Gravitational Stabilization of a Satellite Using a Bounded Control of a Movable Mass

Sergey P. Bezglasnyi

Abstract — **The plane motion of an axisymmetric satellite with a movable mass on its axis of symmetry is examined. A restricted continuous law for controlling the movable mass is proposed, which solves a problem of the gravitational stabilization to in-plane perturbations of two diametrically opposite relative equilibrium positions of the satellite in a circular orbit using the swing-by technique. The problem of reorientation of the satellite by moving it from one stable equilibrium position to the other is solved.**

.Index Terms — **Satellite, orbit, movable mass, gravitational torque, swing-by technique, stabilization, Lyapunov function.**

I. Introduction

The problem of the stability of the relative equilibrium positions and different motions of a satellite about the centre of mass in a Keplerian orbit under the action of gravitational, aerodynamic, and other torques has been the subject of publications by investigators [1–8]. We solve below the problem of control and stabilization of an axisymmetric satellite with a point mass (load) under the action of gravitational torque. The load can move along the axis of symmetry to the principle of swing action. Swings are modeled by a single-mass [9] or two-mass [10–12] pendulum of variable length. Their models can be used to solve applied problems. For example, the swing-by technique has been used in paper [13]. It solves the problem of orbital maneuvering of a satellite with using a space tether system. Note [14] is devoted to the study of two problems of the satellite in a circular orbit. They are the problem of gravitational stabilization with respect to in-plane perturbations of the relative equilibrium position and the problem of reorientation of the satellite under the condition that the mass of the load is considerably less than the mass of the satellite. But in paper [14] was constructed unbounded control law such that the moving mass can move beyond the confines of the satellite. It presents severe difficulties in actual practice.

In the present paper we investigate the plane motion of an axisymmetric satellite with a movable mass for any value of the movable mass. We annex the condition of the constraints of load's motion.

In Section 2 the equations of the plane motion of a satellite with a movable mass about a common centre of mass in elliptical and circular orbits under the action of a gravitational torque are obtained. In Section 3 bounded control law is proposed, and the problem of gravitational stabilization (damping of the in-plane oscillations) in the vicinity of a position of relative equilibrium of the satellite is solved. In Section 4 the problem of the turning the satellite through an angle π is examined. We realize the control process of the "swinging" of the satellite in the vicinity of its stable equilibrium position and moving it into a diametrically opposite, asymptotically stable position.

We solve the problem by using the second method of classical stability theory. We construct the corresponding Lyapunov functions. The asymptotic convergence of the solutions is confirmed by the results of numerical simulation of the motion of the system.

Manuscript received December 08, 2015; revised January 27, 2016. This work was supported by the grant of the Ministry of Education and Science of Russian Federation (project no. 9.540.2014/K).

S. P. Bezglasnyi, Associate Professor, is with the Samara State Aerospace University, Samara, 443086 Russia (corresponding author to provide phone: 8-917-104-1316; e-mail: bezglasnsp@rambler.ru).

II. Problem Definition and Motion Equations

We consider the motion of a satellite about its centre of mass at the point O. The satellite moves in a central Newtonian gravitational field. The model of the satellite is an axisymmetric rigid body (carrier) of mass m_1 with a point load of mass m_2. Load can move along axis of symmetry of carrier (Fig. 1). The centre of mass of the carrier is located on its dynamic axis of symmetry at the point O_1. We use l and d to denote the distances from the point O_1 to the load m_2 and to the centre of mass O_2 of the entire satellite. The following equality holds

$$m_1 d = m_2 (l - d) \tag{1}$$

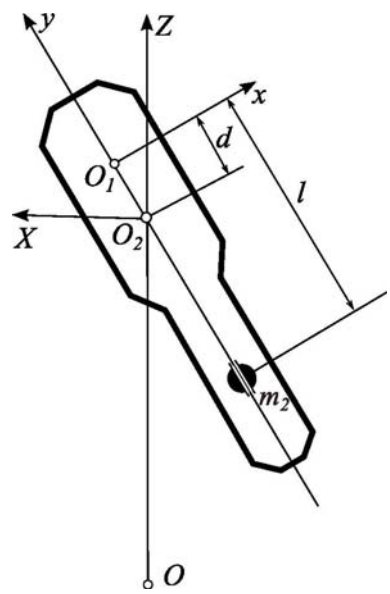

Fig. 1. Satellite.

We annex the orbital system of coordinates O_2XYZ. Let the axis O_2X is directed along a tangent to the satellite orbit, the axis O_2Y is perpendicular to the orbital plane. The axis O_2Z completes the system of coordinates as a set of three axes at right angles. The coordinates system O_1xyz is connected to the satellite. It coincide with its principal central axes of inertia. The orientation of the connected system of coordinates relative to the orbital system of coordinates is specified using the Euler angles ψ, θ and φ. Suppose A, B and C, where $B < A = C$ are the principal central moments of inertia of the satellite.

We derive the equation of plane motion of a satellite with a movable mass about the common centre of mass for any value of the mass m_1 and the mass m_2. We denote $m = m_1 m_2 \left(m_1 + m_2\right)^{-1}$. We denote A_2, B_2 and C_2 moments of inertia of a satellite with a movable load about the axes passing through the common centre of mass O_2 and parallel to the axes of the frame O_1xyz. The following equality holds:

$$\begin{aligned} &A_2 = C_2 = A + m_1 d^2 + m_2 (l-d)^2 = A + ml^2, \\ &B_2 = B = const. \end{aligned} \tag{2}$$

It is well known [2] that exist the plane motions of a satellite $\psi = \pi$, $\theta = \frac{\pi}{2}$, $r = \dot{\varphi} + \dot{\nu}$, $p = q = 0$. They are relative motions about the centre of mass in an elliptical orbit. They occur under the action of the gravitational torque

$$M_z = \frac{3n^2}{(1-e^2)^3}(1+e\cos\nu)^3 (B_2 - A_2)\sin\varphi\cos\varphi .$$

Here p, q and r are the components of the angular velocity of rotation of the satellite. The dot denotes a derivative with respect to time t, $n = \text{const} > 0$ is the mean motion of the centre of mass of the satellite, ν is the true anomaly, e is the orbit eccentricity. The gravitational torque M_z is the torque about the perpendicular to the orbital plane axis passing through the point O_2. Using equality (1.2) we write the angular momentum: $K_z = C_2 r = (A + ml^2)(\dot{\varphi} + \dot{\nu})$. Then we have the equation of plane motions of a satellite with a movable mass in the following form:

$$\begin{aligned} &(A + ml^2)(\ddot{\varphi} + \ddot{\nu}) + 2ml\dot{l}(\dot{\varphi} + \dot{\nu}) = \\ &= \frac{3n^2}{(1-e^2)^3}(1+e\cos\nu)^3 (B - A - ml^2)\sin\varphi\cos\varphi \end{aligned} \tag{3}$$

Here $l = l(\varphi, \dot{\varphi})$. We denote $k_1 = 1 + e\cos\nu$. We treat the true anomaly as a new variable [2]. Then we can rewrite the equation of planar motions of a satellite with a movable mass in a Keplerian orbit under the action of the gravitational torque:

$$\begin{aligned} &k_1\varphi'' + 2\left(\frac{mll'}{A+ml^2}k_1 - e\sin\nu\right)\varphi' = \\ &= 3\frac{B-A-ml^2}{A+ml^2}\sin\varphi\cos\varphi + 2e\sin\nu - \frac{2mll'}{A+ml^2}k_1 \end{aligned} \tag{4}$$

The prime denotes the derivative with respect to ν. Equation (1.4) is true for any value of the mass m_1 and m_2. For motion in a circular orbit $e = 0$ and $k_1 = 1$.

III. Control of Damping Motions of Satellite

We state and solve the problem of the asymptotic damping of in-place oscillations of a satellite about relative equilibrium position $\varphi = \varphi' = 0$ using the swing-by technique. We treat the distance from the centre of mass of the carrier body O_1 to the movable mass m_2 as a control. We obtain the solution by the second method of stability theory. We define the control in the form:

$$l = \begin{cases} l_0 + a\varphi'\sin\varphi, & \text{if } a\varphi'\sin\varphi \in (-b,b) \quad a \geq b > 0; \\ l_0 + b\cdot sign(\sin\varphi)\cdot sign\varphi', & a\varphi'\sin\varphi \notin (-b,b). \end{cases} \tag{5}$$

Here l_0, a, $b = \text{const} > 0$. We write the derivative by the equalities

$$l' = \begin{cases} a\varphi''\sin\varphi + a\varphi'^2\cos\varphi, & \text{if } a\varphi'\sin\varphi \in (-b,b); \\ 0, & \text{if } a\varphi'\sin\varphi \notin (-b,b). \end{cases} \tag{6}$$

About the relative equilibrium position $\varphi = \varphi' = 0$ we rewrite the equation (4) for $e = 0$ and $k_1 = 1$ according to formulas (5) and (6):

$$\begin{aligned} &\varphi''(A + ml(l_0 + 3a\varphi'\sin\varphi + 2a\sin\varphi)) = \\ &= -2mla\cos\varphi(\varphi'+1)\varphi'^2 - 1.5(A + ml^2 - B)\sin 2\varphi \end{aligned} \tag{7}$$

Equation (7) has zero solution $\varphi = \varphi' = 0$, which corresponds to the relative equilibrium position of the satellite. We choose the Lyapunov function:

$$\begin{aligned} &V = \frac{A + ml_0(l_0 + 3a\varphi'\sin\varphi + 4a\sin\varphi)}{2}\varphi'^2 + \\ &+ \frac{3}{4}\left(A - B + ml_0(l_0 + \frac{a}{2}\varphi'\sin\varphi) + \frac{4}{3}p\varphi\right)(1-\cos 2\varphi) \approx \\ &\approx \frac{A + ml_0^2}{2}\varphi'^2 + \frac{3}{2}\left(A - B + ml_0^2\right)\varphi^2 \end{aligned} \tag{8}$$

We determine the coefficient $p = \text{const} > 0$ later. In the vicinity of relative equilibrium position $\varphi = \varphi' = 0$ the function $V(\varphi, \varphi')$ is positive-definite. We will calculate the total derivative of the function $V = V(\varphi, \varphi')$ with respect to time. Since [2] $\dot{\nu} = n$ in circular orbit. We calculate the derivative of the Lyapunov function in view of (8) and expand its expression in series in the variables φ and φ'. Then we discard terms of higher than the fourth order respect to variables φ and φ'. We obtain

$$\begin{aligned} &\frac{1}{n}\dot{V} \approx -\frac{1}{2}F\varphi'^4 - \left(\frac{15}{4}F + \frac{15}{2}F\frac{H}{G}\right)\varphi^2\varphi'^2 - 4\frac{F^2}{G}\varphi\varphi'^3 - \\ &-12\frac{F^2}{G^2}H\varphi^3\varphi' - \frac{9}{4}\frac{F}{G}H\varphi^4 - 6\frac{F}{G}H\varphi^2\varphi' + 4Fp\varphi^2\varphi' \end{aligned} \tag{9}$$

where

$$F = mal_0 > 0,\ G = A + ml_0^2 > 0,\ H = A - B + ml_0^2 > 0 \tag{10}$$

We chose p according to the equation

$$p = \frac{3}{2}\frac{H}{G}, \tag{11}$$

then rewrite (9):

$$\begin{aligned} &\frac{G}{nF}\dot{V} = -G\varphi'^2\left(\frac{1}{2}\varphi'^2 + \frac{15}{4}\varphi^2 + 4\frac{F}{G}\varphi\varphi'\right) - \\ &-3H\varphi^2\left(\frac{5}{2}\varphi'^2 + \frac{3}{4}\varphi^2 + 4\frac{F}{G}\varphi\varphi'\right) \end{aligned} \tag{12}$$

ISBN: 978-988-19253-8-1
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

By Sylvester's criterion [16] satisfied homogeneous form (12) by the condition $8mal_0 < \sqrt{30}\left(A + ml_0^{\,2}\right)$ is negative-definite. By Lyapunov's asymptotic stability theorem [16] we have the follow result. Relative equilibrium position $\varphi = \varphi' = 0$ of the satellite in a circular orbit is asymptotically stable. The results of integrating the equations of motion confirm the conclusions drawn.

The phase portrait of system with control (5) is shown in Fig. 2. Numerically integration the equation of motion was carried out at the following numerical values of the parameters of the system: $m_1 = 500$ kg, $m_2 = 50$ kg, $A = 100$ $\text{kg}\cdot\text{m}^2$, $B = 10$ $\text{kg}\cdot\text{m}^2$, $l_0 = 1$ m, $a = 0.7$ $\text{m}\cdot\text{rad}$, $b = 0.2$ $\text{m}\cdot\text{rad}$, and the initial values: $\varphi(\nu_0) = 1$ rad, $\varphi'(\nu_0) = 0.2$. The integration was performed in the range $\nu \in [0;50]$ rad. The phase trajectory displays the asymptotic decay of the amplitude and speed of the oscillations of the satellite about the zero equilibrium position. The dependence of the distance l on the angle of deflection φ of the satellite is shown in Fig. 3. It demonstrates its asymptotic convergences to the value l_0.

IV. Swinging and Reorientation of the Satellite

We apply a control law of the form (5) to the problem of the swinging of a satellite from an arbitrary neighborhood of the relative equilibrium position and its diametrical reorientation.

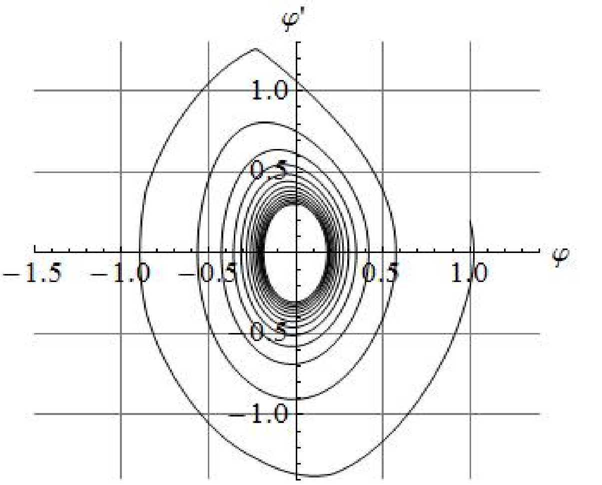

Fig. 2. Phase portrait.

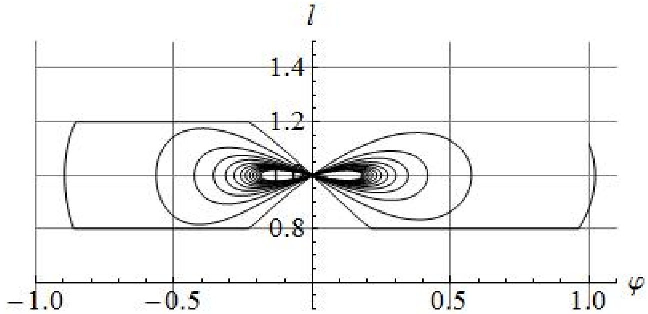

Fig. 3. Value $l(\varphi)$.

We now assume that in control law (5) the parameter

$$a = \text{const} < 0, \quad |a| > b \tag{13}$$

The equation of controlled motion of the satellite maintains the form (7). The function (8) is positive-definite in the vicinity of equilibrium $\varphi = \varphi' = 0$. Derivative of this function on time owing to the equation (7), when choosing a value p according to equation (11) in designations (10) similar (12):

$$\begin{aligned} \frac{G}{n|F|}\dot{V} &= G\varphi'^2\left(\frac{1}{2}\varphi'^2 + \frac{15}{4}\varphi^2 + 4\frac{F}{G}\varphi\varphi'\right) + \\ &+ 3H\varphi^2\left(\frac{5}{2}\varphi'^2 + \frac{3}{4}\varphi^2 + 4\frac{F}{G}\varphi\varphi'\right) \end{aligned} \tag{14}$$

Here $F = mal_0 < 0$. By Sylvester's criterion homogeneous form (14) will be positive-definite, for any parameters of the satellite.

According to Lyapunov's first instability theorem, [16] relative equilibrium position $\varphi = \varphi' = 0$ of the satellite in a circular orbit is unstable Thus, control (5) with a negative value of the parameter a implements the swinging of the satellite about the local vertical

We investigate now the behavior of the satellite with control (5) for positive and negative values of the parameter a in the vicinity of diametrically opposite equilibrium position $\varphi = \pi$, $\varphi' = 0$. We introduce the deflection $\varphi = \pi + x$, and write the equation of perturbed motion.

$$\begin{aligned} & x''(A + ml\ (l_0 - 3ax'\sin x - 2a\sin x)) = \\ & = 2mla\cos x(x'+1)x'^2 - \frac{3}{2}(A + ml^2 - B)\sin 2x \end{aligned} \tag{15}$$

Suppose $a = \text{const} > 0$. Then equation (15) with control (5) is identical to equation (7) with control (5) where $a = \text{const} < 0$. Therefore, the zero solution $x = x' = 0$ of equation (15) is unstable according to the result obtained in Section 3.

Now suppose $a = \text{const} < 0$ and condition (13) is satisfied. Equation of perturbed motion (15) with control (5) and condition (13) is identical to equation (7) with $a = \text{const} > 0$. Therefore, the zero solution $x = x' = 0$ of equation (15) is asymptotically stable, by the result obtained in Section 3.

Thus, control (5) under condition (13) implements the diametrical reorientation of the satellite. After swinging about the relative equilibrium position, at which the axis of dynamic symmetry of the satellite coincides with the local vertical, the satellite swings through an angle π. Then it performs asymptotically decaying oscillations in the vicinity of its opposite position of relative equilibrium in the orbit.

This process is clearly illustrated by the graphs of the corresponding numerical calculations. The change of an angle and the phase portrait of system (7) with control (5) under condition (13) are shown in Fig. 4, 5. Numerically integration the equation of motion was carried out at the following numerical values of the parameters of the system: $m_1 = 500$ kg, $m_2 = 50$ kg, $A = 100$ $\text{kg}\cdot\text{m}^2$, $B = 10$ $\text{kg}\cdot\text{m}^2$, $l_0 = 1$ m, $a = -0.7$ $\text{m}\cdot\text{rad}$, $b = 0.2$ $\text{m}\cdot\text{rad}$ and the initial values: $\varphi(\nu_0) = 0.3$ rad, $\varphi'(\nu_0) = 0.1$. The integration was performed in the range $\nu \in [0;50]$ rad.

The figures 4 and 5 reflect the process of swinging about zero equilibrium position $\varphi = \varphi' = 0$ followed by an asymptotic approach to the new equilibrium position $\varphi = \pi$,

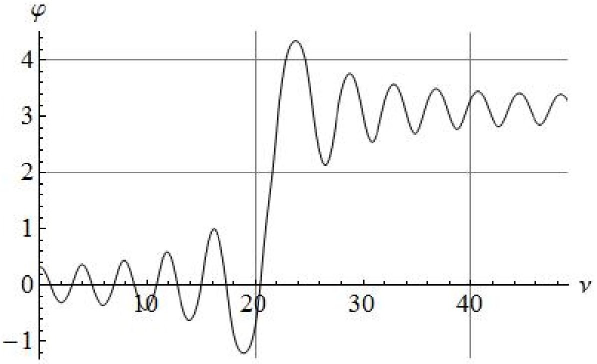

Fig. 4. Change $\varphi(\nu)$ at the turn of the satellite.

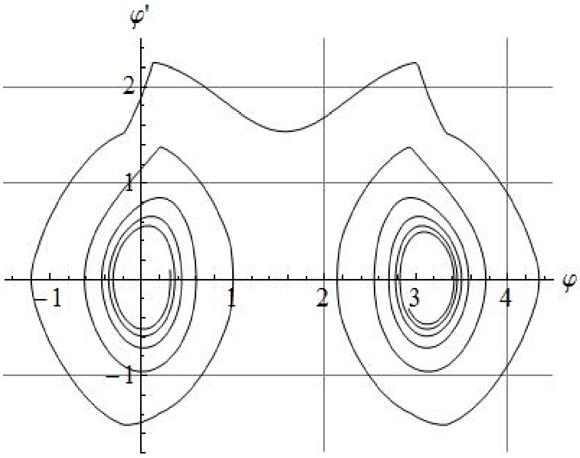

Fig. 5. Phase portrait.

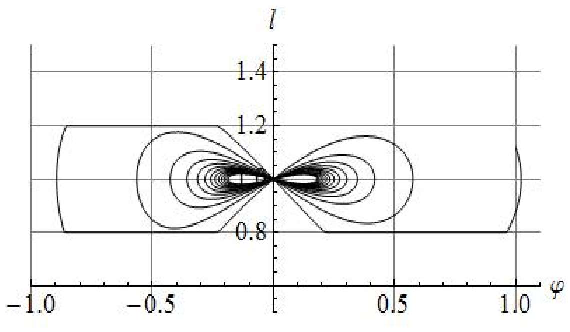

Fig. 6. Value $l(\varphi)$.

$\varphi' = 0$.

Fig. 6 shows the behavior of the distance l as a function of the angle φ. Initially, as the satellite swings, the deviations of the distance l from the value l_0 in the vicinity of equilibrium $\varphi = 0$ increase periodically. After the turning of the satellite its transit into the vicinity of position $\varphi = \pi$, the distance l converges asymptotically to l_0. We note that the value l remains limited.

V. Conclusion

In this paper the equations of the controlled plane motion of a satellite with a movable mass about a common centre of mass in elliptical orbit for all values of the masses are obtained. Bounded control laws are proposed. Two problems are solved. First problem is the task of gravitational stabilization (damping of the in-plane oscillations) in the vicinity of a position of relative equilibrium of the satellite. Second problem is the task of the diametrical reorientation the satellite under the action of a gravitational torque. For the proposed control Lyapunov function is constructed. This function proves asymptotic stability and instability of the studied movements.

References

[1] F. L. Chernous'ko, "On satellite motion relative on the center of mass under the action of gravitational moments" *J. Appl. Math. Mech.*, vol. 27, no. 3, 1963, pp. 708–722.
[2] V. V. Beletskii, *"Motion of an artificial satellite about its center of mass in a gravitational field"*. Moskow: Izd MGU;1966.
[3] V. A. Sarychev, S. A. Mirer, "Relative equilibria of a satellite subjected to gravitational and aerodynamic torques". *Celest. Mech. Dyn. Astr.,* vol. 76, no. 1, 2000, pp. 55–68.
[4] A. P. Markeev, B. S. Bardin, "On stability of planar oscillations and rotations of a satellite in a circular orbit". *Celest. Mech. Dyn. Astron.,* vol. 85, no. 1, 2003, pp. 51–66.
[5] S. Yu. Sadov, "The stability of the resonant rotation of a satellite about the centre of mass in the orbital plane." *Cosmic Research*, vol. 44, no. 2, 2006, pp. 170–181.
[6] L. D. Akulenko, D. D. Leshchenko, A. L. Rachinskaya, "Evolution of the fast rotation of a satellite under the effect of a gravitational torque in a resistive medium." *Mechanics of Solids*, vol. 43, no. 2, 2008, pp. 13–26.
[7] O. V. Kholostova, "The stability of in-plane oscillations of a satellite in a circular orbit." *Mechanics of Solids,* vol. 43, no. 2, 2008, pp. 27–42.
[8] S. P. Bezglasnyi, "Active orientation of a gyrostat with a variable moments of inertia." *J. Appl. Math. Mech.*, vol. 78, no. 6, 2014, pp. 551-559.
[9] K. Magnus *Oscillations*. Moskow: Mir; 1982.
[10] L. D. Akulenko, "Parametric control of the oscillations and rotations of a compound pendulum (swing)." *J. Appl. Math. Mech.*, vol. 57, no. 2, 1993, pp. 301–310.
[11] E. K. Lavrovskii, A. M. Formalskii, "Optimal control of the pumping and damping of a swing." *J. Appl. Math. Mech.*, vol. 57, no. 2, 1993, pp. 311–320.
[12] V. S. Aslanov, S.P. Bezglasnyi, "Stability and instability of controlled motions of a two-mass pendulum of variable length." *Mechanics of Solids,* vol. 47, no. 3, 2012, pp. 285-297.
[13] S. P. Bezglasnyi, E. E. Piyakina, "Parametric control of maneuver of a space tether system." *Cosmic Research*, vol. 53, no. 4, 2015, pp. 323-329.
[14] V. S. Aslanov, S. P. Bezglasnyi, "Gravitational stabilization of a satellite using a movable mass." *J. Appl. Math. Mech.*, vol. 76, no. 4, 2012, pp. 405–412.
[15] S. P. Bezglasnyi, E. E. Piyakina, A. A. Talipova, "Bounded control of two-mass pendulum." *Automatic Control Processes,* vol 34, no. 4, 2013, pp. 35–41.
[16] I. G. Malkin, *Theory of Stability of Motion*. Washington: US Atomic Energy Commission; 1952.