An Iterative Process for Solving Fixed Point Problems for Weak Contraction Mappings

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***Abstract*—The aim of this work is to investigate a strong convergence theorem of the three step iterative process for the class of weak contraction mappings in Banach spaces. The analytical proof is supported via two numerical examples. The first example claims that the new process converges faster than all of Mann iterative process, Ishikawa iterative process and Agarwal iterative processes via the rate of convergence in the sense of Berinde. The second example compares the behavior of new iterative process with all of Mann iterative process, Ishikawa iterative process and Agarwal iterative process by using a numerical result.**

***Index Terms*—Picard iterative process, Mann iterative process, Ishikawa iterative process, rate of convergence; Mean valued theorem.**

I. Introduction and Preliminaries

THERE are several methods to face, from a theoretical aspect, to numerous problems which arise from real-world environment. Due to their possible applications, throughout the last years, the fixed point theory becomes the most interesting branch in mathematics. It is well-known that several mathematical and real-word problems are naturally formulated as a fixed point problem, that is, a problem for finding a point x in a domain of an appropriate mapping T such that

$$Tx = x. \tag{1}$$

A point x satisfying the condition (1) is called a fixed point of the mapping T. Furthermore, fixed point theory has been effectively applied in various topics, including differential equation, integral equation, matrix equation, convex minimization and split feasibility, as well as for finding zeros of contractive mappings.

Example 1. Let $a, b \in \mathbb{R}$ with $a < b$ and $C[a,b]$ be a collection of all continuous real-valued functions defined on the closed interval $[a,b]$. For a given mappings $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$, a solution of the following nonlinear integral equation:

$$x(c) = \phi(c) + \int_a^b K(c,r,x(r))dr, \tag{2}$$

where $x \in C[a,b]$, is equivalently with the fixed point problem for the mapping $T : C[a,b] \to C[a,b]$ defined by

$$(Tx)(c) = \phi(c) + \int_a^b K(c,r,x(r))dr$$

for all $x \in C[a,b]$ and $c \in [a,b]$.

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Throughout this work, for a given mapping T, we denote by $Fix(T)$ the set of all fixed points of T.

For a given self mapping T on a nonempty set X, *the fixed point iteration* (or *the Picard iteration*, or *the Richardson iteration*, or *the method of successive substitution*) is defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{P_n}$$

where x_0 is an arbitrary point but fixed in X. This iteration is investigated in the following famous result called the Banach contraction mapping principle.

Theorem 2 ([3]). *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping, i.e., a mapping for which there exists a constant $k \in [0,1)$ such that*

$$d(Tx,Ty) \leq kd(x,y) \tag{3}$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$ and the iterative process (P_n) converges to the fixed point x^*. Moreover, for each $x \in X$, we have

$$d(T^n x, x^*) \leq \frac{k^n}{1-k} d(x,Tx)$$

for all $n \in \mathbb{N}$.

If we drop k in the condition (3), then T is called a nonexpansive mapping. In approximation fixed points of nonexpansive mappings, the Picard iterative process (P_n) has not been successfully employed. In order to claim this fact, we give the following well-known example.

Example 3. Let $T : [0,1] \to [0,1]$ be defined by $Tx = 1 - x$ for all $x \in [0,1]$. Then T is a nonexpansive mapping with a usual metric and it has a unique fixed point $x^* = \frac{1}{2}$. It is easy to see that the Picard iterative process (P_n) with $x_0 \neq \frac{1}{2}$ as follows:

$$1 - x_0, \ x_0, \ 1 - x_0, \ \ldots$$

Also, it does not converge to the fixed point x^* of T.

From the above example, several iterative processes are needed to introduce. Next, we give some concepts of these processes. Throughout this paper, unless otherwise specified, let E be a normed space and $T : E \to E$ be a given mapping.

In 1953, the Mann iterative process $\{x_n\}$ was introduced by Mann [11] and it is defined by the following:

$$\left.\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \ldots \end{array}\right\} \tag{M_n}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in the interval $[0,1]$.

Remark 4. If $\alpha_n = \alpha \in [0,1]$ for all $n = 0, 1, 2, \ldots$, then the iterative process (M_n) reduces to the Krasnoselskij iterative process. Also, if $\alpha_n = 1$ for all $n = 0, 1, 2, \ldots$, then the iterative process (M_n) becomes to the Picard iterative process (P_n).

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Afterward, Ishikawa [9] introduced an iterative process $\{x_n\}$ defined by

$$\left.\begin{array}{l} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \quad n = 0,1,2,\ldots, \end{array}\right\} \quad (I_n)$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are two sequences in the interval $[0,1]$.

Remark 5. If $\beta_n = 0$ for all $n = 0,1,2,\ldots$, then the Ishikawa iterative process (I_n) reduces to the Mann iterative process (M_n).

In 2007, Agarwal et al. [1] introduced an iterative process $\{x_n\}$ defined by

$$\left.\begin{array}{l} x_0 \in E, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n, \quad n = 0,1,2,\ldots, \end{array}\right\} \quad (ARS_n)$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are two sequences in the interval $[0,1]$.

Recently, Sintunavarat and Pitea [2] introduced the new iterative process $\{x_n\}$ defined by

$$\left.\begin{array}{l} x_0 \in C, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n y_n, \\ x_{n+1} = (1-\alpha_n)T z_n + \alpha_n T y_n, \quad n = 0,1,2,\ldots, \end{array}\right\} \quad (S_n)$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ are real control sequences in the interval $[0,1]$.

Now, we again consider the contractive condition (3). It is east to see that the contractive condition (3) implies the continuity of T. So it is inherent to ask that there exist contractive conditions which do not imply the continuity of T. In order to solve this question, Kannan [10] introduced the following contractive condition:

- there exists $k \in [0,1/2)$ such that

$$d(Tx,Ty) \leq k[d(x,Tx) + d(y,Ty)] \quad (4)$$

for all $x,y \in X$.

Example 6. Let $X = \mathbb{R}$ be a usual metric space and $T : X \to X$ be defined by

$$Tx = \begin{cases} 0, & x \leq 2 \\ -\frac{1}{2}, & x > 2. \end{cases}$$

Then T is not continuous on X and T satisfies contractive condition (4) with $k = \frac{1}{5}$.

Afterward, the similar contractive condition of (4) was introduced by Chatterjea [7] as follows:

- there exists $k \in [0,1/2)$ such that

$$d(Tx,Ty) \leq k[d(x,Ty) + d(y,Tx)] \quad (5)$$

for all $x,y \in X$.

Note that conditions (3) and (4), (3) and (5), respectively, are independent contractive conditions (see more details in [12]).

In 1972, Zamfirescu [13] obtained a stupendous fixed point result by merging the contractive conditions (3), (4) and (5) as follows:

Theorem 7. *Let (X,d) be a complete metric space and $T : X \to X$ be a Zamfirescu mapping, i.e., there exist the real numbers a,b and c satisfying $a \in [0,1)$ and $b,c \in [0,1/2)$ such that for each pair $x,y \in X$, at least one of the following is true:*

(Z_1) $d(Tx,Ty) \leq ad(x,y)$;
(Z_2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$;
(Z_3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point x^* and the Picard iterative process (P_n) converges to x^* for arbitrary but fixed $x_0 \in X$.

In 2004, Berinde [4] introduced a very remarkable kind of contractive condition as follows:

Definition 8. Let (X,d) be a metric space. A mapping $T : X \to X$ is called a weak contraction mapping if there exist $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + Ld(x,Tx) \quad (6)$$

for any $x,y \in X$.

Remark 9. It is easy to see that each Zamfirescu mapping is a weak contraction mapping. So any mappings satisfying the contractive condition (3) or (4) or (5) is also a weak contraction mapping.

In the next year, Berinde [6] proved the strong convergence theorem for approximating fixed points of weak contraction mappings on normed linear spaces using the Ishikawa iterative process (I_n). Afterward, by using the iterative process (ARS_n), Hussain et al. [8] established the following result:

Theorem 10 ([8]). *Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ be a weak contraction mapping. Suppose that $\{x_n\}$ is defined by the iterative process (ARS_n) and $x_0 \in C$, where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ are sequences in the interval $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $Fix(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to the fixed point of T.*

They also gave some example to show that iterative process (ARS_n) is faster than the iterative process (M_n) and the iterative process (I_n) in the sense of Berinde [5] (see in Definition 11).

Definition 11 ([5]). Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l := \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

$(\mathcal{R}_1)$ If $l = 0$, then it can be said that $\{a_n\}$ converges faster to a than $\{b_n\}$ to b.
$(\mathcal{R}_2)$ If $0 < l < \infty$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

The purpose of this work is to prove the strong convergence theorem for weak contraction mappings on a nonempty closed convex subset of normed spaces by using iterative process (S_n). We also provide two examples to illustrate the convergence behavior of the our process and numerically compare the convergence of the our iteration process with the existing processes.

II. Theoretical result

In this section, we prove the strong convergence theorem for weak contraction mappings on a nonempty closed convex subset of normed spaces by using iterative process (S_n).

Theorem 12. *Let C be a nonempty closed convex subset of a Banach space E and $T : C \to C$ be a weak contraction mapping with constants $\delta \in [0,1)$ and $L \geq 0$. Suppose that $\{x_n\}$ is defined by the iterative process (S_n) such that $\sum_{n=0}^{\infty} \beta_n \gamma_n = \infty$. If $Fix(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to the fixed point of T.*

Proof: Since $Fix(T) \neq \emptyset$, we may assume that there exists $w \in Fix(T)$. For each $n \in \{0,1,2,\ldots\}$, by using (S_n), we get

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)Tz_n + \alpha_n Ty_n - w\| \\ &= \|(1-\alpha_n)(Tz_n - w) + \alpha_n(Ty_n - w)\| \\ &\leq (1-\alpha_n)\|Tz_n - w\| + \alpha_n\|Ty_n - w\| \\ &\leq (1-\alpha_n)\delta\|z_n - w\| + \alpha_n\delta\|y_n - w\| \\ &\leq (1-\alpha_n)\|z_n - w\| + \alpha_n\|y_n - w\|. \end{aligned} \tag{7}$$

Using (S_n) again, for each $n \in \{0,1,2,\ldots\}$, we have

$$\begin{aligned} \|y_n - w\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - w\| \\ &= \|(1-\beta_n)(x_n - w) + \beta_n(Tx_n - w)\| \\ &\leq (1-\beta_n)\|x_n - w\| + \beta_n\|Tx_n - w\| \\ &\leq (1-\beta_n)\|x_n - w\| + \beta_n\delta\|x_n - w\| \\ &= (1-(1-\delta)\beta_n)\|x_n - w\| \\ &\leq (1-(1-\delta)\beta_n\gamma_n)\|x_n - w\| \end{aligned} \tag{8}$$

and

$$\begin{aligned} \|z_n - w\| &= \|(1-\gamma_n)x_n + \gamma_n y_n - w\| \\ &= \|(1-\gamma_n)(x_n - w) + \gamma_n(y_n - w)\| \\ &\leq (1-\gamma_n)\|x_n - w\| + \gamma_n\|y_n - w\| \\ &\leq (1-\gamma_n)\|x_n - w\| + \gamma_n(1-(1-\delta)\beta_n\gamma_n)\|x_n - w\| \\ &= (1-(1-\delta)\beta_n\gamma_n)\|x_n - w\|. \end{aligned} \tag{9}$$

From (7), (8) and (9), we have

$$\begin{aligned} \|x_{n+1} - w\| &\leq [(1-\alpha_n)(1-(1-\delta)\beta_n\gamma_n) \\ &\quad + \alpha_n(1-(1-\delta)\beta_n\gamma_n)]\,\|x_n - w\| \\ &= (1-(1-\delta)\beta_n\gamma_n)\|x_n - w\| \end{aligned} \tag{10}$$

for all $n \in \{0,1,2,\ldots\}$. By (10), we inductively obtain that

$$\|x_{n+1} - w\| \leq \prod_{k=0}^{n}[1-(1-\delta)\beta_k\gamma_k]\|x_0 - w\|, \text{ for } n = 0,1,2,\ldots \tag{11}$$

From the fact that $0 \leq \delta < 1$, $0 \leq \beta_n\gamma_n \leq 1$ and $\sum_{n=1}^{\infty} \beta_n\gamma_n = \infty$, it results that

$$\lim_{n\to\infty}\left(\prod_{k=0}^{n}[1-(1-\delta)\beta_k\gamma_k]\right) = 0$$

which by (11) implies

$$\lim_{n\to\infty}\|x_{n+1} - w\| = 0.$$

This means that $\lim_{n\to\infty} x_n = w \in Fix(T)$. ∎

III. Numerical results

In this section, we give two examples to show that our iterative process (S_n) is faster than the iterative processes (M_n), (I_n) and (ARS_n) for some weak contraction mapping. In the first example, we also prove that our iteration (S_n) is faster than the above mentioned iterative processes in the sense of rate of convergence in Definition 11.

Example 13. Let $E = \mathbb{R}$ be a usual normed, $C = [0,1]$ and $T : C \to C$ be defined by

$$Tx = \frac{x}{2}$$

for all $x \in C$. It is easy to see that T satisfies the condition (6) with a unique fixed point $w := 0$. Let

$$\alpha_n = \beta_n = \gamma_n := \begin{cases} 0, & n = 0,1,\ldots,15; \\ \frac{4}{\sqrt{n}}, & n = 16,17,\ldots \end{cases}$$

Also, it clear that sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ satisfy all the conditions of Theorems 10 and 12. Next, we will prove that our corresponding iterative process (S_n) is faster than all of Mann iterative process (M_n), Ishikawa iterative process (I_n) and Agarwal iterative process (ARS_n) with the initial point $x_0 \neq 0$.

For $n = 16, 17, \ldots$, we have

$$\begin{aligned} M_n &= (1-\alpha_n)x_n + \alpha_n Tx_n \\ &= \left(1 - \frac{4}{\sqrt{n}}\right)x_n + \frac{4}{\sqrt{n}}\frac{1}{2}x_n \\ &= \left(1 - \frac{2}{\sqrt{n}}\right)x_n \\ &\vdots \\ &= \prod_{i=16}^{n}\left(1 - \frac{2}{\sqrt{i}}\right)x_{16}, \end{aligned} \tag{12}$$

$$\begin{aligned} I_n &= (1-\alpha_n)x_n + \alpha_n T\left((1-\beta_n)x_n + \beta_n Tx_n\right) \\ &= \left(1 - \frac{4}{\sqrt{n}}\right)x_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(1 - \frac{2}{\sqrt{n}}\right)x_n \\ &= \left(1 - \frac{2}{\sqrt{n}} - \frac{4}{n}\right)x_n \\ &\vdots \\ &= \prod_{i=16}^{n}\left(1 - \frac{2}{\sqrt{i}} - \frac{4}{i}\right)x_{16}, \end{aligned} \tag{13}$$

$$\begin{aligned} ARS_n &= (1-\alpha_n)Tx_n + \alpha_n T\left((1-\beta_n)x_n + \beta_n Tx_n\right) \\ &= \left(1 - \frac{4}{\sqrt{n}}\right)\frac{1}{2}x_n + \frac{4}{\sqrt{n}}\frac{1}{2}\left(1 - \frac{2}{\sqrt{n}}\right)x_n \\ &= \left(\frac{1}{2} - \frac{4}{n}\right)x_n \\ &\vdots \\ &= \prod_{i=16}^{n}\left(\frac{1}{2} - \frac{4}{i}\right)x_{16}, \end{aligned} \tag{14}$$

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$$
\begin{aligned}
S_n &= (1-\alpha_n)T\left((1-\gamma_n)x_n+\gamma_n\left((1-\beta_n)x_n+\beta_n Tx_n\right)\right) \\
&\quad +\alpha_n T\left((1-\beta_n)x_n+\beta_n Tx_n\right) \\
&= \left(1-\frac{4}{\sqrt{n}}\right)\frac{1}{2}\left[\left(1-\frac{4}{\sqrt{n}}\right)x_n+\frac{4}{\sqrt{n}}\left(1-\frac{2}{\sqrt{n}}\right)x_n\right] \\
&\quad +\frac{4}{\sqrt{n}}\frac{1}{2}\left(1-\frac{2}{\sqrt{n}}\right)x_n \\
&= \left(1-\frac{4}{\sqrt{n}}\right)\frac{1}{2}\left(1-\frac{8}{n}\right)x_n+\frac{2}{\sqrt{n}}\left(1-\frac{2}{\sqrt{n}}\right)x_n \\
&= \left(\frac{1}{2}-\frac{8}{n}+\frac{16}{n\sqrt{n}}\right)x_n \\
&\ \ \vdots \\
&= \prod_{i=16}^{n}\left(\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}\right)x_{16}. \qquad (15)
\end{aligned}
$$

For $n = 16, 17, \ldots$, we obtain

$$
\begin{aligned}
\left|\frac{S_n-w}{M_n-w}\right| &= \left|\frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}\right)x_{16}}{\prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}\right)x_{16}}\right| \\
&= \prod_{i=16}^{n}\frac{\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}}{1-\frac{2}{\sqrt{i}}} \\
&= \prod_{i=16}^{n}\left(1-\frac{\frac{1}{2}-\frac{2}{\sqrt{i}}+\frac{8}{i}-\frac{16}{i\sqrt{i}}}{1-\frac{2}{\sqrt{i}}}\right) \\
&\leq \prod_{i=16}^{n}\left(1-\frac{1}{i}\right) \\
&= \frac{15}{16}\cdot\frac{16}{17}\cdot\frac{17}{18}\cdots\frac{n-1}{n} \\
&= \frac{15}{n}, \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{S_n-w}{I_n-w}\right| &= \left|\frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}\right)x_{16}}{\prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}-\frac{4}{i}\right)x_{16}}\right| \\
&= \frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}\right)}{\prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}-\frac{4}{i}\right)} \\
&\leq \frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{4}{i}\right)}{\prod_{i=16}^{n}\left(1-\frac{2}{\sqrt{i}}-\frac{4}{i}\right)} \\
&= \prod_{i=16}^{n}\frac{\frac{1}{2}-\frac{4}{i}}{1-\frac{2}{\sqrt{i}}-\frac{4}{i}} \\
&= \prod_{i=16}^{n}\left(1-\frac{\frac{1}{2}-\frac{2}{\sqrt{i}}}{1-\frac{2}{\sqrt{i}}-\frac{4}{i}}\right) \\
&\leq \prod_{i=16}^{n}\left(1-\frac{1}{i}\right) \\
&= \frac{15}{16}\cdot\frac{16}{17}\cdot\frac{17}{18}\cdots\frac{n-1}{n} \\
&= \frac{15}{n}, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{S_n-w}{ARS_n-w}\right| &= \left|\frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}\right)x_{16}}{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{4}{i}\right)x_{16}}\right| \\
&= \frac{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}\right)}{\prod_{i=16}^{n}\left(\frac{1}{2}-\frac{4}{i}\right)} \\
&= \prod_{i=16}^{n}\frac{\frac{1}{2}-\frac{8}{i}+\frac{16}{i\sqrt{i}}}{\frac{1}{2}-\frac{4}{i}} \\
&= \prod_{i=16}^{n}\left(1-\frac{\frac{4}{i}-\frac{16}{i\sqrt{i}}}{\frac{1}{2}-\frac{4}{i}}\right) \\
&\leq \prod_{i=16}^{n}\left(1-\frac{1}{i}\right) \\
&= \frac{15}{16}\cdot\frac{16}{17}\cdot\frac{17}{18}\cdots\frac{n-1}{n} \\
&= \frac{15}{n}. \qquad (18)
\end{aligned}
$$

From (16), (17) and (18), we get

$$\lim_{n\to\infty}\left|\frac{S_n-w}{M_n-w}\right| = 0,$$

$$\lim_{n\to\infty}\left|\frac{S_n-w}{I_n-w}\right| = 0$$

and

$$\lim_{n\to\infty}\left|\frac{S_n-w}{ARS_n-w}\right| = 0.$$

This implies that our iterative process (S_n) converges faster than the Mann iterative process (M_n), the Ishikawa iterative process (I_n) and the Agarwal iterative process (ARS_n) to the fixed point 0 of T.

For the initial point $x_0 = 0.7$, our corresponding iterative process (S_n), the Agarwal iterative process (ARS_n), the Ishikawa iterative process (I_n), the Mann iterative process (M_n), respectively, and its behaviors is given in Figure 1.

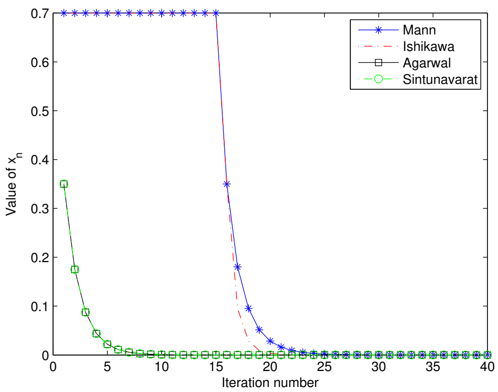

Fig. 1. Behavior of of the Mann iterative process (M_n), the Ishikawa iterative process (I_n), the Agarwal iterative process (ARS_n), and the Sintunavarat iterative process (S_n) for the function given in Example 13.

Next, we compare the behavior of iterative process (S_n) with respect to the Mann iterative process (M_n), the Ishikawa iterative process (I_n) and the Agarwal iterative process (ARS_n) by using a numerical result.

Example 14. Let $E = \mathbb{R}$ be a usual normed, $C = [1, 200]$ and $T : C \to C$ be defined by

$$Tx = \sqrt{x^2 - 8x + 40}$$

for all $x \in C$. By the mean valued theorem, we can compute that T satisfies the condition (6). It is easy to see that T has a unique fixed point $w := 5$. Let

$$\alpha_n = \beta_n = \gamma_n := \frac{1}{2}$$

for all $n = 0, 1, 2, \ldots$ Also, it clear that sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}$ satisfy all the conditions of Theorems 10 and 12.

For the initial point $x_0 = 200$, our corresponding iterative process (S_n), the Agarwal iterative process (ARS_n), the Ishikawa iterative process (I_n), the Mann iterative process (M_n), respectively, and its behaviors are given in Figures 2 and 3.

Step	Iteration (M_n)	Iteration (I_n)	Iteration (ARS_n)	Iteration (S_n)
1	198.0306	197.0462	195.0768	194.5846
2	196.0615	194.0932	190.1556	189.1717
3	194.0928	191.1408	185.2366	183.7613
4	192.1243	188.1892	180.3197	178.3536
5	190.1562	185.2385	175.4051	172.9488
6	188.1884	182.2885	170.4930	167.5471
7	186.2210	179.3393	165.5835	162.1486
8	184.2539	176.3910	160.6768	156.7536
9	182.2872	173.4436	155.7730	151.3624
10	180.3209	170.4971	150.8723	145.9752
⋮	⋮	⋮	⋮	⋮
36	129.3581	94.3736	26.6285	12.0452
37	127.4060	91.4745	22.3050	8.5957
38	125.4546	88.5789	18.1415	6.2591
39	123.5039	85.6868	14.2209	5.2682
40	121.5541	82.7987	10.6933	5.0416
⋮	⋮	⋮	⋮	⋮

Fig. 2. Comparative results of Example 14

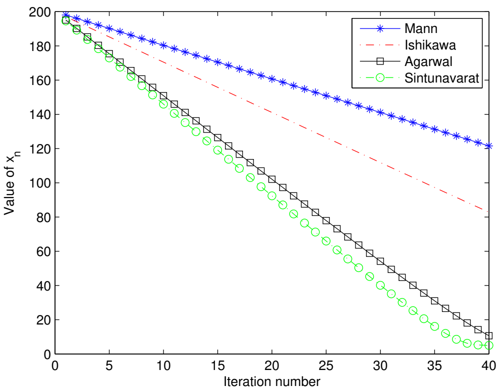

Fig. 3. Behavior of of the Mann iterative process (M_n), the Ishikawa iterative process (I_n), the Agarwal iterative process (ARS_n), and the Sintunavarat iterative process (S_n) for the function given in Example 14.

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