Construction of an Appropriate Membership Function Based on Size of Fuzzy Set and Mathematical Programming

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***Abstract*— This paper proposes an extended approach to construct an appropriate membership function as objectively as possible. It is important to set an appropriate membership function to obtain a reasonable optimal solution for real-world decision making. The main academic contribution of our proposed approach is to integrate a general continuous and nonlinear function minimizing the subjectivity into an interval estimation by a heuristic method under a given probability density function based on real-world data. Our approach is an extend approach of Civanlar and Trussell's study. One of two main steps of our proposed approach is to set membership values which a decision maker confidently judges whether an element is included in the given set or not. Another is to obtain other values objectively by solving Civanlar and Trussell's mathematical programming problem with a nonlinear membership function. In this paper, the given membership function is approximately transformed into a piecewise linear membership function, and the appropriate values of parameters in the piecewise linear membership function are determined.**

***Index Terms*— Membership function, Interactive approach, Piecewise linear function, Mathematical programming**

I. INTRODUCTION

UNCERTAINTY is generally represented as random variables using received data. On the other hand, with respect to ambiguity and fuzziness derived from human cognitive behavior and subjectivity, it is often difficult to set it as a random variables Therefore, it is important to mathematically formulate another uncertainty. One of standard mathematical approaches is Fuzzy theory. The key mathematical element of fuzzy theory is to construct a membership function for the given set. Many approaches to develop the membership functions for fuzzy sets have been shown in a survey by Gottwald [7]. However, it is difficult to select an appropriate membership function's shape and to set the membership values statistically in previous standard approaches. Therefore, heuristic methods have been used, which are not mathematically and statistically guaranteed.

There are some studies that compare various heuristic approaches in terms of appropriateness. Chameau and Santamarina [2] prepared questionnaires to compare four approaches to construct membership functions: point estimation, interval estimation, membership function exemplification, and pairwise comparison as possible candidates for practical applications. Chameau and Santamarina concluded that the interval estimation approach is better than the others. In addition, Yoshikawa [12] discussed the influence of procedures for an interactive identification method on the forms of membership functions. His proposed approach is also based on questionnaires for the interval estimation related to the degree of membership. In real-world decision making, a decision maker can confidently set the intervals with membership values 0 and 1, because she\he can explain how the intervals are set. Thus, the interval estimation method was found to offer a number of advantages that make it suitable for practical applications.

However, heuristic methods have some disadvantages. If a decision maker subjectively sets membership functions using the heuristic methods subjectively from membership values 0 to 1, the optimal solution in the mathematical programming problem with this membership function is dependent on his/her subjectivity. Therefore, other people may not accept this decision due to lack of objectivity. In order to overcome this disadvantage of constructing membership functions, some researchers have proposed more rigorous approaches in terms of statistics. For instance, they adopted a transformation from a probability distribution to a possibility distribution (for instance, Bharathi and Sarma [2]).

In this paper, we focus on the study of Civanlar and Trussell [5]. They proposed a more rigorous method to define the membership function for a certain group of fuzzy sets by minimizing the size of fuzzy set based on the probability density function derived from real-world data and fuzzy entropy. This approach has been extended by Cheng and Cheng [4] and Nieradka and Butkiewicz [11]. Cheng and Cheng [3] also proposed an automatic determination approach of the membership function based on the maximum entropy principle. The advantage of fuzzy entropy approaches is that the decision maker can construct the membership function based on statistics and mathematical programming. However, almost all parameters are automatically determined by these approaches, and hence, human cognitive behavior and subjectivity may not be sufficiently reflected in the resulting membership function.

In order to overcome each disadvantage of previous approaches and heuristic method based on subjective interval estimation, we developed some constructing approaches of appropriate membership functions using both fuzzy Shannon

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entropy and the subjective interval estimation derived from human cognitive behavior and subjectivity [8, 9, 10]. However, our previous approaches do not include interactivity between the decision maker and the designer of the membership function. In addition, it is difficult to obtain the optimal membership function directly and efficiently, because the mathematical programming problem in our proposed approach is a little complex nonlinear programming problem. In real-world application, it is better to obtain the optimal decision rapidly, and hence, it is also better to construct an appropriate membership function efficiently and interactively. Therefore, we develop an interactive approach to obtain the membership function under some natural assumptions.

This paper is organized as follows. In Section II, we introduce a piecewise linear function as a general membership function to apply our constructing approach to any nonlinear membership functions. Furthermore, as a basis of our proposed approach, Civanlar and Trussell's mathematical programming problem is introduced. In Section III, we formulate a mathematical programming problem to extend Civanlar and Trussell's model with the piecewise linear function. under a constraint to the total average membership value derived from the given probability density function (pdf). In order to solve our proposed model efficiently, we introduce a histogram-based pdf as a natural assumption and apply a standard nonlinear programming approach. Using these conditions and nonlinear programming approaches, we develop an interactive approach to construct the appropriate membership function. Finally, in Section IV, we conclude this paper and discuss future research efforts.

II. Mathematical Definition for Appropriate Membership Function

We introduce mathematical definitions to construct an appropriate membership function integrating the decision maker's interval estimation with a given PDF to develop the efficient algorithm. The important mathematical element of our proposed approach is to minimize the size of a fuzzy set for an appropriate membership function considering a piecewise linear function by solving a mathematical programming problem. Therefore, definitions of piecewise function and mathematical programming problem to obtain appropriate parameters are introduced as follows.

A. Piecewise Linear Functions as a General Function

In real-world decision making, it is difficult to initially determine a specific membership function such as Gaussian membership function and exponential function. In this paper, an approach using the piecewise linear function is constructed to obtain membership functions without setting the specific membership function based on statistics and information theory. Almost all nonlinear membership functions are approximated by some piecewise linear membership functions, and hence, it is natural to introduce the piecewise linear membership function as a general membership function. In this paper, we divide received data into *n* groups; $I_k = \{a_{k-1} \le x \le a_k\}, (k = 1,2,\ldots,n)$, and a piecewise linear function is as follows:

$$f(x) = \frac{c_k - c_{k-1}}{a_k - a_{k-1}}(x - a_k) + c_{k-1}, (x \in I_k)$$

If the membership function is determined using this linear membership function, each parameter c_k in the above-mentioned function is represented as membership value μ_k. In this paper, we assume that probability density function (pdf) $p(x)$ derived from received data is a histogram, i.e., $p(x) = p_k, \forall x \in I_k$. In this case, it is natural to assume that we divide intervals including all data into same width, that is, $a_k - a_{k-1} = a_{k+1} - a_k = T$ holds with respect to I_k and I_{k+1}. If $f(x)$ is a smoothly monotonous increasing function on I_k under these assumptions, it is also acceptable that the difference between $\mu_k - \mu_{k-1}$ and $\mu_{k+1} - \mu_k$ is equal to or approximately equal to 0. That is, $\mu_k - \mu_{k-1} \approx \mu_{k+1} - \mu_k$ is obtained where "$\approx$" means "approximately equal". This formula can also be represented in the following form:

$$\mu_{k+1} - 2\mu_k + \mu_{k-1} \approx 0$$

In the case $\mu_k - \mu_{k-1} = \mu_{k+1} - \mu_k = \frac{1}{n}$ any k under assumption $a_k - a_{k-1} = a_{k+1} - a_k = T$, i.e., $\mu_k = \frac{k}{n}$, $(k = 1,2,\ldots,n)$. The piecewise linear membership function is equivalently transformed into the following linear function which is basis of triangle and trapezoidal membership functions:

$$\mu_B = \frac{1}{nT}(x - a_n) + 1 \tag{1}$$

B. Civanlar and Trussell's Model

As an objective approach to construct an appropriate membership function, Civanlar and Trussell [5] proposed a more objective approach to define the membership function for a certain group of fuzzy sets minimizing fuzziness of membership function based on the probability density function derived from real-world data. They obtain the membership function $\mu(x)$ by solving the following mathematical programming problem:

$$\begin{aligned} &\text{Minimize } \frac{1}{2}\int_{-\infty}^{\infty} \mu^2(x)dx \\ &\text{subject to } \int_{-\infty}^{\infty} \mu(x)p(x)dx \ge c, \\ &\qquad 0 \le \mu(x) \le 1, \forall x \in \Re \end{aligned} \tag{2}$$

The objective function is to minimize fuzziness of membership function $\mu(x)$, that is, it means that the size of a fuzzy set is as small as possible [6]. In problem (3), $p(x)$ is the probability density function. In addition, as a key constraint, the total expected value of the membership function $E(x)=\int_{-\infty}^{\infty}\mu(x)p(x)dx$ is more than the target value c, i.e., $\int_{-\infty}^{\infty}\mu(x)p(x)dx \ge c$. If c is a large value close to 1, the constructed membership function is similar to a function whose value is 1 under $p(x)>0$. On the other hand, if c is a small value close to 0, the constructed membership function is similar to a function whose value is 0 in almost all x. Therefore, it is important to flexibly and interactively set a value of parameter c according to the decision maker's feelings. Furthermore, the discrete case of Civanlar and Trussell's model is defined as follows:

$$\begin{aligned}&\text{Minimize } \frac{1}{2}\sum_{k=1}^{n}\mu_k^2\\&\text{subject to } \sum_{k=1}^{n}\mu_k p_k \ge c,\\&\qquad 0\le \mu_k \le 1, (k=1,2,\dots,n)\end{aligned} \tag{3}$$

III. Interactive Approach for Appropriate Membership Function

We propose an interactive approach to construct the appropriate membership function in this section. We estimate intervals initially set by an examinee using two ranges; an event or a condition is (i) entirely within human experience and (ii) never within human experience, respectively. For instance, when we ask the decision maker to specify a range of temperatures for which she/he feels "entirely comfortable," the response will be from 20 to 25 degrees Celsius. Next, when we ask her/him to specify a range of "never comfortable" temperatures, she/he will also answer less than 14 degrees Celsius or more than 30 degrees Celsius. These questions will not be a burden, and hence, it is not difficult to provide these two ranges. Furthermore, if we introduce these two ranges in the proposed approach, it is possible to integrate subjectivity based on personal cognitive behavior with objectivity derived from statistics and information theory. We assume that each endpoint is initially set by the decision maker, i.e., $\mu_0=0$ and $\mu_n=1$.

A. *Mathematical Programming Problem of Our Proposed Approach*

The main objective function of our approach is to minimize the size of fuzzy set based on mathematical programming problem (3). The only available quantitative data is histogram-based pdf $p(x)=p_k, \forall x\in I_k$, $(k=1,2,\dots,n)$ derived from real-world data.

In addition, from the interview or questionnaire to the decision maker, membership values of some important groups m_k are interactively set as interval range $[\mu_k^L,\mu_k^U]$ where μ_k^L and μ_k^U are the lower and upper values. If there are no data from the decision maker, μ_k^L and μ_k^U can be set as 0 and 1, respectively. Therefore, from these initial setting, the appropriate membership function is obtained as optimal solutions of the following problem:

$$\begin{aligned}&\text{Minimize } \frac{1}{2}\sum_{k=1}^{n}\left(\mu_k-\frac{k}{n}\right)^2\\&\text{subject to } \sum_{k=1}^{n}\mu_k p_k \ge c,\\&\qquad \mu_k^L\le \mu(x)\le \mu_k^U, (k=1,2,\dots,n)\end{aligned} \tag{4}$$

To simplify the following mathematical discussion, we assume $\sum_{k=1}^{n}\mu_k p_k = c$. Problem (4) is a second-order convex programming problem. The Lagrange function of problem (4) is obtained as follows:

$$\begin{aligned}L=&\frac{1}{2}\sum_{k=1}^{n}\left(\mu_k-\frac{k}{n}\right)^2+\lambda\left(c-\sum_{k=1}^{n}\mu_k p_k\right)\\&+\sum_{k=1}^{n}\xi_k\left(\mu_k-\mu_k^U\right)+\sum_{k=1}^{n}\eta_k\left(\mu_k^L-\mu_k\right)\end{aligned} \tag{5}$$

From this Lagrange function, we can obtain the following KKT condition:

$$\begin{cases}\dfrac{\partial L}{\partial \mu_k}=\left(\mu_k-\dfrac{k}{n}\right)-\lambda p_k+\xi_k-\eta_k=0\\ \sum_{k=1}^{n}\mu_k p_k=c\\ \xi_k\left(\mu_k-\mu_k^U\right)=0, \eta_k\left(\mu_k^L-\mu_k\right)=0,\\ (k=1,2,\dots,n)\end{cases} \tag{5}$$

where $\xi_k\left(\mu_k-\mu_k^U\right)=0$ and $\eta_k\left(\mu_k^L-\mu_k\right)=0$ are complemetary conditions. If $\mu_k=\mu_k^U$, $\eta_k=0, \xi_k\ge 0$ and the following inequality holds:

$$\lambda \ge \frac{1}{p_k}\left(\mu_k^U-\frac{k}{n}\right) \tag{6}$$

In a way similar to case $\mu_k=\mu_k^U$, if $\mu_k=\mu_k^L$, $\xi_k=0, \eta_k\ge 0$ and the following inequality holds:

$$\lambda \le \frac{1}{p_k}\left(\mu_k^L-\frac{k}{n}\right) \tag{7}$$

Consequently, we obtain the following optimal membership values according to value of λ :

$$\begin{aligned}&\mu_k=\begin{cases}\mu_k^U & U_k\le\lambda\\ \mu_k^* & L_k<\lambda<U_k, (k=1,2,\dots,n)\\ \mu_k^L & \lambda\le L_k\end{cases}\\&L_k=\frac{1}{p_k}\left(\mu_k^L-\frac{k}{n}\right), U_k=\frac{1}{p_k}\left(\mu_k^U-\frac{k}{n}\right)\end{aligned} \tag{8}$$

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From first equation in (5), μ_k^* under $L_k < \lambda < U_k$ is obtained as follows:

$$\mu_k^* = \frac{k}{n} + \lambda p_k \tag{9}$$

If $\mu_k, (k = 1,2,\ldots,K), (K \le n)$ are equal to μ_k^*, the optimal value λ^* is obtained as the following form derived from $\sum_{k=1}^{n} \mu_k p_k = c$:

$$\begin{aligned} &\sum_{k=1}^{K} p_k \left(\frac{k}{n} + \lambda p_k \right) = c \\ &\Leftrightarrow \lambda \sum_{k=1}^{K} p_k^2 + \frac{1}{n} \sum_{k=1}^{K} k p_k = c \\ &\Leftrightarrow \lambda = \frac{c - \frac{1}{n} \sum_{k=1}^{K} k p_k}{\sum_{k=1}^{K} p_k^2} \end{aligned} \tag{10}$$

Therefore, the optimal membership value μ_k^* is obtained as follows:

$$\mu_k^* = \frac{k}{n} + p_k \left(\frac{c - \frac{1}{n} \sum_{k=1}^{K} k p_k}{\sum_{k=1}^{K} p_k^2} \right) \tag{11}$$

We find that each optimal solution in (8) is dependent on λ, it is particularly important which range $[L_k, U_k]$ is included in λ. If we find that the number of indices of μ_k^* is obtained, each optimal membership value is also automatically obtained. Consequently, we develop the following interactive algorithm for the appropriate membership function.

Interactive approach to obtain the membership function

STEP1: Receive probability density function p_k and set values of c. If possible, set μ_k^L and μ_k^U. Go to STEP2.

STEP2: Arrange L_k and U_k, $(k = 1,2,\ldots,n)$ in ascending order, and redefine parameter s_j, $(j = 1,2,\ldots,N)$ where N is the total number of different indices of L_k and U_k, $(k = 1,2,\ldots,n)$ Set $s_0 \leftarrow 0$, $j \leftarrow 0$. Go to STEP3.

STEP3: Set $\lambda \leftarrow s_j$ and solve equations (5) under λ considering each range whose membership value is μ_k^L, μ_k^U or μ_k^*. Go to STEP4.

STEP4: Calculate $\sum_{k=1}^{n} \mu_k p_k$. If $\sum_{k=1}^{n} \mu_k p_k < c$, then set $j \leftarrow j+1$, and return to STEP3. If $\sum_{k=1}^{n} \mu_k p_k \ge c$, set $L_\lambda \leftarrow s_{j-1}$, $U_\lambda \leftarrow s_j$ and fix indices of μ_k^L, μ_k^U and μ_k^*.

STEP5: Obtain each optimal membership value μ_k^* from formula (9) under $L_\lambda \le \lambda \le U_\lambda$. Go to STEP6.

STEP6: If the decision maker is satisfied with the current membership function, terminate this algorithm. If not, reset parameter set values of c, and μ_k^L, μ_k^U, if possible. Return to STEP2.

In our interactive approach, if the decision maker is not satisfied with the optimal membership values, she/he can change some parameters flexibly. Furthermore, the main element of this algorithm is to check each range whose membership value is μ_k^L, μ_k^U or μ_k^*.on current value of λ straightforwardly. We can check the above ranges on linear complexity, because we initially arrange L_k and U_k, $(k = 1,2,\ldots,n)$ in ascending order. Therefore, the optimal solution is easily and efficiently obtained using our approach.

IV. Conclusion

In this paper, we have developed an interactive approach of an appropriate membership function by extending previous useful approach for a piecewise linear membership function under the given probability density function. The proposed approach has been formulated as a more general mathematical programming problem than previous approaches, but it was difficult to solve the proposed problem directly. Furthermore, we have developed the efficient constructing algorithm to obtain the optimal membership values using nonlinear programming approaches under some natural assumptions. Our proposed approach includes advantages of both statistics and interactive approach based on subjectivity. In addition, our approach can be applied to various types of membership functions in the real world. Therefore, the membership function will be more statistically and objectively obtained.

As a future work, we will develop more efficient and versatile approaches for original problem (4) without introducing any assumptions. Particularly, the application of this paper's approach is restricted to monotonous increasing or decreasing pdfs. With respect to membership functions, it needs to hold monotonicity from membership value 0 to 1. Therefore, we need to improve our approach to apply more general situations. Furthermore, we will consider our proposed approach to real-world decision making.

References

[1] B. Bharathi and V.V.S. Sarma, "Estimation of fuzzy membership from histograms", Information Sciences, 35, pp. 43-59, 1985.

[2] J.L. Chameau and J.C. Santamarina, "Membership function I: Comparing methods of Measurement", International Journal of Approximate Reasoning, 1, pp.287-301, 1987.

[3] H.D. Cheng and J.R. Cheng, “Automatically detemine the membership function based on the maximum entropy principle”, Information Sciences, 96, pp.163-182, 1997.

[4] H.D. Cheng and Y.H. Cheng, “Thresholding based on fuzzy partition of 2D histogram”, Proceedings of IEEE International Conference on Pattern Recognition, 2, pp. 1616-1618, 1997.

[5] M.R. Civanlar and H.J. Trussell, “Constructing membership functions using statistical data”, Fuzzy Sets and Systems, 18, pp. 1-13, 1986.

[6] E. Czogała, S. Gottwald, and W. Pedrycz, “Contribution to application of energy measure of fuzzy sets”, Fuzzy Sets and Systems, 8, pp. 205-214, 1982.

[7] S. Gottwald, “A note on measures of fuzziness”, Elektron Informationsverarb Kybernet, 15, pp. 221-223, 1979.

[8] T. Hasuike, H. Katagiri, and H. Tsubaki, “A constructing algorithm for appropriate piecewise linear membership function based on statistics and information theory”, Procedia Computer Science, 60, pp. 994-1003, 2015.

[9] T. Hasuike, H. Katagiri, and H. Tsubaki, “An interactive algorithm to construct an appropriate nonlinear membership function using information theory and statistical method”, Procedia Computer Science, 61, pp. 32-37, 2015.

[10] T. Hasuike, H. Katagiri, and H. Tsubaki, “Constructing an appropriate membership function integrating fuzzy Shannon entropy and human’s interval estimation”, ICIC Express Letters, 8(3), pp. 809-813, 2014.

[11] G. Nieradka and B. Butkiewicz, “A method for automatic membership function estimation based on fuzzy measures”, Proceedings of IFSA2007, LNAI4529, pp. 451-460, 2007.

[12] A. Yoshikawa, “Influence of procedure for interacive identification method on forms of identified membership functions”, Japan Society for Fuzzy Theory and Intelligent Informatices (in Japanese), 19(1), pp. 69-78, 2007