Proceedings of the International MultiConference of Engineers and Computer Scientists 2015 Vol II,
IMECS 2015, March 18 - 20, 2015, Hong Kong

Some Properties of Bipolar Max-min Fuzzy Relational Equations

Chia-Cheng Liu, Yung-Yih Lur, Yan-Kuen Wu*

***Abstract*—In the literature, the solution set of a system of bipolar fuzzy relational equations with max-min composition can be characterized by a finite set of maximal and minimal solution pairs. However, some researches presented that determining the consistency of a system of bipolar fuzzy relational equations is NP-complete. Namely, the solution procedure for solving a system of bipolar fuzzy relational equations with max-min composition contained the high computation complexity. This study proposes some properties to reduce the difficulty of solving the bipolar fuzzy relational equations with max-min composition. Numerical examples illustrate that the proposed properties can be easily used to detect the case of an empty solution set than the method provided by the previous work.**

***Index Terms*—bipolar fuzzy relational equalities, max-min composition, NP-complete.**

I. Introduction

IN the literature, a system of fuzzy relational equations usually formulates in a matrix form as follows:

$$x \circ A = b,$$

where $x = (x_i)_{1\times m}$, $A = [a_{ij}]_{m\times n}$ and $b = (b_j)_{1\times n}$ are all defined over [0, 1]. The operation "$\circ$" represents a well-defined algebraic composition for matrix multiplication.

In the generalized theory of uncertainty, finding solution of fuzzy relational equations can be categorized to the concept of granular precisiation by Zadeh [20], which has played an important role in fuzzy modeling. Many different equations exist is based on a specific composition of fuzzy relations. The first study of fuzzy relational equations was conducted by Sanchez [16], in which max-min composition was considered. Since then, fuzzy relational equations based on various compositions have been investigated. Fuzzy relational equations with max-min composition and max-product compositions are commonly seen in the literature. Both compositions are special cases of the max-triangular-norm (max-t-norm). Di Nola et al. [4] showed that the solution set of fuzzy relational equations with continuous max-t-norm composition can be completed determined by the maximum solution and a finite number of minimal solutions. Finding all minimal solutions of fuzzy relational equations was found to be closely associated with the covering problem, which is NP-hard [2], [3], [15]. Lin et al. [14] presented that all systems of max-continuous u-norm fuzzy relational equations, e.g., max-product, max-continuous Archimedean t-norm and max-arithmetic mean are essentially equivalent, because they all are equivalent to the covering problem.

The typical framework of linear optimization subject to a system of fuzzy relational equations with different algebraic operations has been proposed in the literature. By far the most explored setting is that of finding a minimizer of a linear objective function and using the max-min composition [1], [7]. It almost becomes a standard approach to translate this type of problem into a corresponding 0-1 integer programming problem which is then solved by means of the branch and bound method [5], [18]. Some research efforts have been devoted to a more general operator of linear optimization with the max-min composition is replaced by a max-t-norm composition [9], [13], [17], a max-average composition [10], [19], or a max-star composition, et al. [8], [11].

Recently, Freson et al. [6] considered a generation of the linear optimization problem subject to a system of bipolar fuzzy relational equations with max-min composition. They wanted to pursue the idea of taking into account antagonistic effects for this new optimization problem. For instance, consider a supplier who wants to optimize its public awareness and attributes a degree of appreciation to their products. Such a degree of appreciation can be denoted by a real number x_i in the unit interval [0, 1] whose complement $\bar{x}_i = 1-x_i$ in [0, 1] stands for the degree of disappreciation. Generally, when the positive effect x_i increases, the negative effect $\bar{x}_i = 1-x_i$ will fall. It is called the bipolar character. It is clear that the bipolar fuzzy relational equations contain the decision vector and its negation simultaneously. Motivated by Freson et al. [6], Li and Liu [12] considered the linear optimization problem with bipolar max-Łukasiewicz equation constraints and transformed this problem into a 0-1 integer linear programming problem.

A system of bipolar fuzzy relational equations with max-min composition proposed by Freson et al. [6] can be formulated in the matrix form as follows:

$$x \circ A^+ \vee \bar{x} \circ A^- = b$$

where $x = (x_i)_{1\times m}$, $\bar{x} = (\bar{x}_i)_{1\times m}$, $A^+ = [a^+_{ij}]_{m\times n}$, $A^- = [a^-_{ij}]_{m\times n}$ and $b = (b_j)_{1\times n}$ are all defined over [0, 1]. The notation "$\vee$" denotes max operation and the operation "$\circ$" represents the max-min composition. $\bar{x}_i = 1 - x_i$ denotes the bipolar character.

Solving the bipolar fuzzy relational equations with max-min composition is to find a set of solution vectors $x = (x_i)_{i\in\mathcal{I}}$ such that

$$\max_{i\in\mathcal{I}} \max\{\min(x_i, a^+_{ij}), \min(\bar{x}_i, a^-_{ij})\} = b_j, j \in \mathcal{J}, \quad (1)$$

where index sets $\mathcal{I} = \{1, 2, \cdots, m\}$ and $\mathcal{J} = \{1, 2, \cdots, n\}$, respectively.

For investigating the solution set of a system of bipolar fuzzy relational equations to (1), Freson et al. [6] first analyzed each single equation by a piecewise linear function. Based on the analyzed results which obtained from all of equations, then they structured the solution set of (1) by

Chia-Cheng Liu and Yung-Yih Lur are with the Department of Industrial Management, Vanung University, Taoyuan, Taiwan, R.O.C., Email: liuht@vnu.edu.tw and yylur@vnu.edu.tw.

Yan-Kuen Wu is with the Department of Business Administration, Vanung University, Taoyuan, Taiwan, R.O.C., Email: ykw@vnu.edu.tw.

ISBN: 978-988-19253-9-8
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

taking proper intersections and unions. They also figured out that the solution set of a system of bipolar fuzzy relational equations can be determined by a finite set of maximal and minimal solution pairs. However, Li and Liu [12] presented that determining the consistency of a system of bipolar fuzzy relational equations is NP-complete. That is to say, the proposed solution procedure by Freson et al. [6] for solving (1) contained the high computation complexity. To improve the difficulty of solving this problem, this study proposes some properties for the bipolar fuzzy relational equations with max-min composition. Numerical examples illustrate that the proposed properties can be easily used to detect the case of an empty solution set of (1) than the method provided by previous work.

II. Some properties

In order to investigate the property of the solution set, denoted by $X(A^+, A^-, b)$, of bipolar fuzzy relational equations with max-min composition in (1), some definitions describe as follows:

Definition 1. Let $x^1 = (x_i^1)_{1\times m}$ and $x^2 = (x_i^2)_{1\times m}$ be two vectors. For any vector x^1 and x^2, $x^1 \leq x^2$ if and only if $x_i^1 \leq x_i^2$ for all $i \in \mathcal{I}$.

Definition 2. For a feasible solution $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b) \neq \emptyset$ in (1), x_i is called a *binding variable* for the jth bipolar fuzzy relational equation if $\min(x_i, a_{ij}^+) = b_j$ or $\min(\bar{x}_i, a_{ij}^-) = b_j$ holds for some $j \in \mathcal{J}$. The set $J(x_i) := \{j \in \mathcal{J} | \min(x_i, a_{ij}^+) = b_j, \forall j \in \mathcal{J}\}$ and $J(\bar{x}_i) := \{j \in \mathcal{J} | \min(\bar{x}_i, a_{ij}^-) = b_j, \forall j \in \mathcal{J}\}$ denote the binding set of binding variable x_i.

Note that a feasible solution for bipolar fuzzy relational equations with max-min composition in (1) is to find a set of vector $x = (x_i)_{i\in\mathcal{I}}$ that satisfies all equations. By Definition 2, to find a solution for (1) can be considered the selection of binding variables from the binding set to satisfy all equations. Furthermore, if x_i is binding in the jth equation, then there exists $J(x_i) \neq \emptyset$ or $J(\bar{x}_i) \neq \emptyset$ and the following equations hold:

$$\min(x_i, a_{ij}^+) = b_j \text{ or } \min(\bar{x}_i, a_{ij}^-)\} = b_j.$$

To quickly find the binding variable of bipolar fuzzy relational equations for (1), all equations in (1) can be separated into two sub-equations according to variables x_i and $\bar{x}_i$ as follows:

$$\max_{i\in\mathcal{I}}\{\min(x_i, a_{ij}^+)\} = b_j, j \in \mathcal{J} \tag{2}$$

and

$$\max_{i\in\mathcal{I}} \min\{(\bar{x}_i, a_{ij}^-)\} = b_j, j \in \mathcal{J}. \tag{3}$$

Note that these two sub-equations of (2) and (3) are general fuzzy relational equations with max-min composition. Hence, their corresponding maximum solutions can be computed explicitly by the following operation:

$$\begin{aligned} \tilde{x} &= (\tilde{x}_i)_{i\in\mathcal{I}} = [\min_{j\in\mathcal{J}}(a_{ij}^+ \diamond b_j)]_{i\in\mathcal{I}}, \\ \text{and } \tilde{\bar{x}} &= (\tilde{\bar{x}}_i)_{i\in\mathcal{I}} = [\min_{j\in\mathcal{J}}(a_{ij}^- \diamond b_j)]_{i\in\mathcal{I}}, \end{aligned} \tag{4}$$

where

$$a_{ij}^+ \diamond b_j = \begin{cases} 1 & \text{if } a_{ij}^+ \leq b_j, \\ b_j & \text{if } a_{ij}^+ > b_j, \end{cases}$$

$$\text{and } a_{ij}^- \diamond b_j = \begin{cases} 1 & \text{if } a_{ij}^- \leq b_j, \\ b_j & \text{if } a_{ij}^- > b_j. \end{cases}$$

Lemma 1. If in the jth equation we have $a_{ij}^+ < b_j$ and $a_{ij}^- < b_j$ for each $i \in \mathcal{I}$ in (1), then the solution set $X(A^+, A^-, b)$ is empty.

Proof. Due to $a_{ij}^+ < b_j$ and $a_{ij}^- < b_j$ for each $i \in \mathcal{I}$, it implies

$$\min(x_i, a_{ij}^+) < b_j \text{ and } \min(\bar{x}_i, a_{ij}^-) < b_j, \forall i \in \mathcal{I}.$$

This result leads to

$$\max_{i\in\mathcal{I}}\{\min(x_i, a_{ij}^+), \min(\bar{x}_i, a_{ij}^-)\} < b_j$$

and, hence, no variable x_i can satisfy the jth equation of (1).

Lemma 2. Let $\tilde{x}_i$ and $\tilde{\bar{x}}_i$ be the corresponding maximum solution of (2) and (3) that yielded by (4). If $\tilde{x}_i + \tilde{\bar{x}}_i < 1$ exists for some $i \in \mathcal{I}$, then the solution set $X(A^+, A^-, b)$ of (1) is empty.

Proof. Suppose, on the contrary, that the solution set $X(A^+, A^-, b)$ is nonempty. Then there exists a feasible solution $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b)$ and $\min(x_i, a_{ij}^+) = b_j$ or $\min(\bar{x}_i, a_{ij}^-) = b_j$ for some $i \in \mathcal{I}$. Now assuming that $x_i = \tilde{x}_i$, we then have $\bar{x}_i = 1 - x_i = 1 - \tilde{x}_i$ by the bipolar character. Since $\tilde{x}_i + \tilde{\bar{x}}_i < 1$, it implies $\tilde{\bar{x}}_i < 1 - \tilde{x}_i = \bar{x}_i$. This result implies that $\tilde{\bar{x}}_i$ is not the corresponding maximum solution of (3), a contradiction.

By Lemma 2, one can obtain the necessary condition for the existence of a solution such that the solution set $X(A^+, A^-, b) \neq \emptyset$ in (1) is $\tilde{x}_i + \tilde{\bar{x}}_i \geq 1$ for all $i \in \mathcal{I}$.

Example 1. The following bipolar fuzzy relational equations with max-min composition proposed by Freson et al. [6]. We use this example to detect the case of an empty solution set by verifying Lemma 2.

$$\begin{cases} \max\{\min(x_1, 0.90), \min(x_2, 0.70), \min(\bar{x}_1, 0.60), \\ \qquad \min(\bar{x}_2, 0.80)\} = 0.70, \\ \max\{\min(x_1, 0.80), \min(x_2, 0.60), \min(\bar{x}_1, 0.30), \\ \qquad \min(\bar{x}_2, 0.75)\} = 0.25. \end{cases}$$

For this example, computing the corresponding maximum solutions by (4) can obtain

$$\tilde{x}_1 = \min\{0.7, 0.25\} = 0.25, \ \tilde{x}_2 = \min\{1, 0.25\} = 0.25,$$
$$\tilde{\bar{x}}_1 = \min\{1, 0.25\} = 0.25, \text{ and } \tilde{\bar{x}}_2 = \min\{0.7, 0.25\} = 0.25.$$

Clearly, Lemma 2 is satisfied by $\tilde{x}_1 + \tilde{\bar{x}}_1 < 1$ or $\tilde{x}_2 + \tilde{\bar{x}}_2 < 1$ and therefore the solution set is empty.

Definition 3. Let $\tilde{x}_i$ and $\tilde{\bar{x}}_i$ be the corresponding maximum solution of (2) and (3) that yielded by (4). Two index sets define as follows:

$$I_j := \{i \in \mathcal{I} | \min(\tilde{x}_i, a_{ij}^+) = b_j, i \in \mathcal{I}\}$$

and

$$\bar{I}_j := \{i \in \mathcal{I} | \min(\tilde{\bar{x}}_i, a_{ij}^-) = b_j, i \in \mathcal{I}\}, \forall j \in \mathcal{J}.$$

For Definition 3, index sets I_j and $\bar{I}_j$ denote that the possible variables of x may be selected as a binding variable in the jth equation.

Lemma 3. If index sets with $I_j = \bar{I}_j = \emptyset$ for some $j \in \mathcal{J}$ exists, then the solution set $X(A^+, A^-, b)$ of (1) is empty.

Proof. For any solution $x = (x_i)_{i\in\mathcal{I}} \in X(A^+, A^-, b)$, all equations must be satisfied. Index sets with $I_j = \bar{I}_j = \emptyset$ show that no any possible variables of x can be selected as a binding variable in the jth equation. Hence the solution set $X(A^+, A^-, b)$ of (1) is empty.

Example 2. The following bipolar fuzzy relational equations with max-min composition proposed by Freson et al. [6]. We use this example to detect the case of an empty solution set by verifying Lemma 3.

$$\begin{cases} \max\{\min(x_1, 0.85), \min(x_2, 0.23), \min(\bar{x}_1, 0.60), \\ \qquad \min(\bar{x}_2, 0.80)\} = 0.70, \\ \max\{\min(x_1, 0.70), \min(x_2, 0.10), \min(\bar{x}_1, 0.30), \\ \qquad \min(\bar{x}_2, 0.80)\} = 0.35. \end{cases}$$

For this example, computing the corresponding maximum solutions by (4) can obtain

$$\tilde{x}_1 = \min\{0.70, 0.35\} = 0.35,\ \tilde{x}_2 = \min\{1, 1\} = 1,$$
$$\tilde{\bar{x}}_1 = \min\{1, 1\} = 1, \text{ and } \tilde{\bar{x}}_2 = \min\{0.70, 0.35\} = 0.35.$$

Computing index sets I_j and $\bar{I}_j$ for $j \in \{1, 2\}$ by Definition 3, they are

$$I_1 = \emptyset, \bar{I}_1 = \emptyset, I_2 = \{1\}, \text{ and } \bar{I}_2 = \{2\}.$$

Since $I_1 = \bar{I}_1 = \emptyset$, it leads that the solution set of Example 2 is empty by Lemma 3.

Note that the method proposed by Freson et al. [6] needs to generate the set of possible feasible solution pairs for solving Example 2 as follows:

$$\{(0.35, 1), (0.30, 1), (0.35, 0.70), (0, 1), (0.30, 0.70), \\ (0.35, 0.65), (0, 0.70), (0.30, 0.65), (0, 0.65)\}.$$

According to their method, all of the nine pairs of this set are used to search possible feasible solutions for Example 2. Their method obtained that none of the pairs of this set satisfies both equations and hence the solution set is empty. For detecting the possibility of the bipolar fuzzy relational equations with max-min composition, Lemma 3 is more simple than the method proposed by Freson et al. [6].

III. Conclusion

In this study, we propose some properties to detect the case of an empty solution set of bipolar fuzzy relational equations with max-min composition. Numerical examples illustrate that the proposed properties can be easily used to detect the case of an empty solution set than the method provided by previous work.

Acknowledgment

This work is supported under grants no. MOST 103-2410-H-238-004 and MOST 103-2115-M-238-001, Ministry of Science and Technology, Taiwan, R.O.C.

References

[1] C.-W. Chang and B.-S. Shieh, “Linear optimization problem constrained by fuzzy max-min relation equations,” *Information Sciences*, vol. 234, pp. 71-79, 2013.

[2] L. Chen and P.-P. Wang, “Fuzzy relation equations (i): the general and specialized solving algorithms,” *Soft Computing*, vol. 6, no. 6, pp. 428-435, 2002.

[3] L. Chen and P.-P. Wang, “Fuzzy relation equations (ii): The branch-point-solutions and the categorized minimal solutions,” *Soft Computing*, vol. 11, no. 1, pp. 33-40, 2007.

[4] A. Di Nola, S. Sessa, W. Pedrycz and E. Sanchez, *Fuzzy Relational Equations and Their Applications in Knowledge Engineering,* Dordrecht: Kluwer Academic Press, 1989.

[5] S.-C. Fang and G. Li, “Solving fuzzy relation equations with a linear objective function,” *Fuzzy Sets and Systems*, vol. 103, pp. 107-113, 1999.

[6] S. Freson, B. De Baets and H. De Meyer, “Linear optimization with bipolar maxVmin constraints,” *Information Sciences*, vol. 234, pp. 3-15, 2013.

[7] A. Ghodousian and E. Khorram, “Fuzzy linear optimization in the presence of the fuzzy relation inequality constraints with max-min composition,” *Information Sciences*, vol. 178, pp. 501-519, 2008.

[8] A. Ghodousian and E. Khorram, “Solving a linear programming problem with the convex combination of the max-min and the max-average fuzzy relation equations,” *Applied Mathematics and Computation*, vol. 180, pp. 411-418, 2006.

[9] S.-M. Guu and Y.-K. Wu, “Minimizing a linear objective function under a max-t-norm fuzzy relational equation constraint,” *Fuzzy Sets and Systems* vol. 161, pp. 285-297, 2010.

[10] E. Khorram and A. Ghodousian, “Linear objective function optimization with fuzzy relation equation constraints regarding max-av composition,” *Applied Mathematics and Computation*, vol. 173, pp. 872-886, 2006.

[11] E. Khorram, A. Ghodousian and A. A. Molai, “Solving linear optimization problems with max-star composition equation constraints,” *Applied Mathematics and Computation*, vol. 178, pp. 654-661, 2006.

[12] P. Li and Y. Liu, “Linear optimization with bipolar fuzzy relational equation constraints using Łukasiewicz triagular norm,” *Soft Computing*, vol. 18, pp. 1399-1404, 2014.

[13] P. Li and S.-C. Fang, “On the resolution and optimization of a system of fuzzy relational equations with sup-T composition,” *Fuzzy Optimization and Decision Making*, vol. 7, pp. 169-214, 2008.

[14] J.-L. Lin, Y.-K. Wu and S.-M. Guu, “On fuzzy relational equations and the covering problem,” *Information Sciences*, vol. 181, pp. 2951-2963, 2011.

[15] A. V. Markovskii, “On the relation between equations with max-product composition and the covering problem,” it Fuzzy Sets and Systems, vol. 153, pp. 261-273, 2005.

[16] E. Sanchez, “Resolution of composite fuzzy relation equations,” *Information and Control*, vol. 30, pp. 38-48, 1976.

[17] B.-S. Shieh, “Minimizing a linear objective function under a fuzzy max-t-norm relation equation constraint,” *Information Sciences*, vol. 181, pp. 832-841, 2011.

[18] Y.-K. Wu and S.-M. Guu, “Minimizing a linear function under a fuzzy max-min relational equation constraint,” *Fuzzy Sets and Systems*, vol. 150, pp. 147-162, 2005.

[19] Y.-K. Wu, “Optimization of fuzzy relational equations with max-av composition,” *Information Sciences*, vol. 177, pp. 4216-4229, 2007.

[20] L. A. Zadeh, “Toward a generalized theory of uncertainty (GTU)-an outline,” *Information Sciences*, vol. 172, pp. 1-40, 2005.