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The Wigner System Coupled with Hartree-type Nonlinearity

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***Abstract*—This paper is concerned with the generalized Wigner system, which models quantum mechanical (charged) particles-transport under the influence of a Hartree-type nonlinearity. In three dimensions, existence and uniqueness of the local mild solution are established on weighted-L^2 space. The main difficulty in establishing mild solution is to derive a priori estimates on the Hartree-type nonlinearity. The proof is based on semigroup theory and splitting the singular kernel $\frac{1}{|x|^\alpha}$ with $0 < \alpha \leq 1$. This result generalizes the previous result by Manzini [14], which dealt with the solution with $\alpha = 1$.**

***Index Terms*—Wigner system, Hartree-type nonlinearity, semigroup theory, singular kernel.**

I. Introduction

THE following Hartree (generalized Schrödinger) equations

$$i\hbar\psi_t(t,x) = -\frac{\hbar^2}{2}\Delta\psi(t,x) + V(t,x)\psi(t,x),\ x \in R^n, \quad (1)$$

$$V(t,x) = \frac{\lambda}{|x|^\alpha} *_x \rho(t,x),\ 0 < \alpha < n \quad (2)$$

have arisen in quantum mechanics and been studied by many researchers in the recent years, see [7], [12] and therein for more details. This model describes the time-evolution of a complex-valued wave function $\psi(t,x)$, under the influence of a Hartree-type nonlinearity (see [9], [17] for a broader introduction). Above, $\frac{\lambda}{|x|^\alpha}$ denotes a given real-valued interaction kernel, and $*$ denotes the convolution. Where $x \in R^n$ is the position, t is the time, $\hbar$ denotes the Planck constant and $\lambda \in R$. The macroscopic density $\rho = \rho(t,x)$ is now given by the zeroth order moment in the kinetic variable v, i.e., by the physical observables from both the wave function ψ and the Winger function w, namely,

$$\rho(t,x) = |\psi|^2 = \int_{R^n} w(t,x,v)dv. \quad (3)$$

The Wigner transform of $\psi(t,x)$ is

$$w(t,x,v) = \frac{1}{(2\pi)^n}\int_{R^n} \psi(t, x-\frac{\hbar}{2}y)\overline{\psi}(t, x+\frac{\hbar}{2}y)\exp(iv\cdot y)dy \quad (4)$$

where $\overline{\psi}$ denotes the complex conjugate of ψ. A direct calculation by applying the Wigner transform (4) to the generalized Schrödinger equations (1)-(2) shows that $w(t,x,v)$ satisfies the so-called Wigner equation, see e.g. [13], [19],

$$w_t + (v\cdot\nabla_x)w - \Theta_\hbar[V]w = 0. \quad (5)$$

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Where the Wigner function $w = w(t,x,v)$ is a probabilistic quasi-distribution function of particles at time $t \geq 0$, located at $x \in R^n$ with velocity $v \in R^n$. The operator $\Theta_\hbar[V]w$ in the equation (5) is a pseudo-differential operator, as in [3], [4], [10], formally defined by

$$\Theta_\hbar[V]w(t,x,v) = \frac{i}{(2\pi)^n\hbar}\int_{R^n}\int_{R^n}\delta[V](t,x,\eta)w(t,x,v')e^{i(v-v')\eta}dv'd\eta, \quad (6)$$

$$\delta[V](t,x,\eta) = V\left(t, x+\frac{\hbar\eta}{2}\right) - V\left(t, x-\frac{\hbar\eta}{2}\right).$$

It is clear that the equations (5)-(6) coupled with (2) contain the Wigner-Poisson equation (e.g., $n = 3, \alpha = 1$). Over the past years, there have been many mathematical studies of mild or classical solution for the Wigner-Poisson equation, which models the charge transport in a semiconductor device under the Poisson potential. For instance, it has been studied in the whole space $R_x^3 \times R_v^3$ (see [11] and the references therein), in a bounded spatial domain with periodic [5], or absorbing [1], or time-dependent inflow [14], [15], boundary conditions, and on a discrete lattice [8], [18].

The present paper is devoted to investigating the system (5)-(6) coupled (2) and establishing certain mathematical results on the existence and uniqueness of the mild solution, with the following initial boundary conditions

$$w(t,0,x_2,x_3,v) = w(t,v,l,x_2,x_3), \quad (7)$$

$$w(t,x_1,0,x_3,v) = w(t,x_1,l,x_3,v), \quad (8)$$

$$w(t,x_1,x_2,0,v) = w(t,x_1,x_2,l,v), \quad (9)$$

$$w(t=0,x,v) = w_0(x,v),\ l > 1, \quad (10)$$

which is very difficult: first, the function $V(t,x)$ does not satisfy $\Delta_x V = \rho$ as in Wigner-Poisson equation (see [15] and so on); and the second is to derive priori estimates on the nonlinear operators $\Theta_\hbar[V]w$. Therefore, the mathematical analysis must be done on the some new methods such as splitting the singular kernel $\frac{1}{|x|^\alpha}$ with $0 < \alpha \leq 1$.

On the other hand, the natural choice of the functional setting for the study of the Wigner-Poisson or Wigner-Poisson-Fokker-Planck problem is the Hilbert space $L^2(R_x^n \times R_v^n)$, see [6], [13]. However, it can be immediately observed that the density $\rho(t,x)$, given by (3), is not well-defined for any $w(t,x,v)$ belonging to this space. In other words, the nonlinear term $\Theta_\hbar[V]w$ is not defined pointwise in t on the state space of the Wigner function. Therefore, in Section II we introduce a Hilbert space $X = L^2([0,l]^3 \times R_v^3, (1+|v|^2)^2dxdv)$ (see also [3], [5], [15]), such that, the existence of the density $\rho(t,x)$ is granted for any $w \in X$.

With the above notations, the main result of this paper can be described as the following theorem:

Theorem 1 Let $0 < \alpha \leq 1$, for every $w_0 \in X$, the equations (5)-(6) coupled with the equation (2), with initial

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boundary conditions (7)-(10), has a unique mild solution $w \in C([0, t_{\max}), X)$.

Remark 1 It is straightforward to extend our results to the high dimensional case ($n > 3$) with $0 < \alpha \leq n-2$.

Our paper is structured as follows: In section II we introduce a weighted space for the Wigner function w that allows to define the nonlinear term $\Theta_\hbar[V]w$. In section III, we obtain a local-in-time mild solution on the weighted L^2 space via the the Lumer-Phillips theorem [16] in three dimensions.

II. The functional setting and preliminaries

In this section we shall discuss the functional analytic preliminaries for studying the nonlinear Wigner equations (5)-(6) coupled (2). Or more accurately, we shall introduce an appropriate state space for the Wigner function w which allows to control the nonlinear term $\Theta_\hbar[V]w$, which will be considered as a perturbation of the generator A defined in (19). This is one of the key ingredients for proving the Theorem 1.

We would show that the pseudo-differential operator $\Theta_\hbar[V]w$ is (local) bounded in the weighted L^2 space, in symbols:

$$X := L^2(I_x \times R_v^3, (1+|v|^2)^2 dx dv),\ I_x = [0, l]_x^3, \quad (11)$$

endowed with the following scalar product

$$\langle f, g\rangle_X := \int_I \int_{R_v^3} f(x,v) \cdot \overline{g(x,v)} (1+v^2)^2 dv dx, \quad (12)$$

for $f, g \in X$. In our calculations, we shall use the following equivalent norm:

$$||f||_X^2 := ||f||_{L^2}^2 + \sum_{i=1}^{3} ||v_i^2 f||_{L^2}^2, \quad (13)$$

The following proposition motivates our choice of the space X for the analysis.

Lemma 1 Let $w \in X$ and $\rho(t,x)$ defined in (3), for all $x \in I$, then ρ belongs to $L^2(I)$ and satisfies

$$||\rho||_{L^2(I)} \leq C||w||_X, \quad (14)$$

ρ also belongs to $L^1(I)$ and satisfies

$$||\rho||_{L^1(I)} \leq C||w||_X. \quad (15)$$

Moreover, for every $p \in [1,2]$, ρ belongs to $L^p(I)$ and satisfies

$$||\rho||_{L^p(I)} \leq C||w||_X. \quad (16)$$

Proof The first assertion follows directly by using Cauchy-Schwartz inequality in v-integral, see also [14], [15]. On the other hand, by Hölder inequality, we have

$$\|\rho\|_{L^1(I)} \leq \int_I \left| \int_{R_v^3} w(t,x,v) dv \right| dx \leq$$

$$\left[\int_I 1^2 dx\right]^{\frac{1}{2}} \left[\int_0^1 \left| \int_{R_v^3} w(t,x,v) dv \right|^2 dx\right]^{\frac{1}{2}} \leq C\|w\|_X.$$

Using the interpolation inequality, we get

$$\|\rho\|_{L^p(I)} \quad \leq \quad \|\rho\|_{L^2(I)}^{\theta} \|\rho\|_{L^1(I)}^{1-\theta} \leq C\|w\|_X.$$

Remark 2 The choice of the $|v|^2$ weight was already seen to be convenient to control the L^2-norm of the density on the whole space R_x^3 [3] and therein, and a bounded or periodic spatial domain [14], [15]. However, the choice of the space X as the state space for our analysis is not optimal, see [15], in the sense that we could obtain an estimate analogous to the (14) even under decreased regularity assumption on the function $\mathcal{F}_{v\to\eta} w$. Precisely, we could assume $w \in L^2(R_x^n \times R_v^n, (1+|v|^2)^k) dx dv)$ with $k > 2n$.

Next, we consider the Lipschitz properties of the pseudo-differential operator $\Theta_\hbar[V]w$ defined by (6). But by the definition of it, the w have to be 0-extended to R_x^3. We will show indeed that this operator is well defined from the space X to itself. Moreover, we can state the following results:

Lemma 2 Let $0 < \alpha \leq 1$, for all $w \in X$, the operator $\Theta_\hbar [V] w$ maps X into itself and there exists $C > 0$ such that

$$||\Theta_\hbar [V] w||_X \leq C||w||_X^2. \quad (17)$$

Proof Indeed, the operator $\Theta_\hbar[V]w$ can be rewritten in a more compact form as,

$$\mathcal{F}_{v\to\eta}(\Theta_\hbar [V] w)(x,\eta) = \frac{i}{\hbar} \delta V(x,\eta) \mathcal{F}_{\eta\to v}^{-1} w(x,\eta), \quad (18)$$

where the symbol $*_v$ is the partial convolution with respect to the variable v, $\mathcal{F}_{v\to\eta}$ is the Fourier transformation with respect to the variable v and $\mathcal{F}_{\eta\to v}^{-1}$ its inverse:

$$\mathcal{F}_{v\to\eta}[f(x,\cdot)](\eta) = \int_{R^n} f(x,v) e^{iv\cdot\eta} dv,$$

$$\mathcal{F}_{\eta\to v}^{-1}[g(x,\cdot)](v) = \frac{1}{(2\pi)^n} \int_{R^n} g(x,v) e^{-iv\cdot\eta} d\eta$$

for suitable functions f and g. Then one has

$$\begin{aligned} &\|\Theta_\hbar [V] w\|_{L^2} \leq C\|\delta V(x,\eta) \mathcal{F}_{\eta\to v}^{-1} w\|_{L^2} \leq \\ &\quad C\|V\|_{L^\infty} \|\mathcal{F}_{\eta\to v}^{-1} w\|_{L^2} \leq C\|V\|_{L^\infty} \|w\|_{L^2}. \end{aligned}$$

Let $k(\cdot) = \frac{1}{|\cdot|^\alpha}$, $k_1 = k(\cdot)|_{|\cdot|\leq 1}$ and $k_2 = k(\cdot)|_{|\cdot|>1}$, so, $k(\cdot) = k_1 + k_2$ with $k_1 \in L^p(R^3)$ for all $p \in [1, \frac{3}{\alpha})$ and $k_2 \in L^q(R^3)$ for all $q \in (\frac{3}{\alpha}, +\infty]$. On the other hand, since $V = \frac{1}{|\cdot|^\alpha} * \rho$, using Hölder's inequality we have

$$\begin{aligned} &\|k_1 * \rho\|_{L^\infty(B)} \leq C\|k_1\|_{L^2(B)} \|\rho\|_{L^2(B)} \leq \\ &\qquad C\|\rho\|_{L^2(B)} \leq C\|\rho\|_{L^2(I)}, \end{aligned}$$

where B is the three dimensional unit ball. Likewise, outside B we get

$$\begin{aligned} &\|k_2 * \rho\|_{L^\infty(R^3\backslash B)} \leq C\|k_2\|_{L^\infty(R^3\backslash B)} \|\rho\|_{L^1(I\backslash B)} \leq \\ &\qquad C\|\rho\|_{L^1(I\backslash B)} \leq C\|\rho\|_{L^1(I)}. \end{aligned}$$

By Lemma 1, we can get

$$\|\Theta_\hbar [V] w\|_{L^2} \leq C\|V\|_{L^\infty} \|w\|_{L^2} \leq C\|w\|_X^2.$$

On the other hand, by [3],

$$v_i^2 \Theta_\hbar [V] w = \frac{1}{4} \Theta_\hbar \left[\partial_i^2 V\right] w + \Theta_\hbar [V] v_i^2 w + \Omega_\hbar [\partial_i V] w$$

with the pseudo-differential operator

$$\Omega_\hbar[\varphi]w = \frac{i}{(2\pi)^n \hbar} \int_{R^n} \int_{R^n} \kappa[\varphi] w(t,x,v') e^{i(v-v')\eta} dv' d\eta,$$

$$\kappa[\varphi] = \varphi\left(t, x + \frac{\hbar\eta}{2}\right) + \varphi\left(t, x - \frac{\hbar\eta}{2}\right).$$

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In the sequel we use the abreviation $\partial_i = \partial_{x_i}$, and get

$$\|v_i^2 \Theta_\hbar [V] w\|_{L^2} \leq \frac{1}{4}\|\Theta_\hbar [\partial_i^2 V] w\|_{L^2} + \|\Omega_\hbar [\partial_i V] w\|_{L^2} + \|\Theta_\hbar [V] v_i^2 w\|_{L^2}.$$

The first two terms can be estimated as follows:

$$\begin{aligned} \|\Theta_\hbar [\partial_i^2 V] w\|_{L^2} \leq C\|\delta(\partial_i^2 V)\mathcal{F}_{v\to\eta} w\|_{L^2} \leq \\ C\|\partial_i^2 V\|_{L^2(R^3)}\|\mathcal{F}_{v\to\eta} w\|_{L^2(I;L^\infty(R^3_\eta))} \leq \\ C\||x|^{-2-\alpha} *_x \rho\|_{L^2(R^3)}\|(1+|v|^2)w\|_{L^2} \leq \\ C(\|\frac{1}{|x|^{2+\alpha}} *_x \rho\|_{L^2(B)} + \|\frac{1}{|x|^{2+\alpha}} *_x \rho\|_{L^2(R^3\setminus B)})\|w\|_X \leq \\ C(\|\rho\|_{L^2(I)} + \|\rho\|_{L^1(I)})\|w\|_X \leq C\|w\|_X^2 \end{aligned}$$

by applying Hölder's inequality, $\frac{3}{2+\alpha} < 2$ with $0 < \alpha \leq 1$ and the Sobolev imbedding $\mathcal{F}_{v\to\eta} w \in W^{2,2}(R^3_\eta) \hookrightarrow L^\infty(R^3_\eta)$.

$$\begin{aligned} \|\Omega_\hbar [\partial_i V] w\|_{L^2} \leq C\|\delta(\partial_i V)\partial_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2} \leq \\ C\|\partial_i V\|_{L^4(R^3)}\|\partial_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2(I;L^4(R^3_\eta))} \leq \\ C\||x|^{-1-\alpha} *_x \rho\|_{L^4(R^3)}\|(1+|v_i|^2)w\|_{L^2} \leq \\ C(\|\frac{1}{|x|^{1+\alpha}} *_x \rho\|_{L^4(B)} + \|\frac{1}{|x|^{1+\alpha}} *_x \rho\|_{L^4(R^3\setminus B)})\|w\|_X \leq \\ C\|\frac{1}{|x|^{1+\alpha}}\|_{L^{\frac{7}{5}}(B)}\|\rho\|_{L^{\frac{28}{15}}(B)}\|w\|_X + \\ C\|\frac{1}{|x|^{1+\alpha}}\|_{L^4(R^3\setminus B)}\|\rho\|_{L^1(I\setminus B)}\|w\|_X \leq \\ C(\|\rho\|_{L^{\frac{28}{15}}(B)} + \|\rho\|_{L^1(I\setminus B)})\|w\|_X \leq C\|w\|_X^2 \end{aligned}$$

by the Sobolev imbedding $\partial_{\eta_i}\mathcal{F}_{v\to\eta} w \in W^{1,2}(R^3_\eta) \hookrightarrow L^4(R^3_\eta)$, $\frac{3}{1+\alpha} > \frac{7}{5}$ and $\frac{3}{1+\alpha} < 4$ with $0 < \alpha \leq 1$, and Lemma 1. We also get

$$\begin{aligned} \|\Theta_\hbar [V] v_i^2 w\|_{L^2} \leq C\|V\partial^2_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2} \leq \\ C\|V\|_{L^\infty(R^3)}\|\partial^2_{\eta_i}\mathcal{F}_{v\to\eta} w\|_{L^2} \leq \\ C(\|\frac{1}{|x|^\alpha} *_x \rho\|_{L^\infty(B)} + \|\frac{1}{|x|^\alpha} *_x \rho\|_{L^\infty(R^3\setminus B)})\|w\|_X \leq \\ C\|\frac{1}{|x|^\alpha}\|_{L^2(B)}\|\rho\|_{L^2(B)}\|w\|_X + \\ C\|\frac{1}{|x|^\alpha}\|_{L^\infty(R^3\setminus B)}\|\rho\|_{L^1(I\setminus B)}\|w\|_X \leq \\ C(\|\rho\|_{L^2(B)} + \|\rho\|_{L^1(I\setminus B)})\|w\|_X \leq C\|w\|_X^2 \end{aligned}$$

by applying Hölder's inequality and Lemma 1. This concludes the proof of result.

Lemma 3 Let $0 < \alpha \leq 1$, for all $w \in X$, the operator $\Theta_\hbar [V] w$ is of class C^∞ in X, and satisfies

$$\begin{aligned} \|\Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_2] w_2\|_X \leq \\ C(\|w_1\|_X + \|w_2\|_X)\|w_1 - w_2\|_X. \end{aligned}$$

Proof For all $w_i \in X, i = 1,2$, setting $\Pi = \Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_2] w_2$, $\Pi_1 = \Theta_\hbar [V_1] w_1 - \Theta_\hbar [V_1] w_2$ and $\Pi_2 = \Theta_\hbar [V_1] w_2 - \Theta_\hbar [V_2] w_2$, we have

$$\|\Pi\|_X \leq \|\Pi_1\|_X + \|\Pi_2\|_X$$

with

$$\begin{aligned} \|\Pi_1\|_X = \|\Theta_\hbar [V_1] (w_1 - w_2)\|_X \leq \\ \|\Theta_\hbar [V_1] (w_1 - w_2)\|_{L^2} + \sum_{i=1}^3 \|v_i^2 \Theta_\hbar [V_1] (w_1 - w_2)\|_{L^2} \leq \\ C\|\delta V[w_1]\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} + \\ C\sum_{i=1}^3 \|\delta\partial_i^2 V[w_1])\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} + \\ C\sum_{i=1}^3 \|\delta(\partial_i V[w_1])\partial_{\eta_i}\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} + \\ C\sum_{i=1}^3 \|\delta V[w_1]\partial^2_{\eta_i}\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} \leq \\ C\|V[w_1]\|_{L^\infty}\|w_1 - w_2\|_{L^2} + \\ C\sum_{i=1}^3 \|\partial_i^2 V[w_1]\|_{L^2}\|\mathcal{F}[w_1 - w_2]\|_{L^2(I;L^\infty(R^3_\eta))} + \\ C\sum_{i=1}^3 \|\partial_i V[w_1]\|_{L^4}\|\partial_{\eta_i}\mathcal{F}[w_1 - w_2]\|_{L^2(I;L^4(R^3_\eta))} + \\ C\sum_{i=1}^3 \|V[w_1]\|_{L^\infty}\|\partial^2_{\eta_i}\mathcal{F}[w_1 - w_2]\|_{L^2} \leq \\ C\|w_1\|_{L^2}\|w_1 - w_2\|_X; \\ \|\Pi_2\|_X = \|\Theta_\hbar [V_1 - V_2] w_2\|_X \leq \\ \|\Theta_\hbar [V_1 - V_2] w_2\|_{L^2} + \\ \sum_{i=1}^3 \|v_i^2 \Theta_\hbar [V_1 - V_2] w_2\|_{L^2} \leq \\ C\|\delta V[w_1 - w_2]\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\ C\sum_{i=1}^3 \|\delta\partial_i^2 V[w_1 - w_2])\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\ C\sum_{i=1}^3 \|\delta(\partial_i V[w_1 - w_2])\partial_{\eta_i}\mathcal{F}_{v\to\eta} w_2\|_{L^2} + \\ C\sum_{i=1}^3 \|\delta V[w_1 - w_2]\partial^2_{\eta_i}\mathcal{F}_{v\to\eta} w_2\|_{L^2} \leq \\ C\|V[w_1 - w_2]\|_{L^\infty}\|w_2\|_{L^2} + \\ C\sum_{i=1}^3 \|\partial_i^2 V[w_1 - w_2]\|_{L^2}\|\mathcal{F} w_2\|_{L^2(I;L^\infty(R^3_\eta))} + \\ C\sum_{i=1}^3 \|\partial_i V[w_1 - w_2]\|_{L^4}\|\partial_{\eta_i}\mathcal{F} w_2\|_{L^2(I;L^4(R^3_\eta))} + \\ C\sum_{i=1}^3 \|V[w_1 - w_2]\|_{L^\infty}\|\partial^2_{\eta_i}\mathcal{F}_{v\to\eta} w_2\|_{L^2} \leq \\ C\|w_2\|_X\|w_1 - w_2\|_{L^2}, \end{aligned}$$

and the assertion is proved.

III. Proof of Theorem 1

In this section, we will prove the main result of the paper. Let we may rewrite the Wigner equation as

$$w_t = Aw + \Theta_\hbar[V]w, \ t > 0, \tag{19}$$
$$w(t = 0) = w_0, \tag{20}$$

where linear operator $A : D(A) \to X$ by

$$Af = -v \cdot \nabla_x w$$

and its domain

$$\begin{aligned} D(A) = \{w \in X | v \cdot \nabla_x w \in X, \\ w(0, x_2, x_3) = w(l, x_2, x_3), \\ w(x_1, 0, x_3) = w(x_1, l, x_3), \\ w(x_1, x_2, 0) = w(x_1, x_2, l), l > 1\}. \end{aligned}$$

Proof of Theorem 1 Indeed, the A generates a C_0 group of isometries $\{S(t), t \in R\}$ on X, given by $S(t)w(x, v) = w(x - vt, v)$, see also [2]. Next, we consider $\Theta_\hbar[V]w$ as a bounded perturbation of the generator A. Since $\Theta_\hbar[V]w$ is locally Lipschitz continuous (see Lemmas 2 and 3 for detail), Theorem 6.1.4 of [16] shows that the problem (19)-(20) coupled with (2) has a unique mild solution for every $w_0 \in X$ on some time interval $[0, t_{\max})$, where $t_{\max}$ denotes the maximal existence time of the mild solution. Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \to t_{\max}} \|w\|_X = \infty.$$

This concludes the proof of result.

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