begin{aligned}
&L_1\delta_k = L_2\sigma_l = \tau,\ \ L_1\sigma_l = q_2\tau,\ \ L_2\delta_k = q_1\tau.\\
&L_2\delta_i = 0,\ \ i = 1, 2, \cdots, J_1 - 1;\ i \ne k.\\
&L_1\sigma_j = 0,\ \ j = 1, 2, \cdots, J_2 - 1;\ j \ne l.\\
&S_M\delta_i = 0,\ \ i = 1, 2, \cdots, J_1 - 1;\ i \ne k;\ M = 1, 2, 3.\\
&S_M\sigma_j = 0,\ \ j = 1, 2, \cdots, J_2 - 1;\ j \ne l;\ M = 1, 2, 3.\\
&0 \le \delta_i \le \frac{1}{2}H_1,\ \ i = 0, 1, \cdots, J_1.\\
&0 \le \sigma_j \le \frac{1}{2}H_2,\ \ j = 0, 1, \cdots, J_2.
\end{aligned}$$

Let

$$\gamma_{ij} = \delta_i + \sigma_j, \tag{25}$$

then we derive

$$\begin{aligned}
&L_1\gamma_{kj} = \tau,\ \ j = 1, 2, \cdots, J_2 - 1;\ j \ne l.\\
&L_2\gamma_{il} = \tau,\ \ i = 1, 2, \cdots, J_1 - 1;\ i \ne k.\\
&L_1\gamma_{kl} = (1 + q_2)\tau,\ \ L_2\gamma_{kl} = (1 + q_1)\tau.\\
&S_M\gamma_{ij} = 0,\ i = 1, 2, \cdots, J_1 - 1;\ j = 1, 2, \cdots, J_2 - 1;\\
&\qquad i \ne k;\ j \ne l;\ M = 1, 2, 3.\\
&0 \le \gamma_{ij} \le \frac{1}{2}H_1 + \frac{1}{2}H_2,\\
&\qquad i = 0, 1, \cdots, J_1;\ j = 0, 1, \cdots, J_2.
\end{aligned}$$

We first show that for $M = 2, 3$, the conclusion of Theorem 1 is valid. Let

$$\begin{aligned}
\xi_{ij} &= C\theta_{ij}(\tau^2 + h_1^2 + h_2^2)\\
&\quad + C\gamma_{ij}(\tau + h_1^2 + h_2^2 + H_1^2 + H_2^2),
\end{aligned} \tag{26}$$

then we have

$$\begin{aligned}
L_1\xi_{kj} &= 2C\tau(\tau^2 + h_1^2 + h_2^2)\\
&\quad + C\tau(\tau + h_1^2 + h_2^2 + H_1^2 + H_2^2),\\
&\qquad j = 1, 2, \cdots, J_2 - 1; j \ne l.\\
L_2\xi_{il} &= 2C\tau(\tau^2 + h_1^2 + h_2^2)\\
&\quad + C\tau(\tau + h_1^2 + h_2^2 + H_1^2 + H_2^2),\\
&\qquad i = 1, 2, \cdots, J_1 - 1; i \ne k.\\
L_1\xi_{kl} &= 2C\tau(\tau^2 + h_1^2 + h_2^2)\\
&\quad + C(1 + q_2)\tau(\tau + h_1^2 + h_2^2 + H_1^2 + H_2^2),\\
or\ L_2\xi_{kl} &= 2C\tau(\tau^2 + h_1^2 + h_2^2)\\
&\quad + C(1 + q_1)\tau(\tau + h_1^2 + h_2^2 + H_1^2 + H_2^2).\\
S_2\xi_{ij} &= 2C\tau(\tau^2 + h_1^2 + h_2^2),\\
S_3\xi_{ij} &= 4C\tau(\tau^2 + h_1^2 + h_2^2),\\
&\qquad i \ne k; j \ne l.
\end{aligned} \tag{27}$$

Hence

$$\begin{array}{ll}
-\xi_{ij} \le e_{ij}^n \le \xi_{ij}, & BPs;\\
-L_1\xi_{kj} \le L_1e_{kj}^n \le L_1\xi_{kj}, & IFPs\ (x_k, y_j, t^n),\ j \ne l;\\
-L_2\xi_{il} \le L_2e_{il}^n \le L_2\xi_{il}, & IFPs\ (x_i, y_l, t^n),\ i \ne k;\\
-L_1\xi_{kl} \le L_1e_{kl}^n \le L_1\xi_{kl}, & or\\
-L_2\xi_{kl} \le L_2e_{kl}^n \le L_2\xi_{kl}; & \\
-S_M\xi_{ij} \le S_Me_{ij}^n \le S_M\xi_{ij}, & IPs.
\end{array} \tag{28}$$

Let $z_{ij} = e_{ij}^n - \xi_{ij}$ and use Lemma 1, we deduce $z_{ij}^n \le 0$, which means $e_{ij}^n \le \xi_{ij}$; then let $z_{ij} = -e_{ij}^n - \xi_{ij}$ and still use Lemma 1, we have $-e_{ij}^n \le \xi_{ij}$; thus $|e_{ij}^n| \le \xi_{ij}$. Notice that $\xi_{ij} \le C(\tau^2 + h_1^2 + h_2^2 + H_1\tau + H_2\tau + H_1^3 + H_2^3 + H_1^2H_2 + H_1H_2^2)$, we have $|e_{ij}^n| \le C(\tau^2 + h_1^2 + h_2^2 + H_1\tau + H_2\tau + H_1^3 + H_2^3 + H_1^2H_2 + H_1H_2^2)$.

For $M = 1$, let

$$\begin{aligned}
\xi_{ij} &= C\theta_{ij}(\tau + h_1^2 + h_2^2)\\
&\quad + C\gamma_{ij}(\tau + h_1^2 + h_2^2 + H_1^2 + H_2^2);
\end{aligned} \tag{29}$$

similarly as before, we can know $|e_{ij}^n| \le C(\tau + h_1^2 + h_2^2 + H_1\tau + H_2\tau + H_1^3 + H_2^3 + H_1^2H_2 + H_1H_2^2)$.

Thus we accomplish the proof of Theorem 1. ¶

V. NUMERICAL TESTS

Numerical tests are given with model problem to verify the results of theoretical analysis and examine the accuracy and stability of the algorithms. In the tests, take $\Omega = (0, 1) \times (0, 1)$, the initial condition and the exact solution as

$$\begin{aligned}
&u_0(x, y) = sin(\pi x)sin(\pi y),\\
&u(x, y) = exp(-2\pi^2 t)sin(\pi x)sin(\pi y).
\end{aligned}$$

Calculations are carried out on different meshes and various CPUs. To focus on the test of properties of the discrete schemes, there is no special consideration on solutions of algebraic systems, and conjugate gradient iteration method is adopted. In fact, computation can be accelerated much faster if the discrete schemes are combined with advanced algebraic system solution techniques.

Table I gives the approximation of the numerical solutions of the domain decomposition algorithm to the exact solution of the problem with different temporal and spatial step lengths and various q_1 and q_2. Herein $T = 0.1$, the interior computation is carried out with implicit scheme. The algorithm still converges when the mesh ratio r rises over $q_1^2/2 = q_2^2/2$. It shows the algorithm has good accuracy and stability property.

Table II shows the parallel efficiency of the algorithm with $T = 0.1$ and $\tau = 0.001$ on various spatial meshes with different CPU numbers. Herein "Time-s" and "Time-p" respectively represent the time needed for serial and parallel computations, "Speedup" and "Eff" respectively denote the speedup ratio and the parallel efficiency of the algorithms. It shows the domain decomposition algorithm has nice expansibility with super-linear speedup ratio and parallel efficiency more than 100%.

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TABLE I
ACCURACY AND STABILITY

h	τ	q_1, q_2	Error-maximum	Error-average
1/30	0.001	3	2.4575e-3	8.3418e-4
	0.005	3	7.9024e-3	3.2947e-3
	0.008	3	1.2545e-2	5.2528e-3
	0.01	3	1.4895e-2	6.2385e-3
1/60	0.001	3	2.2209e-3	8.3323e-4
	0.005	6	7.7529e-3	3.1415e-3
	0.008	6	1.2468e-2	5.0301e-3
	0.01	6	1.4828e-2	5.9843e-3

TABLE II
PARALLEL EFFICIENCY

Mesh	Time-s	CPUs	Time-p	Speedup	Eff
60×60	1.4558	4	0.3719	3.91	0.98
		9	0.1542	9.44	1.05
		16	0.1054	13.81	0.86
120×120	14.0195	4	4.0800	3.43	0.86
		9	1.5325	9.14	1.02
		16	0.6822	20.56	1.29
180×180	52.7525	4	15.4773	3.41	0.85
		9	5.9044	8.93	0.99
		16	2.7726	19.03	1.19
240×240	203.8975	4	38.0000	5.37	1.31
		9	16.2264	12.57	1.40
		16	6.9288	29.43	1.84
		36	2.2928	88.93	2.47
		64	1.0166	200.57	3.13
300×300	504.8729	4	77.7609	6.49	1.62
		9	36.0100	14.02	1.56
		25	8.5481	59.06	2.36
		36	4.6647	108.23	3.01
360×360	898.8346	4	204.8047	4.39	1.10
		16	37.1268	24.21	1.51
		25	20.4300	44.00	1.76
		36	12.0107	74.84	2.08
		64	4.7277	190.12	3.00

VI. CONCLUSION

In this paper, some simple and efficient domain decomposition algorithms for 2-dimensional heat equation are studied. The values at the interface points are calculated by using large step lengths discrete approximation for spatial partial derivatives in two directions alternatively. Three algorithms are considered for sub-domain calculation. The stability and accuracy properties of the new algorithms are analyzed. When the values at the interior points are calculated by implicit scheme, if larger spatial discrete method is used, then the stability conditions can be reduced by $2Q^2$ times (Q stands for the smaller ratio by the larger and usual step increments in the two spatial directions). As to interior evaluations, if implicit scheme is adopted, then weak stability condition and much calculation are needed, and second order spatial and first order temporal convergence properties can be obtained; CN scheme is similar to the former, but has second order accuracy in time step; EI scheme has second order accuracy in both time and space increments at even steps, and needs less computational work, but stricter stability condition. Numerical experiments verify the theoretical conclusions and demonstrate the high efficiency and scalability of the algorithms.

The algorithms above are described on block decomposition of four sub-domains for example. In fact the domain can be divided into several parts in each direction. Generally, it can be divided in to I_1I_2 sub-domains by using $x = x_k(k = k_{s1}, s1 = 1, 2, \cdots, I_1 - 1)$ and $y = y_l(l = l_{s2}, s2 = 1, 2, \cdots, I_2 - 1)$ as interfacial lines. And corresponding parallel schemes can be designed similarly with large step length interface discretizations and similar conclusion can be obtained.

The methods here can be extended to three-dimensional problems. Also they can be adopted to study parallel schemes for diffusion problems with variant coefficients. The techniques in this paper can be combined with the idea of fractional steps to gain more looser stability conditions, see [11] for example with a brief deduction.

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