The Symmetric and Antipersymmetric Solutions of the Matrix Equation $A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C$ and Its Optimal Approximation

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***Abstract*—A matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be symmetric and antipersymmetric matrix if $a_{ij} = a_{ji} = -a_{n-j+1,n-i+1}$ for all $1 \le i, j \le n$. Peng gave the bisymmetric solutions of the matrix equation $A_1X_1B_1+A_2X_2B_2+\ldots+A_lX_lB_l = C$, where $[X_1, X_2, \ldots, X_l]$ is a real matrices group. Based on this work, an adjusted iterative method is proposed to find the symmetric and antipersymmetric solutions of the above matrix equation. When the matrix equation is consistent, for any initial symmetric and antipersymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, the least norm symmetric and antipersymmetric solution group can be obtained. In addition, for a given symmetric and antipersymmetric matrix group $[\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_l]$, the optimal approximation symmetric and antipersymmetric solution group can be obtained. Given numerical examples show that the iterative method is efficient.**

***Index Terms*—Iterative method, Matrix equation, Symmetric and antipersymmetric, Least-norm solution group, Optimal approximation solution**

I. Introduction

A Matrix $A = (a_{ij}) \in R^{n\times n}$ is said to be symmetric and antipersymmetric matrix if $a_{ij} = a_{ji} = -a_{n-j+1,n-i+1}$ for all $1 \le i, j \le n$. Let $R^{m\times n}$, $SR^{n\times n}$ and SA^n denote the set of $m\times n$ real matrices, $n\times n$ real symmetric matrices and $n \times n$ real symmetric and antipersymmetric matrices respectively. $S_n(S_n = (e_n, e_{n-1}, \ldots, e_1))$ denotes the $n\times n$ reverse unit matrix (e_i denotes ith column of $n\times n$ unit matrix). The superscripts T represents the transpose of a matrix. In space $R^{m\times n}$, we define inner products as: $< A, B >= \text{trace}(B^TA)$ for all $A, B \in R^{m\times n}$. Then the norm of A generated by this inner product is Frobenius norm and denoted by $\| A \|$.

Problem I. (See [1]) Given $A_i \in R^{p\times n_i}, B_i \in R^{n_i\times q} (i = 1, 2, \ldots, l)$ and $C \in R^{p\times q}$, find matrix group $[X_1, X_2, \ldots, X_l]$ with $X_i \in SA^{n_i}, i = 1, 2, \ldots, l$, such that

$$A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C. \qquad (1)$$

Problem II. (See [2]) When Problem I is consistent, let S_E denote its solution group set, for given matrix group $[\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_l]$ with $\bar{X}_i \in SA^{n_i}(i = 1, 2, \ldots, l)$, find $[\widehat{X_1}, \widehat{X_2}, \ldots, \widehat{X_l}] \in S_E$ with $\widehat{X_i} \in SA^{n_i}$, such that

$$\begin{aligned} &\| \widehat{X_1} - \bar{X}_1 \|^2 + \| \widehat{X_2} - \bar{X}_2 \|^2 + \ldots + \| \widehat{X_l} - \bar{X}_l \|^2 \\ &= \min_{[X_1,X_2,\ldots,X_l]\in S_E} [\| X_1 - \bar{X}_1 \|^2 + \| X_2 - \bar{X}_2 \|^2 \\ &+ \ldots + \| X_l - \bar{X}_l \|^2]. \end{aligned} \qquad (2)$$

In [3], an iterative method is constructed to find the bisymmetric solutions of matrix equation $A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C$, where $[X_1, X_2, \ldots, X_l]$ is real matrices group. In this paper, by adjusting the algorithm in [3], we can find symmetric and antipersymmetric solutions of the above matrix equation. In electricity, control theory, and processing of digital signals, symmetric and antipersymmetric matrices have wide application.

II. Iterative methods for solving Problem I and II

Firstly we introduce some lemmas used to solve Problem I.

Lemma 2.1.(see [4]) A matrix $X \in SA^n$ if and only if $X = X^T = -S_nXS_n$.

Lemma 2.2. Suppose that a matrix $X \in SR^{n\times n}$, then $X - S_nXS_n \in SA^n$.

Lemma 2.3. Suppose that $A \in R^{n\times n}, X \in SA^n$, then

$$< A, X >= \langle \frac{1}{4}[A + A^T - S_n(A + A^T)S_n], X\rangle.$$

Proof

$$\begin{aligned} &\langle \tfrac{1}{4}[A + A^T - S_n(A + A^T)S_n], X\rangle \\ = &\tfrac{1}{4}[\langle A, X\rangle + \langle A^T, X\rangle - \langle S_nAS_n, X\rangle - \langle S_nA^TS_n, X\rangle] \\ = &\tfrac{1}{4}[\langle A, X\rangle + \langle A^T, X^T\rangle - \langle A, S_nXS_n\rangle - \langle A^T, S_nXS_n\rangle] \\ = &\langle A, X\rangle. \end{aligned} \qquad (3)$$

Next, by adjusting the method in [3], an iterative method to obtain the symmetric and antipersymmetric solution groups of Problem I.

Algorithm 2.1.

Step 1. Input matrices $A_i \in R^{p\times n_i}, B_i \in R^{n_i\times q}, X_i^{(0)} \in SA^{n_i\times n_i} (i = 1, 2, \cdots, l)$ and $C \in R^{p\times q}$;

Step 2. Calculate $R_0 = C - \sum_{i=1}^{l} A_iX_i^{(0)}B_i$;
$Y_{0,i} = A_i^TR_0B_i^T, i = 1, 2, \cdots, l$;
$P_{0,i} = \frac{1}{4}[Y_{0,i} + Y_{0,i}^T - S_{n_i}Y_{0,i}S_{n_i} - S_{n_i}Y_{0,i}^TS_{n_i}]$;
$i = 1, 2, \cdots, l$;
k:=0;

Step 3. If $R_k = 0$, then stop; else, $k := k + 1$;

Step 4. Calculate
$X_i^{(k)} = X_i^{(k-1)} + \frac{\|R_{k-1}\|^2}{\sum_{j=1}^{l}\|P_{k-1,j}\|^2}P_{k-1,i}, i = 1, \cdots, l$;

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$R_k = C - \sum_{i=1}^{l} A_i X_i^{(k)} B_i$
$= R_{k-1} - \frac{\|R_{k-1}\|^2}{\sum_{j=1}^{l}\|P_{k-1,j}\|^2}(\sum_{i=1}^{l} A_i P_{k-1,i} B_i);$
$Y_{k,i} = A_i^T R_k B_i^T, i = 1, 2, \cdots, l;$
$P_{k,i} = \frac{1}{4}[Y_{k,i} + Y_{k,i}^T - S_{n_i} Y_{k,i} S_{n_i} - S_{n_i} Y_{k,i}^T S_{n_i}] + \frac{\|R_k\|^2}{\|R_{k-1}\|^2} P_{k-1,i}, i = 1, 2, \cdots, l;$
Step 5. goto step 3.

Similar with the discussions in [3], we need verify whether Algorithm 2.1 has the following basic properties.

Lemma 2.4 Suppose that the sequences $\{R_i\}$, $\{P_{i,r}\}(R_i \neq 0, i = 0, 1, 2, \cdots, k, r = 1, 2, \cdots, l)$ are generated by Algorithm 2.1, then

$$\begin{array}{c}\langle R_i, R_j\rangle = 0, \sum_{r=1}^{l}\langle P_{i,r}, P_{j,r}\rangle = 0, \\ (i, j = 0, 1, 2, \cdots, k, i \neq j).\end{array} \tag{4}$$

Proof. We verify the conclusion by induction with the method of [3].

Step 1. Show that $< R_0, R_1 >= 0$ and $\sum_{r=1}^{l} < P_{0,r}, P_{1,r} >= 0$, when $k = 1$.

$$\begin{aligned} & < R_0, R_1 >=< R_0, R_0 - \frac{||R_0||^2}{\sum_{r=1}^{l}||P_{0,r}||^2}\sum_{r=1}^{l} A_r P_{0,r} B_r > \\ = \ & ||R_0||^2 - \frac{||R_0||^2}{\sum_{r=1}^{l}||P_{0,r}||^2}\sum_{r=1}^{l} < \tfrac{1}{4}[Y_{0,r} + Y_{0,r}^T \\ & -S_{n_r}(Y_{0,r} + Y_{0,r}^T)S_{n_r}], P_{0,r} >= 0 \end{aligned} \tag{5}$$

and

$$\begin{aligned} & \sum_{r=1}^{l} < P_{0,r}, P_{1,r} > \\ = \ & \sum_{r=1}^{l} < P_{0,r}, \tfrac{1}{4}[Y_{1,r} + Y_{1,r}^T - S_{n_r} Y_{1,r} S_{n_r} \\ & -S_{n_r} Y_{1,r}^T S_{n_r}] + \tfrac{||R_1||^2}{||R_0||^2} P_{0,r} > \\ = \ & \sum_{r=1}^{l} < P_{0,r}, \tfrac{1}{4}[Y_{1,r} + Y_{1,r}^T - S_{n_r} Y_{1,r} S_{n_r} \\ & -S_{n_r} Y_{1,r}^T S_{n_r}] > + \tfrac{||R_1||^2}{||R_0||^2}\sum_{r=1}^{l}||P_{0,r}||^2 = 0. \end{aligned} \tag{6}$$

Step 2. Suppose that (4) holds when $k = s$, then when $k = s + 1$,

$$\begin{aligned} & < R_s, R_{s+1} > \\ = \ & < R_s, R_s - \frac{||R_s||^2}{\sum_{r=1}^{l}||P_{s,r}||^2}\sum_{r=1}^{l} A_r P_{s,r} B_r > \\ = \ & ||R_s||^2 - \frac{||R_s||^2}{\sum_{r=1}^{l}||P_{s,r}||^2}\sum_{r=1}^{l} < \tfrac{1}{4}[Y_{s,r} + Y_{s,r}^T \\ & -S_{n_r}(Y_{s,r} + Y_{s,r}^T)S_{n_r}], P_{s,r} >= 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} & \sum_{r=1}^{l} < P_{s,r}, P_{s+1,r} > \\ = \ & \sum_{r=1}^{l} < P_{s,r}, \tfrac{1}{4}[Y_{s+1,r} + Y_{s+1,r}^T - S_{n_r} Y_{s+1,r} S_{n_r} \\ & -S_{n_r} Y_{s+1,r}^T S_{n_r}] + \tfrac{||R_{s+1}||^2}{||R_s||^2} P_{s,r} > \\ = \ & \sum_{r=1}^{l} < P_{s,r}, \tfrac{1}{4}[Y_{s+1,r} + Y_{s+1,r}^T - S_{n_r} Y_{s+1,r} S_{n_r} \\ & -S_{n_r} Y_{s+1,r}^T S_{n_r}] > + \tfrac{||R_{s+1}||^2}{||R_s||^2}\sum_{r=1}^{l}||P_{s,r}||^2 = 0. \end{aligned} \tag{8}$$

For $j = 1, 2, \cdots, s - 1$, we have that

$$\begin{aligned} & < R_j, R_{s+1} > \\ = \ & < R_j, R_s - \frac{||R_s||^2}{\sum_{r=1}^{l}||P_{s,r}||^2}\sum_{r=1}^{l} A_r P_{s,r} B_r > \\ = \ & -\frac{||R_s||^2}{\sum_{r=1}^{l}||P_{s,r}||^2}\sum_{r=1}^{l} < \tfrac{1}{4}[Y_{j,r} + Y_{j,r}^T \\ & -S_{n_r}(Y_{j,r} + Y_{j,r}^T)S_{n_r}], P_{s,r} >= 0 \end{aligned} \tag{9}$$

and

$$\begin{aligned} & \sum_{r=1}^{l} < P_{j,r}, P_{s+1,r} > \\ = \ & \sum_{r=1}^{l} < P_{j,r}, \tfrac{1}{4}[Y_{s+1,r} + Y_{s+1,r}^T - S_{n_r} Y_{s+1,r} S_{n_r} \\ & -S_{n_r} Y_{s+1,r}^T S_{n_r}] > + \tfrac{||R_{s+1}||^2}{||R_s||^2}\sum_{r=1}^{l} < P_{j,r}, P_{s,r} > \\ = \ & 0. \end{aligned} \tag{10}$$

From steps 1 and 2, the conclusion $< R_i, R_j >= 0$ and $\sum_{r=1}^{l} < P_{i,r}, P_{j,r} >= 0$ hold for all $i, j = 0, 1, 2, \cdots, k(i \neq j)$ by the principle of induction.

Next, we verify that if Problem I is consistent, for any initial symmetric and antipersymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, the matrix group sequences $[X_1^{(k)}, X_2^{(k)}, \ldots, X_l^{(k)}]$ generated by the iterative method converge to a symmetric and antipersymmetric solution group within at most pq iterative steps in the absence of roundoff errors.

Lemma 2.5 Suppose that Problem I is consistent, and $[X_1^*, X_2^*, \cdots, X_l^*]$ is a symmetric and antipersymmetric solution group. For any initial symmetric and antipersymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \cdots, X_l^{(0)}]$, the sequences $\{X_j^{(i)}\}$,$\{R_i\}$, $\{P_{i,j}\}(j = 1, 2, \cdots, l)$ generated by Algorithm 2.1 satisfy

$$\sum_{j=1}^{l} < P_{i,j}, X_j^* - X_j^{(i)} >= ||R_i||^2 (i = 0, 1, 2, \cdots). \tag{11}$$

Proof. We prove the conclusion by induction with the method in [3]. When $i = 0$,

$$\begin{aligned} & \sum_{j=1}^{l} < P_{0,j}, X_j^* - X_j^{(0)} > \\ = \ & \sum_{j=1}^{l} < \tfrac{1}{4}[Y_{0,j} + Y_{0,j}^T - S_{n_j} Y_{0,j} S_{n_j} - S_{n_j} Y_{0,j}^T S_{n_j}], \\ & X_j^* - X_j^{(0)} >= ||R_0||^2. \end{aligned} \tag{12}$$

Suppose that the conclusion holds for $i = s(s \geq 0)$, that is,

$$\sum_{j=1}^{l} < P_{s,j}, X_j^* - X_j^{(s)} >= ||R_s||^2, \tag{13}$$

then when $i = s + 1$,

$$\begin{aligned} & \sum_{j=1}^{l} < P_{s+1,j}, X_j^* - X_j^{(s+1)} > \\ & = \sum_{j=1}^{l} < \tfrac{1}{4}[Y_{s+1,j} + Y_{s+1,j}^T - S_{n_j} Y_{s+1,j} S_{n_j} \\ & -S_{n_j} Y_{s+1,j}^T S_{n_j}] + \tfrac{||R_{s+1}||^2}{||R_s||^2} P_{s,j}, X_j^* - X_j^{(s+1)} > \\ & = ||R_{s+1}||^2. \end{aligned} \tag{14}$$

By the principle of induction, the conclusion $\sum_{j=1}^{l} < P_{i,j}, X_j^* - X_j^{(i)} >= ||R_i||^2$ holds for all $i = 0, 1, 2, \cdots$.

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Theorem 2.1 Suppose that Problem I is consistent, then for any initial symmetric and an antipersymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \cdots, X_l^{(0)}]$, a symmetric and an antipersymmetric solution group can be obtained within at most pq iteration steps for Algorithm 2.1.

Theorem 2.2 Problem I is consistent if and only if there exists a nonnegative integer number k, such that $R_k = 0$ or $P_{k,j} \neq 0$ for some $j \in \{1, 2, \cdots, l\}$.

The proofs of the two theorems are similar with the proofs of Theorem 2.1 and 2.2 in [3], so they are omitted. From Lemma 2.5, if there exists a nonnegative integer number k such that $P_{k,j} = 0$ for all $j \in \{1, 2, \cdots, l\}$, but $R_k \neq 0$, then Problem I is inconsistent. Hence, the solvability of Problem I can be judged automatically by Algorithm 2.1.

Lemma 2.6 Problem I has a symmetric and antipersymmetric solution groups if and only if the following linear matrix equations is consistent

$$\begin{cases} A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C \\ B_1^TX_1A_1^T + B_2^TX_2A_2^T + \cdots + B_l^TX_lA_l^T = C^T \\ A_1S_{n_1}X_1S_{n_1}B_1 + \cdots + A_lS_{n_l}X_lS_{n_l}B_l = -C \\ B_1^TS_{n_1}X_1S_{n_1}A_1^T + \cdots + B_l^TS_{n_l}X_lS_{n_l}A_l^T = -C^T. \end{cases} \tag{15}$$

Proof. Suppose that Problem I has a symmetric and antipersymmetric solution group $[Y_1, Y_2, \cdots, Y_l]$, then $Y_i = Y_i^T = -S_{n_i}Y_iS_{n_i} (i = 1, 2, \cdots, l)$, and

$$\begin{array}{c} A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l = C \\ B_1^TY_1A_1^T + B_2^TY_2A_2^T + \cdots + B_l^TY_lA_l^T = \\ (A_1Y_1^TB_1 + A_2Y_2^TB_2 + \cdots + A_lY_l^TB_l)^T = C^T \\ A_1S_{n_1}Y_1S_{n_1}B_1 + \cdots + A_lS_{n_l}Y_lS_{n_l}B_l = \\ -(A_1Y_1B_1 + \cdots + A_lY_lB_l) = -C \\ B_1^TS_{n_1}Y_1S_{n_1}A_1^T + \cdots + B_l^TS_{n_l}Y_lS_{n_l}A_l^T = \\ -(B_1^TY_1A_1^T + \cdots + B_l^TY_lA_l^T) = -C^T. \end{array} \tag{16}$$

Hence the symmetric and antipersymmetric solution group $[Y_1, Y_2, \cdots, Y_l]$ is a solution group of the linear matrix equations (15), namely, the linear matrix equations (15) is consistent.

Conversely, suppose that the linear matrix equations (15) is consistent, then there exists a matrix group $[\bar{Y}_1, \bar{Y}_2, \cdots, \bar{Y}_l] (\bar{Y}_i \in R^{n_i \times n_i}, i = 1, 2, \cdots, l)$, such that

$$\begin{cases} A_1\bar{Y}_1B_1 + A_2\bar{Y}_2B_2 + \cdots + A_l\bar{Y}_lB_l = C \\ B_1^T\bar{Y}_1A_1^T + B_2^T\bar{Y}_2A_2^T + \cdots + B_l^T\bar{Y}_lA_l^T = C^T \\ A_1S_{n_1}\bar{Y}_1S_{n_1}B_1 + \cdots + A_lS_{n_l}\bar{Y}_lS_{n_l}B_l = -C \\ B_1^TS_{n_1}\bar{Y}_1S_{n_1}A_1^T + \cdots + B_l^TS_{n_l}\bar{Y}_lS_{n_l}A_l^T = -C^T. \end{cases} \tag{17}$$

Let $Y_i = \frac{\bar{Y}_i + \bar{Y}_i^T - S_{n_i}(\bar{Y}_i + \bar{Y}_i^T)S_{n_i}}{4}$, then $Y_i \in SA^{n \times n}$, and

$$\begin{array}{rl} & A_1Y_1B_1 + A_2Y_2B_2 + \cdots + A_lY_lB_l \\ = & A_1 \cdot \frac{\bar{Y}_1 + \bar{Y}_1^T - S_{n_1}(\bar{Y}_1 + \bar{Y}_1^T)S_{n_1}}{4} \cdot B_1 \\ + & A_2 \cdot \frac{\bar{Y}_2 + \bar{Y}_2^T - S_{n_2}(\bar{Y}_2 + \bar{Y}_2^T)S_{n_2}}{4} \cdot B_2 + \cdots \\ + & A_l \cdot \frac{\bar{Y}_l + \bar{Y}_l^T - S_{n_l}(\bar{Y}_l + \bar{Y}_l^T)S_{n_l}}{4} \cdot B_l = C. \end{array} \tag{18}$$

Therefore, $[Y_1, Y_2, \cdots, Y_l]$ is a symmetric and antipersymmetric solution group of Problem I.

Remark From the proof process of Lemma 2.6, any symmetric and antipersymmetric solution groups of Problem I must be the solution groups of the linear matrix equations (15). Let S_E' denote the solution group set of the linear matrix equations (15), then $S_E \subseteq S_E'$, where S_E is the solution group set of Problem I. Similar with the discussions in [3], checking $[X_1^*, X_2^*, \cdots, X_l^*]$ being the least Frobenius norm symmetric and antipersymmetric solution group of problem I is equivalent to verify that $[X_1^*, X_2^*, \cdots, X_l^*]$ is the least Frobenius norm symmetric and antipersymmetric solution group of the linear matrix equation (15).

Lemma 2.7 (see [3]) Suppose that the consistent systems of the linear equations $Ax = b$ has a solution $x^* \in R(A^T)$, the x^* is an unique least Frobenius norm solution of the systems of linear equations.

Next, we verify that if let the initial symmetric and antipersymmetric matrices be $X_j^{(0)} = A_j^THB_j^T + B_jH^TA_j - S_{n_j}(A_j^THB_j^T + B_jH^TA_j)S_{n_j}, j = 1, 2, \ldots, l$, where H is arbitrary, then the symmetric and antipersymmetric solution group $[X_1^*, X_2^*, \ldots, X_l^*]$ obtained by the iterative method is the least Frobenius norm symmetric and antipersymmetric solution group.

Theorem 2.3 Suppose that Problem I is consistent. If we choose the initial symmetric and antipersymmetric matrices $X_j^{(0)} = A_j^THB_j^T + B_jH^TA_j - S_{n_j}(A_j^THB_j^T + B_jH^TA_j)S_{n_j}, j = 1, 2, \cdots, l$. H is arbitrary. Let $X_1^{(0)} = 0, X_2^{(0)} = 0, \cdots, X_l^{(0)} = 0$, then the symmetric and antipersymmetric solution group $[X_1^*, X_2^*, \cdots, X_l^*]$ obtained by Algorithm 2.1 is the unique least Frobenius norm symmetric and antipersymmetric solution group of Problem I.

Proof. With the proof method in [3], in term of Theorem 2.1, if we take $X_i^{(0)} = A_i^THB_i^T + B_iH^TA_i - S_{n_i}(A_i^THB_i^T + B_iH^TA_i)S_{n_i}, (i = 1, 2, \cdots, l)$, We can obtain the symmetric and antipersymmetric solution group $[X_1^*, X_2^*, \cdots, X_l^*]$ of problem I in finite iteration steps, and $X_i^* (i = 1, 2, \cdots, l)$ can be expressed as

$$X_i^* = A_i^TYB_i^T + B_iY^TA_i - S_{n_i}(A_i^TYB_i^T + B_iY^TA_i)S_{n_i}, \tag{19}$$

where $Y \in R^{p \times q}$. Next, we will verify that $[X_1^*, X_2^*, \cdots, X_l^*]$ is the unique least Frobenius norm symmetric and antipersymmetric solution group of Problem I. From the Remark, we only need show that $[X_1^*, X_2^*, \cdots, X_l^*]$ is the least Frobenius norm symmetric and antipersymmetric solution group of the linear matrix equation (15).

For matrix $A \in R^{m \times n}$, let $\text{vec}(A)$ denote the following $mn-$vector containing all the entries of matrix A:

$$\text{vec}(A) = (A(:,1)^T, A(:,2)^T, \cdots, A(:,n)^T) \in R^{mn}, \tag{20}$$

where $A(:,i)$ denote ith column of matrix A. $A \otimes B$ denote the Kronecker product of matrices A and B. Then linear matrix equation (15) is equivalent to the system of linear matrix equations

$$\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 & \cdots & B_l^T \otimes A_l \\ A_1 \otimes B_1^T & A_2 \otimes B_2^T & \cdots & A_l \otimes B_l^T \\ (B_1^TS_{n_1}) \otimes (A_1S_{n_1}) & (B_2^TS_{n_2}) \otimes (A_2S_{n_2}) & \cdots & (B_l^TS_{n_l}) \otimes (A_lS_{n_l}) \\ (A_1S_{n_1}) \otimes (B_1^TS_{n_1}) & (A_2S_{n_2}) \otimes (B_2^TS_{n_2}) & \cdots & (A_lS_{n_l}) \otimes (B_l^TS_{n_l}) \end{pmatrix} \cdot \begin{pmatrix} \text{vec}(X_1) \\ \text{vec}(X_2) \\ \vdots \\ \text{vec}(X_l) \end{pmatrix} = \begin{pmatrix} \text{vec}(C) \\ \text{vec}(C^T) \\ -\text{vec}(C) \\ -\text{vec}(C^T) \end{pmatrix}. \tag{21}$$

Noting that

$$\begin{pmatrix} \text{vec}(X_1^*) \\ \text{vec}(X_2^*) \\ \vdots \\ \text{vec}(X_l^*) \end{pmatrix} =$$

$$\begin{pmatrix} \text{vec}(A_1^TYB_1^T + B_1Y^TA_1 - S_{n_1}(A_1^TYB_1^T + B_1Y^TA_1)S_{n_1}) \\ \text{vec}(A_2^TYB_2^T + B_2Y^TA_2 - S_{n_2}(A_2^TYB_2^T + B_2Y^TA_2)S_{n_2}) \\ \vdots \\ \text{vec}(A_l^TYB_l^T + B_lY^TA_l - S_{n_l}(A_l^TYB_l^T + B_lY^TA_l)S_{n_l}) \end{pmatrix}$$

$$
= \begin{pmatrix} B_1 \otimes A_1^T & A_1^T \otimes B_1 & (S_{n_1}B_1)\otimes(S_{n_1}A_1^T) & (S_{n_1}A_1^T)\otimes(S_{n_1}B_1) \\ B_2 \otimes A_2^T & A_2^T \otimes B_2 & (S_{n_2}B_2)\otimes(S_{n_2}A_2^T) & (S_{n_2}A_2^T)\otimes(S_{n_2}B_2) \\ & & \vdots & \\ B_l \otimes A_l^T & A_l^T \otimes B_l & (S_{n_l}B_l)\otimes(S_{n_l}A_l^T) & (S_{n_l}A_l^T)\otimes(S_{n_l}B_l) \end{pmatrix} \begin{pmatrix} \mathrm{vec}(Y) \\ \mathrm{vec}(Y^T) \\ -\mathrm{vec}(Y) \\ -\mathrm{vec}(Y^T) \end{pmatrix}
$$

$$
= \begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 & \cdots & B_l^T \otimes A_l \\ A_1 \otimes B_1^T & A_2 \otimes B_2^T & \cdots & A_l \otimes B_l^T \\ (B_1^T S_{n_1}) \otimes (A_1 S_{n_1}) & (B_2^T S_{n_2}) \otimes (A_2 S_{n_2}) & \cdots & (B_l^T S_{n_l}) \otimes (A_l S_{n_l}) \\ (A_1 S_{n_1}) \otimes (B_1^T S_{n_1}) & (A_2 S_{n_2}) \otimes (B_2^T S_{n_2}) & \cdots & (A_l S_{n_l}) \otimes (B_l^T S_{n_l}) \end{pmatrix}^T \begin{pmatrix} \mathrm{vec}(Y) \\ \mathrm{vec}(Y^T) \\ -\mathrm{vec}(Y) \\ -\mathrm{vec}(Y^T) \end{pmatrix}
$$

$$
\in R\left(\begin{pmatrix} B_1^T \otimes A_1 & B_2^T \otimes A_2 & \cdots & B_l^T \otimes A_l \\ A_1 \otimes B_1^T & A_2 \otimes B_2^T & \cdots & A_l \otimes B_l^T \\ (B_1^T S_{n_1}) \otimes (A_1 S_{n_1}) & (B_2^T S_{n_2}) \otimes (A_2 S_{n_2}) & \cdots & (B_l^T S_{n_l}) \otimes (A_l S_{n_l}) \\ (A_1 S_{n_1}) \otimes (B_1^T S_{n_1}) & (A_2 S_{n_2}) \otimes (B_2^T S_{n_2}) & \cdots & (A_l S_{n_l}) \otimes (B_l^T S_{n_l}) \end{pmatrix}^T \right).
$$

Hence, from Lemma 2.7, $[\mathrm{vec}(X_1^*), \mathrm{vec}(X_2^*), \cdots, \mathrm{vec}(X_l^*)]$ is the unique least Frobenius norm symmetric and antipersymmetric solution group of the matrix equation (21). Since vec operator is isomorphic, $[X_1^*, X_2^*, \cdots, X_l^*]$ is the unique least Frobenius norm symmetric and antipersymmetric solution group of the linear marix equation (15), thus it is also the unique least Frobenius norm symmetric and antiperwymmetric solution group of Problem I.

When Problem I is consistent, its symmetric and antipersymmetric solution group set S_E is nonempty, then

$$
\begin{array}{rl} & A_1X_1B_1 + A_2X_2B_2 + \cdots + A_lX_lB_l = C \\ \Leftrightarrow & A_1(X_1 - \bar{X}_1)B_1 + \cdots + A_l(X_l - \bar{X}_l)B_l \\ = & C - A_1\bar{X}_1B_1 - A_2\bar{X}_2B_2 - \cdots - A_l\bar{X}_lB_l. \end{array} \tag{22}
$$

Let

$$
\widetilde{X_1} = X_1 - \bar{X}_1, \widetilde{X_2} = X_2 - \bar{X}_2, \cdots, \widetilde{X_l} = X_l - \bar{X}_l
$$

and

$$
\widetilde{C} = C - A_1\bar{X}_1B_1 - A_2\bar{X}_2B_2 - \cdots - A_l\bar{X}_lB_l,
$$

then Problem II is equivalent to find the least Frobenius norm symmetric and antipersymmetric solution group of the linear matrix equation

$$
A_1\widetilde{X_1}B_1 + A_2\widetilde{X_2}B_2 + \cdots + A_l\widetilde{X_l}B_l = \widetilde{C}. \tag{23}
$$

By using the Algorithm 2.1, let initial matrices $\widetilde{X_j}^{(0)} = A_j^T H B_j^T + B_j H^T A_j - S_{n_j}(A_j^T H B_j^T + B_j H^T A_j)S_{n_j}, j = 1, 2, \cdots, l$, where H is arbitrary, let $\widetilde{X_1}^{(0)} = 0, \widetilde{X_2}^{(0)} = 0, \cdots, \widetilde{X_l}^{(0)} = 0$, we can obtain the unique least Frobenius norm symmetric and antipersymmetric solution group $[\widetilde{X_1}^*, \widetilde{X_2}^*, \cdots, \widetilde{X_l}^*]$ of the linear matrix equation (23). Once above symmetric and antipersymmetric matrix group $[\widetilde{X_1}^*, \widetilde{X_2}^*, \cdots, \widetilde{X_l}^*]$ is obtained, the unique symmetric and antipersymmetric solution group $[\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_l]$ of problem II can be obtained. So $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_l$ can be denoted as $\hat{X}_1 = \widetilde{X_1}^* + \bar{X}_1, \hat{X}_2 = \widetilde{X_2}^* + \bar{X}_2, \cdots, \hat{X}_l = \widetilde{X_l}^* + \bar{X}_l$.

III. Example

Let $n = 3$, $A_1, A_2, B_1, B_2, A_3, B_3$ (see [5]), C as follows:

$$
A_1 = \begin{pmatrix} 1 & 3 & -2 & 5 & -1 & 4 & 2 & 2 \\ 3 & -7 & 1 & -8 & 2 & -9 & 4 & -8 \\ 3 & -2 & 4 & -4 & 5 & -3 & 5 & -4 \\ 11 & 6 & 12 & 7 & 10 & 4 & 12 & 7 \\ -5 & 1 & -5 & 2 & -3 & 3 & -6 & 2 \\ 9 & 4 & 6 & 3 & 7 & 5 & 8 & 3 \\ 8 & -3 & 9 & -3 & 5 & -6 & 7 & -2 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1 & 4 & -1 & 4 & -1 & 4 & -1 \\ 5 & -1 & 5 & -1 & 5 & -1 & 5 \\ -1 & -2 & -1 & -2 & -1 & -2 & -1 \\ 3 & 9 & 3 & 9 & 3 & 9 & 3 \\ 7 & -8 & 7 & -8 & 7 & -8 & 7 \\ -3 & 5 & -3 & 5 & -3 & 5 & -3 \\ 4 & -6 & 4 & -6 & 4 & -6 & 4 \\ 7 & 2 & 7 & 2 & 7 & 2 & 7 \end{pmatrix}, \tag{24}
$$

$$
A_2 = \begin{pmatrix} 3 & -4 & 1 & -3 & 7 & -8 & 2 & -1 & 4 \\ -1 & 3 & -2 & 4 & -2 & 6 & -1 & 2 & -2 \\ 3 & -5 & 4 & -6 & 6 & -4 & 4 & -3 & 1 \\ 3 & -4 & 1 & -7 & 5 & -2 & 3 & -5 & 6 \\ -1 & 3 & -3 & 6 & -2 & 1 & -5 & -6 & -4 \\ 3 & -5 & 2 & -8 & 1 & -2 & 3 & -7 & 1 \\ 1 & 2 & 5 & 2 & 3 & 5 & 1 & 1 & 4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -9 & 4 & -9 & 4 & -9 & 4 & -9 \\ -2 & 3 & -2 & 3 & -2 & 3 & -2 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 \\ 2 & -6 & 2 & -6 & 2 & -6 & 2 \\ 9 & 3 & 9 & 3 & 9 & 3 & 9 \\ 4 & -13 & 4 & -13 & 4 & -13 & 4 \\ 11 & -5 & 11 & -5 & 11 & -5 & 11 \\ -3 & 7 & -3 & 7 & -3 & 7 & -3 \\ 6 & 1 & 6 & 1 & 6 & 1 & 6 \end{pmatrix}, \tag{25}
$$

$$
A_3 = \begin{pmatrix} -1 & 3 & -1 & 3 & -1 & 3 & -1 & 5 & -2 & 2 \\ 3 & -2 & 3 & -2 & 3 & -2 & 3 & -4 & 1 & -1 \\ 5 & -3 & 5 & -3 & 5 & -3 & 5 & -1 & 2 & -3 \\ 3 & -1 & 3 & -1 & 3 & -1 & 3 & -2 & 2 & -1 \\ -1 & 3 & -1 & 3 & -1 & 3 & -1 & 1 & -1 & 2 \\ 5 & -3 & 5 & -3 & 5 & -3 & 5 & -4 & 3 & -3 \\ 3 & -5 & 3 & -5 & 3 & -5 & 3 & -2 & 5 & 6 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -4 & 9 & -4 & -9 & -3 & 7 & -1 \\ 5 & -3 & 5 & -3 & 6 & -1 & -1 \\ -6 & 2 & -6 & 2 & -6 & 2 & -6 \\ -8 & 4 & -8 & 4 & -8 & 4 & -8 \\ -1 & -8 & -1 & -8 & -1 & -8 & -1 \\ -3 & 2 & -3 & 2 & -3 & 2 & -3 \\ -1 & -2 & -1 & -2 & -1 & -2 & -1 \\ 4 & -7 & 4 & -7 & 4 & -7 & 4 \\ 3 & 12 & 3 & 12 & 3 & 12 & 3 \\ 2 & -5 & 2 & -5 & 2 & -5 & 2 \end{pmatrix}, \tag{26}
$$

$$
C = \begin{pmatrix} 1249 & 1971 & 1249 & 3591 & 1205 & 2243 & 703 \\ -3125 & 2699 & -3125 & 2141 & -3214 & 2397 & -2312 \\ -2013 & 2494 & -2013 & 3250 & -2285 & 2118 & -759 \\ -330 & 3174 & -330 & 3354 & -460 & 2954 & 360 \\ -1618 & -3192 & -1618 & -2724 & -1574 & -3000 & -2116 \\ -2725 & -1723 & -2725 & -1831 & -2931 & -2159 & -1435 \\ 38 & 2430 & 38 & 1962 & -104 & 2042 & 1124 \end{pmatrix}. \tag{27}
$$

Let $X_1^{(0)} = 0, X_2^{(0)} = 0, \cdots, X_l^{(0)} = 0$ with the precision of 10^{-10}. Using the Algorithm 2.1, we can obtain the unique least Frobenius norm symmetric and antipersymmetric solution group in 40 iteration steps as follows:

$$
X_1^{(43)} = \begin{pmatrix} -4.8004 & -4.6670 & 18.5424 & -9.9899 & 4.3252 & -19.4939 & 4.6241 & -0.0000 \\ -4.6670 & 13.7171 & -9.3743 & 11.6301 & -7.4657 & 7.9113 & -0.0000 & -4.6241 \\ 18.5424 & -9.3743 & 2.0803 & 10.8122 & 6.3803 & 0.0000 & -7.9113 & 19.4939 \\ -9.9899 & 11.6301 & 10.8122 & 9.9503 & 0.0000 & -6.3803 & 7.4657 & -4.3252 \\ 4.3252 & -7.4657 & 6.3803 & 0.0000 & -9.9503 & -10.8122 & -11.6301 & 9.9899 \\ -19.4939 & 7.9113 & -0.0000 & -6.3803 & -10.8122 & -2.0803 & 9.3743 & -18.5424 \\ 4.6241 & -0.0000 & -7.9113 & 7.4657 & -11.6301 & 9.3743 & -13.7171 & 4.6670 \\ -0.0000 & -4.6241 & 19.4939 & -4.3252 & 9.9899 & -18.5424 & 4.6670 & 4.8004 \end{pmatrix}, \tag{28}
$$

$$
X_2^{(43)} = \begin{pmatrix} -2.1038 & -13.6654 & -4.6350 & 7.2188 & -7.3391 & 0.9022 & -9.7685 & -8.3408 & 0.0000 \\ -13.6654 & -8.2656 & -0.7813 & 6.7475 & -7.3618 & -0.8809 & 0.6753 & -0.0000 & 8.3408 \\ -4.6350 & -0.7813 & 0.4840 & 1.1274 & 14.4273 & 8.9317 & -0.0000 & -0.6753 & 9.7685 \\ 7.2188 & 6.7475 & 1.1274 & 3.9360 & 4.9533 & -0.0000 & -8.9317 & 0.8809 & -0.9022 \\ -7.3391 & -7.3618 & 14.4273 & 4.9533 & -0.0000 & -4.9533 & -14.4273 & 7.3618 & 7.3391 \\ 0.9022 & -0.8809 & 8.9317 & 0.0000 & -4.9533 & -3.9360 & -1.1274 & -6.7475 & -7.2188 \\ -9.7685 & 0.6753 & -0.0000 & -8.9317 & -14.4273 & -1.1274 & -0.4840 & 0.7813 & 4.6350 \\ -8.3408 & -0.0000 & -0.6753 & 0.8809 & 7.3618 & -6.7475 & 0.7813 & 8.2656 & 13.6654 \\ -0.0000 & 8.3408 & 9.7685 & -0.9022 & 7.3391 & -7.2188 & 4.6350 & 13.6654 & 2.1038 \end{pmatrix}, \tag{29}
$$

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$$X_3^{(43)} = \begin{pmatrix} -12.6570 & -7.1179 & -5.5972 & 0.4250 & 0.4193 & -2.3071 & 0.8348 & -16.0000 & 0.0000 & -0.0000 \\ -7.1179 & 3.8188 & -11.7002 & 5.8563 & -12.2392 & 2.3249 & -8.9426 & -6.0000 & -0.0000 & -0.0000 \\ -5.5972 & -11.7002 & 4.3770 & 3.0701 & 4.9764 & 3.3717 & 3.2658 & 0.0000 & 6.0000 & 16.0000 \\ 0.4250 & 5.8563 & 3.0701 & 5.9834 & -0.8825 & 2.8275 & 0.0000 & -3.2658 & 8.9426 & -0.8348 \\ 0.4193 & -12.2392 & 4.9764 & -0.8825 & 0.3284 & 0.0000 & -2.8275 & -3.3717 & -2.3249 & 2.3071 \\ -2.3071 & 2.3249 & 3.3717 & 2.8275 & -0.0000 & -0.3284 & 0.8825 & -4.9764 & 12.2392 & -0.4193 \\ 0.8348 & -8.9426 & 3.2658 & 0.0000 & -2.8275 & 0.8825 & -5.9834 & -3.0701 & -5.8563 & -0.4250 \\ -16.0000 & -6.0000 & -0.0000 & -3.2658 & -3.3717 & -4.9764 & -3.0701 & -4.3770 & 11.7002 & 5.5972 \\ 0.0000 & 0.0000 & 6.0000 & 8.9426 & -2.3249 & 12.2392 & -5.8563 & 11.7002 & -3.8188 & 7.1179 \\ 0.0000 & -0.0000 & 16.0000 & -0.8348 & 2.3071 & -0.4193 & -0.4250 & 5.5972 & 7.1179 & 12.6570 \end{pmatrix}. \tag{30}$$

$$\begin{aligned} & \|R_{43}\| \\ = \ & \|C - A_1X_1^{(43)}B_1 - A_2X_2^{(43)}B_2 - A_3X_3^{(43)}B_3\| \\ = \ & 8.4055 \times 10^{-11}. \end{aligned} \tag{31}$$

Let

$$\bar{X}_1 = \begin{pmatrix} -18 & -6 & 22 & 14 & 6 & -1 & 19 & 0 \\ -6 & 4 & 2 & -22 & -5 & -5 & 0 & -19 \\ 22 & 2 & 20 & 1 & -2 & 0 & 5 & 1 \\ 14 & -22 & 1 & -38 & 0 & 2 & 5 & -6 \\ 6 & -5 & -2 & 0 & 38 & -1 & 22 & -14 \\ -1 & -5 & 0 & 2 & -1 & -20 & -2 & -22 \\ 19 & 0 & 5 & 5 & 22 & -2 & -4 & 6 \\ 0 & -19 & 1 & -6 & -14 & -22 & 6 & 18 \end{pmatrix}, \tag{32}$$

$$\bar{X}_2 = \begin{pmatrix} -28 & -25 & -4 & -1 & 2 & -22 & 4 & -6 & 0 \\ -25 & -4 & -20 & -9 & -8 & 4 & 9 & 0 & 6 \\ -4 & -20 & -12 & -2 & -9 & -23 & 0 & -9 & -4 \\ -1 & -9 & -2 & -18 & -6 & 0 & 23 & -4 & 22 \\ 2 & -8 & -9 & -6 & 0 & 6 & 9 & 8 & -2 \\ -22 & 4 & -23 & 0 & 6 & 18 & 2 & 9 & 1 \\ 4 & 9 & 0 & 23 & 9 & 2 & 12 & 20 & 4 \\ -6 & 0 & -9 & -4 & 8 & 9 & 20 & 4 & 25 \\ 0 & 6 & -4 & 22 & -2 & 1 & 4 & 25 & 28 \end{pmatrix}, \tag{33}$$

$$\bar{X}_3 = \begin{pmatrix} -14 & 12 & -17 & -11 & 13 & -20 & 7 & -17 & -1 & 0 \\ 12 & -4 & -9 & 16 & -19 & 12 & 5 & -1 & 0 & 1 \\ -17 & -9 & 4 & -15 & -2 & 0 & 4 & 0 & 1 & 17 \\ -11 & 16 & -15 & 14 & 14 & -6 & 0 & -4 & -5 & -7 \\ 13 & -19 & -2 & 14 & -22 & 0 & 6 & 0 & -12 & 20 \\ -20 & 12 & 0 & -6 & 0 & 22 & -14 & 2 & 19 & -13 \\ 7 & 5 & 4 & 0 & 6 & -14 & -14 & 15 & -16 & 11 \\ -17 & -1 & 0 & -4 & 0 & 2 & 15 & -4 & 9 & 17 \\ -1 & 0 & 1 & -5 & -12 & 19 & -16 & 9 & 4 & -12 \\ 0 & 1 & 17 & -7 & 20 & -13 & 11 & 17 & -12 & 14 \end{pmatrix}. \tag{34}$$

Using the Algorithm 2.1, we can obtain the unique least Frobenius norm symmetric and antipersymmetric solution group of the linear matrix equation (23) in 43 iteration steps as follows:

$$\widetilde{X_1}^{(49)} = \begin{pmatrix} -22.3594 & 0.3590 & 10.0539 & -7.0761 & -3.9384 & -18.9314 & 0.0191 & -0.0000 \\ 0.3590 & 13.1912 & -3.9789 & 23.4709 & -12.1088 & 11.3756 & 0.0000 & -0.0191 \\ 10.0539 & -3.9789 & -6.2713 & 12.2530 & -4.4749 & 0.0000 & -11.3756 & 18.9314 \\ -7.0761 & 23.4709 & 12.2530 & 34.9120 & -0.0000 & 4.4749 & 12.1088 & 3.9384 \\ -3.9384 & -12.1088 & -4.4749 & 0.0000 & -34.9120 & -12.2530 & -23.4709 & 7.0761 \\ -18.9314 & 11.3756 & 0.0000 & 4.4749 & -12.2530 & 6.2713 & 3.9789 & -10.0539 \\ 0.0191 & -0.0000 & -11.3756 & 12.1088 & -23.4709 & 3.9789 & -13.1912 & -0.3590 \\ 0.0000 & -0.0191 & 18.9314 & 3.9384 & 7.0761 & -10.0539 & -0.3590 & 22.3594 \end{pmatrix}, \tag{35}$$

$$\widetilde{X_2}^{(49)} = \begin{pmatrix} -7.0253 & -19.7409 & -11.4260 & -3.3220 & -8.1438 & -8.0415 & -10.0933 & -13.7059 & 0.0000 \\ -19.7409 & -4.1880 & -18.0239 & 7.9062 & -13.7630 & 14.1348 & 10.1999 & 0.0000 & 13.7059 \\ -11.4260 & -18.0239 & -0.0945 & -6.9317 & 13.0957 & 2.7816 & 0.0000 & -10.1999 & 10.0933 \\ -3.3220 & 7.9062 & -6.9317 & -8.2101 & 2.6929 & 0.0000 & -2.7816 & -14.1348 & 8.0415 \\ -8.1438 & -13.7630 & 13.0957 & 2.6929 & 0.0000 & -2.6929 & -13.0957 & 13.7630 & 8.1438 \\ -8.0415 & 14.1348 & 2.7816 & -0.0000 & -2.6929 & 8.2101 & 6.9317 & -7.9062 & 3.3220 \\ -10.0933 & 10.1999 & -0.0000 & -2.7816 & -13.0957 & 6.9317 & 0.0945 & 18.0239 & 11.4260 \\ -13.7059 & -0.0000 & -10.1999 & -14.1348 & 13.7630 & -7.9062 & 18.0239 & 4.1880 & 19.7409 \\ -0.0000 & 13.7059 & 10.0933 & 8.0415 & 8.1438 & 3.3220 & 11.4260 & 19.7409 & 7.0253 \end{pmatrix}, \tag{36}$$

$$\widetilde{X_3}^{(49)} = \begin{pmatrix} -29.3210 & -3.8856 & -14.5719 & -5.6793 & 5.3683 & -18.4351 & 10.5246 & -33.0000 & -1.0000 & 0.0000 \\ -3.8856 & 21.2442 & -14.7878 & 10.2970 & -18.9837 & 4.4588 & -13.3429 & -7.0000 & 0.0000 & 1.0000 \\ -14.5719 & -14.7878 & 9.7894 & 8.5329 & 10.5066 & 6.4844 & 9.0015 & -0.0000 & 7.0000 & 33.0000 \\ -5.6793 & 10.2970 & 8.5329 & 4.9580 & -0.4648 & 1.7909 & 0.0000 & -9.0015 & 13.3429 & -10.5246 \\ 5.3683 & -18.9837 & 10.5066 & -0.4648 & 1.3763 & 0.0000 & -1.7909 & -6.4844 & -4.4588 & 18.4351 \\ -18.4351 & 4.4588 & 6.4844 & 1.7909 & 0.0000 & -1.3763 & 0.4648 & -10.5066 & 18.9837 & -5.3683 \\ 10.5246 & -13.3429 & 9.0015 & -0.0000 & -1.7909 & 0.4648 & -4.9580 & -8.5329 & -10.2970 & 5.6793 \\ -33.0000 & -7.0000 & 0.0000 & -9.0015 & -6.4844 & -10.5066 & -8.5329 & -9.7894 & 14.7878 & 14.5719 \\ -1.0000 & 0.0000 & 7.0000 & 13.3429 & -4.4588 & 18.9837 & -10.2970 & 14.7878 & -21.2442 & 3.8856 \\ 0.0000 & 1.0000 & 33.0000 & -10.5246 & 18.4351 & -5.3683 & 5.6793 & 14.5719 & 3.8856 & 29.3210 \end{pmatrix}. \tag{37}$$

$\|\widetilde{R_{49}}\| = \|\widetilde{C} - A_1\widetilde{X_1}^{(49)}B_1 - A_2\widetilde{X_2}^{(49)}B_2 - A_3\widetilde{X_3}^{(49)}B_3\| = 7.5659 \times 10^{-11}$.

So, the symmetric and antipersymmetric solution group of Problem II can be obtained as follows:

$$\hat{X_1}^{(49)} = \begin{pmatrix} -40.3594 & -5.6410 & 32.0539 & 6.9239 & 2.0616 & -19.9314 & 19.0191 & -0.0000 \\ -5.6410 & 17.1912 & -1.9789 & 1.4709 & -17.1088 & 6.3756 & 0.0000 & -19.0191 \\ 32.0539 & -1.9789 & 13.7287 & 13.2530 & -6.4749 & 0.0000 & -6.3756 & 19.9314 \\ 6.9239 & 1.4709 & 13.2530 & -3.0880 & -0.0000 & 6.4749 & 17.1088 & -2.0616 \\ 2.0616 & -17.1088 & -6.4749 & 0.0000 & 3.0880 & -13.2530 & -1.4709 & -6.9239 \\ -19.9314 & 6.3756 & 0.0000 & 6.4749 & -13.2530 & -13.7287 & 1.9789 & -32.0539 \\ 19.0191 & -0.0000 & -6.3756 & 17.1088 & -1.4709 & 1.9789 & -17.1912 & 5.6410 \\ 0.0000 & -19.0191 & 19.9314 & -2.0616 & -6.9239 & -32.0539 & 5.6410 & 40.3594 \end{pmatrix}, \tag{38}$$

$$\hat{X_2}^{(49)} = \begin{pmatrix} -35.0253 & -44.7409 & -15.4260 & -4.3220 & -6.1438 & -30.0415 & -6.0933 & -19.7059 & 0.0000 \\ -44.7409 & -8.1880 & -38.0239 & -1.0938 & -21.7630 & 18.1348 & 19.1999 & 0.0000 & 19.7059 \\ -15.4260 & -38.0239 & -12.0945 & -8.9317 & 4.0957 & -20.2184 & 0.0000 & -19.1999 & 6.0933 \\ -4.3220 & -1.0938 & -8.9317 & -26.2101 & -3.3071 & 0.0000 & 20.2184 & -18.1348 & 30.0415 \\ -6.1438 & -21.7630 & 4.0957 & -3.3071 & 0.0000 & 3.3071 & -4.0957 & 21.7630 & 6.1438 \\ -30.0415 & 18.1348 & -20.2184 & -0.0000 & 3.3071 & 26.2101 & 8.9317 & 1.0938 & 4.3220 \\ -6.0933 & 19.1999 & -0.0000 & 20.2184 & -4.0957 & 8.9317 & 12.0945 & 38.0239 & 15.4260 \\ -19.7059 & -0.0000 & -19.1999 & -18.1348 & 21.7630 & 1.0938 & 38.0239 & 8.1880 & 44.7409 \\ -0.0000 & 19.7059 & 6.0933 & 30.0415 & 6.1438 & 4.3220 & 15.4260 & 44.7409 & 35.0253 \end{pmatrix}, \tag{39}$$

$$\hat{X_3}^{(40)} = \begin{pmatrix} -43.3210 & 8.1144 & -31.5719 & -16.6793 & 18.3683 & -38.4351 & 17.5246 & -50.0000 & -2.0000 & 0.0000 \\ 8.1144 & 17.2442 & -23.7878 & 26.2970 & -37.9837 & 16.4588 & -8.3429 & -8.0000 & 0.0000 & 2.0000 \\ -31.5719 & -23.7878 & 13.7894 & -6.4671 & 8.5066 & 6.4844 & 13.0015 & -0.0000 & 8.0000 & 50.0000 \\ -16.6793 & 26.2970 & -6.4671 & 18.9580 & 13.5352 & -4.2091 & 0.0000 & -13.0015 & 8.3429 & -17.5246 \\ 18.3683 & -37.9837 & 8.5066 & 13.5352 & -20.6237 & 0.0000 & 4.2091 & -6.4844 & -16.4588 & 38.4351 \\ -38.4351 & 16.4588 & 6.4844 & -4.2091 & 0.0000 & 20.6237 & -13.5352 & -8.5066 & 37.9837 & -18.3683 \\ 17.5246 & -8.3429 & 13.0015 & -0.0000 & 4.2091 & -13.5352 & -18.9580 & 6.4671 & -26.2970 & 16.6793 \\ -50.0000 & -8.0000 & 0.0000 & -13.0015 & -6.4844 & -8.5066 & 6.4671 & -13.7894 & 23.7878 & 31.5719 \\ -2.0000 & 0.0000 & 8.0000 & 8.3429 & -16.4588 & 37.9837 & -26.2970 & 23.7878 & -17.2442 & -8.1144 \\ 0.0000 & 2.0000 & 50.0000 & -17.5246 & 38.4351 & -18.3683 & 16.6793 & 31.5719 & -8.1144 & 43.3210 \end{pmatrix}, \tag{40}$$

$$\begin{aligned} & \min_{[X_1,X_2,X_3]\in S_E} [\|X_1-\bar{X}_1\|^2 + \|X_2-\bar{X}_2\|^2 + \|X_3-\bar{X}_3\|^2] \\ & = \|\hat{X_1}^{(49)}-\bar{X}_1\|^2 + \|\hat{X_2}^{(49)}-\bar{X}_2\|^2 + \|\hat{X_3}^{(49)}-\bar{X}_3\|^2 \\ & = 3.3704e+04. \end{aligned} \tag{41}$$

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IV. Conclusion

In this paper, an iterative method is constructed to find the symmetric and antisymmetric solutions of matrix equation $A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C$. When the matrix equation is consistent, for any initial symmetric and antipersymmetric matrix group $[X_1^{(0)}, X_2^{(0)}, \ldots, X_l^{(0)}]$, a symmetric and antipersymmetric solution group can be obtained in finite iteration steps, and the least norm symmetric and antipersymmetric solution group can be obtained by choosing a special kind of initial symmetric and antipersymmetric matrix group. In addition, for a given symmetric and antipersymmetric group $[\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_l]$, the optimal approximation symmetric and antipersymmetric solution group can be obtained by finding the least norm symmetric and antipersymmetric solution group of matrix equation $A_1\widetilde{X_1}B_1 + A_2\widetilde{X_2}B_2 + \ldots + A_l\widetilde{X_l}B_l = \widetilde{C}$, where $\widetilde{C} = C - A_1\bar{X}_1B_1 - A_2\bar{X}_2B_2 - \cdots - A_l\bar{X}_lB_l$. Given numerical examples show that the iterative method is efficient.

References

[1] Zhen-yun Peng, "The nearest bisymmetric solutions of linear matrix equations", *J. Comput. Math.*, 22(6), 2004, pp. 873-880.

[2] K. T. Jeseph, "Inverse eigenvalue problem in structural design", *AIAA J.* 30, 1992, pp. 2890-2896.

[3] Zhuo-hua Peng, Xi-yan Hu and Lei Zhang, "The bisymmetric solutions of the matrix equation $A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C$ and its optimal approximation", *Linear Algebra and its Applications*, 426, 2007, pp. 583–595,

[4] Dongxiu Xie and Yanping Sheng, "The least-Squares solutions of inconsistent matrix equation over symmetric and antipersymmetric matrices", *Applid Mathematics Letters*, 16, 2003, pp. 589-598.

[5] Zhuo-hua Peng, Xi-yan Hu and Lei Zhang, "The centrosymmetric solutions of matrix equation $A_1X_1B_1 + A_2X_2B_2 + \ldots + A_lX_lB_l = C$ and its optimal approximation", *Acta Mathematica Scientia*, 29A(1), 2009, pp. 193–207.

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