Realization of Gyrostat Program Motion with Cavity Filled with Viscous Fluid

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***Abstract*—The program spherical motions problem around the center of mass of a single-rotor dynamically symmetrical gyrostat with a spherical cavity, entirely filled with highly viscous fluid, is studied. The active stabilizing controls attached to the gyrostat are constructed by the principle of feedback. They solve stabilizing program motions problem of a gyrostat with fluid. Conditions under which the desirable program motions property of asymptotic stability is possible are received. The task is solved on the base of a method of Lyapunov functions and a method of the limit equations and the limit systems.**

***Index Terms*—Gyrostat, cavity with fluid, Lyapunov function, feedback, stabilization.**

I. Introduction

PROBLEMS about spatial orientation of satellites and aircraft in an orbit have important applied value and are widely considered by authors in many notes. Spatial motions of aircraft concerning the center of masses are modeled by spherical motions of solid bodies or bodies systems, in particular, gyrostats. The basic methods and principles of control of rotational motions of bodies and systems were studied, for example, in notes [1-3]. Modern domestic and foreign scientists actively study tasks of resonance modes and bifurcations of stationary motions of satellites [4, 5], of chaotic motions and methods of their elimination [6, 7], about stabilization of the set program motions of gyrostats of various structure [8, 9]. From the middle of the previous century, various problems of dynamic motions of rigid bodies with cavities filled with fluid were widely researched. Two main ways of research and the first results of the theory of rigid bodies with cavities filled with liquid are represented in papers [10, 11]. The author of paper [10] suggested a model, describing the motions of rigid body with a cavity entirely filled with highly viscous fluid. In the model the influence of fluid on the motion of the body is described using the kinematic characteristics of the body. This approach is widely used in papers of modern scientists [12-14].

This paper is devoted to a study of spherical motions around a center of mass of a single-rotor gyrostat with a spherical cavity filled with highly viscous fluid. The gyrostat motion equations are obtained by virtue of the method from note [10]. The active program and stabilizing controls attached to the gyrostat are constructed by the principle of feedback. The problem of stabilization of program motions of a gyrostat with fluid to asymptotically stable is solved. The presented results are received on the base of a method of Lyapunov functions of the classical stability theory and of a method of the limit equations and limit systems [15]. The asymptotic convergence of the solutions is confirmed and illustrate by the results of numerical simulation of the motion of the gyrostat.

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II. Statement of the Problem and Motion Equations

We research spherical motion of a gyrostat. It is modeled by a system of two dynamically symmetric connected bodies with common axis of rotation. The first body is the carrier. It has a cavity filled with highly viscous fluid. The second body is the rotor. $A_1 = B_1$ and C_1 are main inertia moments of a carrier with fluid, $A_2 = B_2$ and C_2 are main inertia moments of a rotor. Here $OXYZ$ is fixed coordinate system, $Oxyz$ is related to a carrier coordinate system. The fixed point O of a gyrostat coincides with the system's center of mass and is located on an axis of dynamic symmetry of both bodies (Fig. 1). The rotation of a rotor around the carrier is described with the rotation angle σ counted around Oz axis.

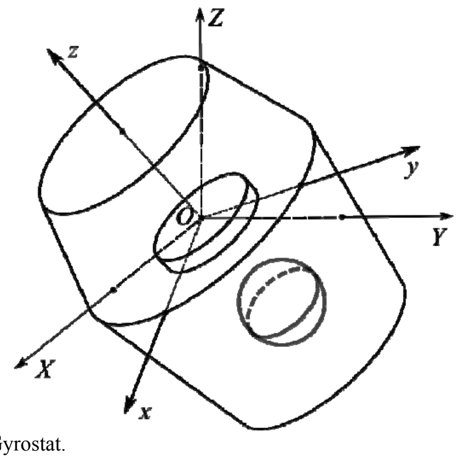

Fig. 1. Gyrostat.

Motion equations of a single-rotor gyrostat with a cavity filled with fluid are projected on axis in the related coordinate system, and they are following [14] in the form:

$$\begin{aligned} A\dot{p} + (C-B)qr + C_2 q\sigma &= m_x, \\ B\dot{q} + (A-C)pr - C_2 p\sigma &= m_y, \\ C\dot{r} + (B-A)pq + C_2\dot{\sigma} &= m_z, \end{aligned} \tag{1}$$

Here $\boldsymbol{\omega}=\left(p,q,r\right)^{\mathrm{T}}$ is the vector of absolute angular velocity of the carrier in the coordinate system $Oxyz$. $A=A_1+A_2$, $B=B_1+B_2$ and $C=C_1+C_2$ are main inertia moments of a gyrostat calculated in the coordinate system $Oxyz$. Symbol $(\ )^{\mathrm{T}}$ means transposition. The angular velocity $\sigma=\sigma(t)$ of rotor rotation is a determined continuous function of time.

Right parts $\mathbf{m}=\left(m_x,m_y,m_z\right)^{\mathrm{T}}$ of equations (1) are projections on axes of the frame $Oxyz$ of the force torques acting on the carrier from the cavity with fluid. According to model suggested in note [10] they are calculated in form:

$$\mathbf{m}=-\frac{\rho}{\nu}P\begin{bmatrix}\ddot{p}+q\dot{r}-r\dot{q}\\ \ddot{q}+r\dot{p}-p\dot{r}\\ \ddot{r}+p\dot{q}-q\dot{p}\end{bmatrix}. \tag{2}$$

Here $\left(\dot{p},\dot{q},\dot{r}\right)^{\mathrm{T}}=\dot{\boldsymbol{\omega}}$ is the vector of angular acceleration of the carrier body, $P=8\pi a^7/525$ is coefficient, that is considering the form (sphere with radius a) of the cavity, ρ is density and ν is kinematic viscosity of fluid. We assume that the cavity is filled with highly viscous fluid: $\nu^{-1} \ll 1$.

Following the paper [10, 14] we have vector components of angular acceleration

$$\begin{aligned}
&\dot{p}\approx -A^{-1}\left[\left(C-B\right)qr+C_2q\sigma\right],\\
&\dot{q}\approx -B^{-1}\left[\left(A-C\right)pr-C_2p\sigma\right],\\
&\dot{r}\approx -C_1^{-1}\left(B-A\right)pq,\\
&\ddot{p}\approx A^{-1}\left[\left(B-C\right)\left(\dot{q}r+q\dot{r}\right)-C_2\left(\dot{q}\sigma+q\dot{\sigma}\right)\right],\\
&\ddot{q}\approx B^{-1}\left[\left(C-A\right)\left(\dot{p}r+p\dot{r}\right)+C_2\left(\dot{p}\sigma+p\dot{\sigma}\right)\right],\\
&\ddot{r}\approx C_1^{-1}\left(A-B\right)\left(\dot{p}q+p\dot{q}\right).
\end{aligned} \tag{3}$$

After replacing right parts (2) of (3) the force torques acting on the carrier from the cavity with fluid are calculated in form:

$$\begin{cases}
m_x=-\varepsilon P\left\{\dfrac{(C-B+A)(A-C)r^2p}{AB}+\dfrac{(C-B-A)(A-B)q^2p}{AC}\right\};\\
m_y=-\varepsilon P\left\{\dfrac{(A-B-C)(C-B)r^2q}{AB}+\dfrac{(A-C+B)(A-B)p^2q}{BC}\right\};\\
m_z=-\varepsilon P\left\{\dfrac{(A-B+C)(C-B)q^2r}{AC}+\dfrac{(A-B-C)(A-C)p^2r}{BC}\right\}.
\end{cases} \tag{4}$$

We constructed the gyrostat motion equations in the Lagrange equations form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial\mathbf{q}}=\mathbf{Q} \tag{5}$$

Here $\mathbf{q}=\left(\psi,\theta,\varphi\right)^{\mathrm{T}}$ is the general coordinates vector, and $\dot{\mathbf{q}}=\left(\dot{\psi},\dot{\theta},\dot{\varphi}\right)^{\mathrm{T}}$ is the general velocities vector. Values $\varphi,\ \psi,\ \theta$ are Euler variables. The kinetic energy of the system is

$$2T=Ap^2+Bq^2+Cr^2+2C_2r\sigma+C_2\sigma^2. \tag{6}$$

We designate the general forces $\mathbf{Q}=\mathbf{Q}^e+\mathbf{Q}^p+\mathbf{Q}^s$. Here $\mathbf{Q}^e$ is the force torque acting on the carrier from the cavity with fluid. It has coordinates

$$\begin{cases}
Q_\psi^e=m_x\sin\theta\sin\varphi+m_y\sin\theta\cos\varphi+m_z\cos\theta,\\
Q_\theta^e=m_x\cos\varphi-m_y\sin\varphi,\\
Q_\varphi^e=m_z.
\end{cases} \tag{7}$$

Torque $\mathbf{Q}^p$ is the program control, torque $\mathbf{Q}^s$ is the stabilizing control.

Equations (5) are dynamic gyrostat motion equations. They are closed, for example, with Euler kinematic equations

$$\begin{cases}
p=\dot{\theta}\cos\varphi+\dot{\psi}\sin\theta\sin\varphi;\\
q=-\dot{\theta}\sin\varphi+\dot{\psi}\sin\theta\cos\varphi;\\
r=\dot{\varphi}+\dot{\psi}\cos\theta.
\end{cases} \tag{8}$$

Let the gyrostat moves according to the law $\mathbf{r}(t)=\left(\psi^*(t),\theta^*(t),\varphi^*(t)\right)^{\mathrm{T}}$. Here $\psi^*(t),\ \theta^*(t),\ \varphi^*(t)$ are the determined continuous functions of time. We call the function $\mathbf{r}(t)$ the program motion of the gyrostat.

We now state the task about realization and stabilization of program motion of gyrostat. Namely, we have to find the attached to the carrier control torques $\mathbf{Q}^p$ and $\mathbf{Q}^s$ making the program motion $\mathbf{r}(t)$ asymptotically stable.

We solve this task and we construct active control using principle of feedback on the base Lyapunov method of stability theory.

The kinetic energy (6) may be presented as the following sum of the components $T=T_2+T_1+T_0$. Here $T_0=T_0(t,\mathbf{q})=C_2\sigma^2(t)$ is a scalar function. The component $T_1=\mathbf{B}^{\mathrm{T}}(t,\mathbf{q})\dot{\mathbf{q}}$ is a linear form of the general velocities $\dot{\mathbf{q}}$. Vector $\mathbf{B}(t,\mathbf{q})$ has the coordinates $b_1=2C_2\sigma(t)\cos\theta,$ $b_2=0,\ b_3=2C_2\sigma(t)$. The last component $T_2=0.5\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}(\mathbf{q})\dot{\mathbf{q}}$ is a quadratic form of the velocities. The matrix $\mathbf{A}(\mathbf{q})$ is bounded and positive definite. It has the elements

$$\begin{aligned}
&a_{11}=A\sin^2\theta\sin^2\varphi+B\sin^2\theta\cos^2\varphi+C\cos^2\theta,\\
&a_{12}=a_{21}=\left(A-B\right)\sin\theta\sin\varphi\cos\varphi,\ a_{33}=C,\\
&a_{22}=A\cos^2\varphi+B\sin^2\varphi,\\
&a_{13}=a_{31}=C\cos\theta,\ a_{23}=a_{32}=0.
\end{aligned}$$

As a result we obtain the motion equation (5) in the form

$$\mathbf{A}\ddot{\mathbf{q}}+\boldsymbol{\Lambda}\left[\frac{\partial\mathbf{B}}{\partial\mathbf{q}^{\mathrm{T}}}-\frac{\partial\mathbf{B}^{\mathrm{T}}}{\partial\mathbf{q}}\right]\dot{\mathbf{q}}+\frac{\partial\mathbf{B}}{\partial t}=\mathbf{Q}^e+\mathbf{Q}^p+\mathbf{Q}^s. \tag{9}$$

Here $\boldsymbol{\Lambda}=\boldsymbol{\Lambda}(\mathbf{q},\dot{\mathbf{q}})$ is the vector with coordinates

$$\Lambda_i=\sum_{j,k=1}^{3}\frac{\partial a_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j-\frac{1}{2}\sum_{j,k=1}^{3}\frac{\partial a_{kj}}{\partial q_i}\dot{q}_k\dot{q}_j,\ \ (i=\overline{1,3}). \tag{10}$$

III. Program and Stabilization Controls

We calculate the program control torque according direct substitution of function $\mathbf{r}(t)$ in the motion equations (9):

$$\begin{aligned}
\mathbf{Q}^p=\ &\mathbf{A}\left(\mathbf{r}(t)\right)\ddot{\mathbf{r}}+\left[\frac{\partial\mathbf{B}\left(\mathbf{r}(t)\right)}{\partial\mathbf{r}^{\mathrm{T}}}-\frac{\partial\mathbf{B}^{\mathrm{T}}\left(\mathbf{r}(t)\right)}{\partial\mathbf{r}}\right]\dot{\mathbf{r}}+\\
&+\boldsymbol{\Lambda}\left(\mathbf{r}(t),\dot{\mathbf{r}}(t)\right)+\partial\mathbf{B}\left(\mathbf{r}(t)\right)/\partial t-\mathbf{Q}^e\left(\mathbf{r}(t),\dot{\mathbf{r}}(t)\right)
\end{aligned} \tag{11}$$

The program control torque (11) realizes the program motion $\mathbf{r}(t)$ of the gyrostat. We mean, that the function $\mathbf{r}(t)$ is the solution of the equation (9). But in the presence of initial deviations or actions of small perturbations we construct the additional stabilizing torque $\mathbf{Q}^s$ making the program motion $\mathbf{r}(t)$ asymptotically stable.

Let us introduce the new generalized coordinates (deflections) $\overline{x}$ according to equality $\mathbf{x} = \mathbf{q} - \mathbf{r}(t) = (\psi - \psi^*(t), \theta - \theta^*(t), \varphi - \varphi^*(t))^{\mathrm{T}} = (x_1, x_2, x_3)^{\mathrm{T}}$.

Then we rewrite the equation (9) as

$$\mathbf{A}(\ddot{\mathbf{r}} + \ddot{\mathbf{x}}) + \mathbf{\Lambda} + \mathbf{\Lambda}' + \mathbf{\Lambda}'' + \left[\frac{\partial \mathbf{B}}{\partial \mathbf{x}^{\mathrm{T}}} - \frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial \mathbf{x}}\right](\dot{\mathbf{r}} + \dot{\mathbf{x}}) + \partial \mathbf{B}/\partial t = \mathbf{Q}^e + \mathbf{Q}^p + \mathbf{Q}^s. \quad (12)$$

Here program control $\mathbf{Q}^p$ has the form (11), and $\mathbf{\Lambda}$, $\mathbf{\Lambda}'$, $\mathbf{\Lambda}''$ analogously (10) are the vectors with components

$$\Lambda_i = \sum_{j,k=1}^{3} \frac{\partial a_{ij}}{\partial x_k} \dot{x}_k \dot{x}_j - \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial a_{kj}}{\partial x_i} \dot{x}_k \dot{x}_j,$$

$$\Lambda_i' = \sum_{j,k=1}^{3} \frac{\partial a_{ij}}{\partial x_k} \dot{x}_k \dot{r}_j - \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial a_{kj}}{\partial x_i} \dot{x}_k \dot{r}_j + \sum_{j,k=1}^{3} \frac{\partial a_{ij}}{\partial x_k} \dot{r}_k \dot{x}_j - \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial a_{kj}}{\partial x_i} \dot{r}_k \dot{x}_j,$$

$$\Lambda_i'' = \sum_{j,k=1}^{3} \frac{\partial a_{ij}}{\partial x_k} \dot{r}_k \dot{r}_j - \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial a_{kj}}{\partial x_i} \dot{r}_k \dot{r}_j, \quad (i = \overline{1,3}).$$

We choose the Lyapunov function

$$V(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} \mathbf{x}^{\mathrm{T}} \mathbf{C} \mathbf{x} + \frac{1}{2} \dot{\mathbf{x}}^{\mathrm{T}} \mathbf{A} \dot{\mathbf{x}}, \quad (13)$$

And we construct the stabilization control in the form

$$\mathbf{Q}^s = -\mathbf{C}\mathbf{x} - \mathbf{D}\dot{\mathbf{x}} + \mathbf{A}\ddot{\mathbf{r}} + \mathbf{\Lambda}'' + \mathbf{\Lambda}' - \frac{1}{2}\left(\frac{\partial \mathbf{A}}{\partial \mathbf{x}^{\mathrm{T}}} \dot{\mathbf{r}}\right)\dot{\mathbf{x}} + \frac{\partial \mathbf{B}}{\partial t} + \left[\frac{\partial \mathbf{B}}{\partial \mathbf{x}^{\mathrm{T}}} - \frac{\partial \mathbf{B}^{\mathrm{T}}}{\partial \mathbf{x}}\right]\dot{\mathbf{r}} - \mathbf{Q}^e - \mathbf{Q}^p. \quad (14)$$

Here matrices $\mathbf{C}$ and $\mathbf{D}$ are bounded and positive definite. We calculate the total derivative of the function (13) with respect to time according the equation (12) with controls (11) and (14):

$$\frac{dV}{dt} \cong -\dot{\mathbf{x}}^{\mathrm{T}} \mathbf{D} \dot{\mathbf{x}} \le -d_0 \left\| \dot{\mathbf{x}}^2 \right\| \le 0 \quad (0 < d_0 = \text{const}) \quad (15)$$

The derivative (15) of Lyapunov function (13) is negative definite determined by speeds. The set on which the derivative is equal to zero, is a set $\{\dot{\mathbf{x}} = 0\}$. The system limit to system (12), with (11), (14) on a set $\{\dot{\mathbf{x}} = 0\}$ has no other decisions, except $\mathbf{x} = 0$. Therefore on the basis of the theorem from paper [15] we receive, that program motion $\mathbf{r}(t) = (\psi^*(t), \theta^*(t), \varphi^*(t))^{\mathrm{T}}$ of the gyrostat with fluid is asymptotically stable.

IV. Numerical Simulation

To illustrate the analytical results we integrate numerically the equations of control motion of the gyrostat. We use "Wolfram Mathematica 7.0". We model the carrier and rotor by a rigid bodies with inertia moments $A = B = 20,\ C = 30,\ C_2 = 10$ kg/m^2. Let the angular velocity of rotor rotation about the carrier is $\sigma = 1$ s^{-1}. The program motion is

$\psi^*(t) = 6\cos(2t),\ \ \theta^*(t) = \pi/2 + \sin(2t),\ \ \varphi^*(t) = \sin(10t)$ rad.

We assume that the initial deviations at $t = 0$ are $\overline{x}(0) = (0.1,\ 0.07,\ 0.05)^{\mathrm{T}}$ rad and $\dot{\overline{x}}(0) = (0.1,\ 0.1,\ 0.1)^{\mathrm{T}}$ rad/s. The integration was performed over the time interval $[0, 30]$ s. Let the coefficients of the matrices $\mathbf{C}$ and $\mathbf{D}$ are $c_{ii} = d_{ii} = 10$, $c_{ij} = d_{ij} = 0,\ i \ne j,\ i, j = 1,2,3$.

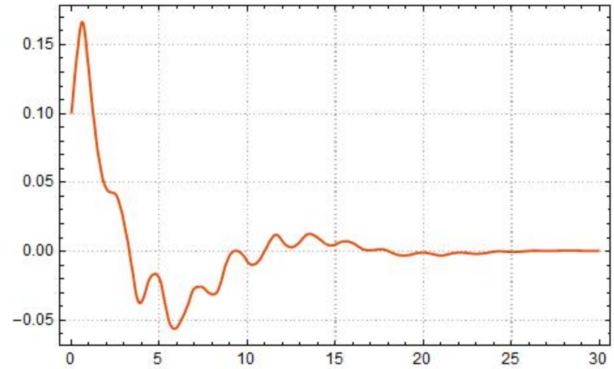

Fig. 2. Value $x_1(t)$.

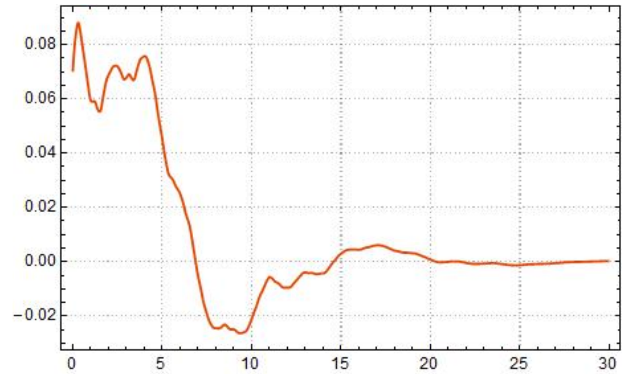

Fig. 3. Value $x_2(t)$.

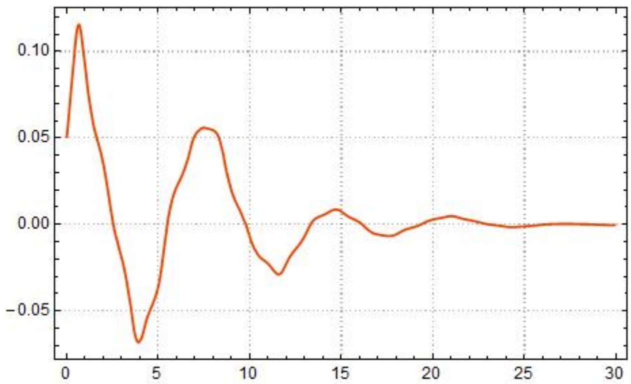

Fig. 4. Value $x_3(t)$.

Figures 2-4 present graphs of the behavior of the components of the vector $\overline{x}(t)$. They are the deviations of the general coordinates $\mathbf{q} = (\psi, \theta, \varphi)^{\mathrm{T}}$ of the gyrostat for its program motion $\mathbf{r}(t) = (\psi^*(t), \theta^*(t), \varphi^*(t))^{\mathrm{T}}$. This motion occurs under the action of programmed torque (11) and stabilizing torque (14). The graphs illustrate the asymptotical stable of the obtained solutions.

V. CONCLUGION

This paper describes mathematic model of movement around the center of mass of single-rotor dynamically symmetrical gyrostat with a spherical cavity filled with viscous fluid. The problem about realization and stabilization of gyrostat program motions is solved. The active program and stabilizing controls acting to the gyrostat by the principle of feedback are constructed. The task is solved on the base of a method of Lyapunov functions and a method of the limit equations and the limit systems. The asymptotic convergence of the solutions is confirmed and illustrate by the results of numerical simulation of the motion of the gyrostat.

The results of this paper further develop results from notes [10, 14] and can be used for projecting control systems for objects with cavities filled with highly viscous fluids.

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