Small and Large Time Behavior of the Quantum Fokker-Planck System

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Abstract—This paper is concerned with the Quantum Fokker-Planck equation, which is a kinetic model for quantum mechanical (charged) particles-transport under the influence of diffusive effects. In this work, we study small and large time behavior of solutions, respectively. Though this setting is fairly introductory, our method of proof, which uses a priori estimates, can be extended to prove analogous results for transport equations, and nonlinear equations even when other tools, such as semigroup methods or the use of explicit fundamental solutions, are unavailable.

Index Terms—Quantum Fokker-Planck system, small time behavior, large time behavior.

I. INTRODUCTION

THE Quantum Fokker-Planck (called QFP in the sequel for simplicity) equation is a parabolic partial differential equation. From the physical point of view, the QFP equation is a kinetic model for quantum mechanical (charged) particles-transport under the influence of diffusive effects, for example, in the description of quantum Brownian motion [8], quantum optics [9], and semiconductor device simulations [10].

In this paper, we are interested in the small and large time behavior of solutions of the following version of the QFP system, namely

\[ f_t + v \cdot \nabla_x f - x \cdot \nabla_v f = \beta \nabla_v \cdot (v f) + \sigma \Delta_v f + 2 \gamma \nabla_x \cdot (\nabla_v f) + \alpha \Delta_x f, \]

\[ f(x, v, 0) = f_0(x, v), \]

where the Wigner function \( f = f(t, x, v) \) is a probabilistic quasi-distribution function of particles at time \( t \geq 0 \), located at \( x \in \mathbb{R}^n \) with velocity \( v \in \mathbb{R}^n \). \( \beta \geq 0 \) is a friction parameter and the parameters \( \sigma > 0, \alpha > 0, \gamma \geq 0 \) constitute the phase-space diffusion matrix of the system, which are necessary in order to make the model consistent with quantum physics, that is,

\[ \alpha \sigma \geq \gamma^2 + \beta^2 \frac{n^2}{40}. \]

which guarantees that the system is quantum mechanically correct, see [1], [2] for detail.

In last years, mathematical studies of QFP type equations mainly focused on the Cauchy problem, such as [1], [2], [3], [4], [5], [6], where different aspects regarding the derivation of the model were taken into consideration. In the present work we shall be mainly interested in the asymptotic behavior as \( t \ll 1 \) or \( t \to \infty \) of solutions to (1)-(3). There is only a few results for this problem. For instance, if \( f_0 \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n}) \) Sparber et al. [7] proved that the solution of (1)-(3) converged exponentially towards the steady state by the use of explicit fundamental solutions and entropy-methods. If \( f_0 \in L^2(\mathbb{R}^{2n}), (1 + A(x, v))^m \) with \( A(x, v) = \frac{1}{2}(|x|^2 + 2x \cdot v + 3|v|^2) \) for some \( m > 0 \), Arnold et al. [12] proved that the (1)-(3) with \( \beta = 2, \sigma = \alpha = 1, \gamma = 0 \) admitted a unique stationary solution that converged towards the steady state with an exponential rate.

However, if \( f_0 \) only belongs to \( L^2(\mathbb{R}^{2n}) \), the cited work can not be applied. In order to do so, we must introduce new tools, which uses a priori estimates, can be extended to prove analogous results for problems with time-dependent coefficients, transport equations, and nonlinear equations even when other tools, such as semigroup methods or the use of explicit fundamental solutions, are unavailable.

We are in a position to describe one of the main results:

**Theorem 1** For every \( f_0 \in L^2(\mathbb{R}^{2n}) \), let the equation (1)-(3) satisfy (4). If \( \frac{\alpha \sigma}{4} \geq \gamma \sqrt{1 + (\frac{\sigma n^2}{40})^2} \):

- For any \( t \in [0, 1] \), any solution of (1)-(3) satisfies
  \[ \sup_{0 \leq t \leq 1} \|f(t)\|_{L^2(\mathbb{R}^{2n})} \leq C, \|\nabla_x f(t)\|_{L^2(\mathbb{R}^{2n})} \leq C t^{-\frac{1}{2}}; \]

- If \( v \in \mathbb{R}^n \) and \( x \in \Omega \) (\( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \)), \( f \) obeys the prescribed boundary conditions on \( \partial \Omega \)
  \[ f(t, x, v) = f(t, y, v), \forall x, y \in \partial \Omega, v \in \mathbb{R}^n, t \geq 0. \]

Setting \( \lambda = \min \{\alpha C_\Omega^{-1}, \beta C_\Omega^{-1}\} - n \beta \geq 0 \), any solution of (1)-(3) satisfies
  \[ \|f(t)\|_{L^2} \leq \|f(0)\|_{L^2} e^{-\frac{1}{2} \lambda t}, \]

where \( C_\Omega \) is defined by Lemma 4. Moreover, If \( \lambda > 0 \), then one has \( f \to 0 \) as \( t \to \infty \).

II. PRELIMINARY

In the ensuing, we shall analyze the basic properties of the solutions of (1)-(3).

**Lemma 1** Let the coefficients of the equation (1)-(3) satisfy (4).

- If \( \gamma = 0 \), then one has
  \[ \frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq \frac{n \beta}{2} \|f\|_{L^2}^2 - \alpha \|\nabla_x f\|_{L^2}^2 - \sigma \|\nabla_v f\|_{L^2}^2. \]

- If \( \gamma > 0 \), then one has
  \[ \frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq \frac{n \beta}{2} \|f\|_{L^2}^2 - \theta \|\nabla_x f\|_{L^2}^2 - \theta \|\nabla_v f\|_{L^2}^2, \]

where \( \theta = \frac{\alpha \sigma}{2} - \gamma \sqrt{1 + (\frac{\sigma n^2}{40})^2} \).

**Proof:** The first assertion follows directly by applying the integration by parts. Next we shall prove the second

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assertion. Multiplication of the equation (1)-(3) by \( f \), we have
\[
\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq \sum_{i=1}^{n} \int \left\{-v_i f_{x_i} + x_i f_v + \beta(v_i(f))_{x_i} f_{x_i} + \sigma f_{x_i} f_v + 2\gamma f_{x_i} + \alpha f_{x_i, x_i} f_{x_i}\right\} dx dv.
\]

Using the integration by parts, it is clear that
\[
- \int \int v_i f_{x_i} f_{x_i} dx dv = \int \int (v_i f)_{x_i} f dx dv
\]
\[
= \int \int v_i f_{x_i} f_{x_i} dx dv,
\]
hence \( \int v_i f_{x_i} f_{x_i} dx dv = 0 \).

Analogy, we have \( \int x_i f_{x_i} f_v dx dv = 0 \) and
\[
\int \int \beta(v_i(f))_{x_i} f_{x_i} dx dv = \int \int \beta f^2 + \beta v_{x_i} f_{x_i} dx dv
\]
\[
= \int \int \beta f^2 - \beta(v_i(f))_{x_i} f_{x_i} dx dv,
\]
thus
\[
\int \int \beta(v_i(f))_{x_i} f_{x_i} dx dv = \frac{\beta}{2} \int \int f^2 dx dv.
\]

By [1] and the interpolation inequality
\[
\int \int f_{x_i} f_{x_i} dx dv \leq \frac{\epsilon}{2} \|f_{x_i}\|_{L^2}^2 + \frac{1}{2\epsilon} \|f_{x_i}\|_{L^2}^2,
\]
we obtain
\[
\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq \frac{n\beta}{2} \|f\|_{L^2}^2 - \sigma \|\nabla f\|_{L^2}^2 + \epsilon \gamma \|\nabla f\|_{L^2}^2
\]
\[
+ \frac{\gamma}{\epsilon} \|\nabla f\|_{L^2}^2 - \alpha \|\nabla f\|_{L^2}^2
\]
\[
\leq \frac{n\beta}{2} \|f\|_{L^2}^2 + \frac{\gamma}{\epsilon} \|\nabla f\|_{L^2}^2
\]
\[
+ (\epsilon \gamma - \alpha) \|\nabla f\|_{L^2}^2.
\]

Note that if \( \epsilon = \frac{\alpha - \sigma}{2\gamma} + \sqrt{1 + \left(\frac{\alpha - \sigma}{2\gamma}\right)^2} \) then one has
\[
\frac{1}{2} \frac{d}{dt} \|f(t)\|_{L^2}^2 \leq \frac{n\beta}{2} \|f\|_{L^2}^2 - \theta \|\nabla f\|_{L^2}^2 - \theta \|\nabla f\|_{L^2}^2,
\]
where \( \theta = \frac{\alpha - \sigma}{2} - \sqrt{1 + \left(\frac{\alpha - \sigma}{2\gamma}\right)^2} \).

**Lemma 2** Let the coefficients of the equation (1)-(3) satisfy (4), then one has
\[
\frac{1}{2} \frac{d}{dt} \|\nabla f(t)\|_{L^2}^2 \leq \frac{\beta n}{2} \|\nabla f(t)\|_{L^2}^2
\]
\[
- \min\{\alpha, \sigma\} \|\Delta f(t)\|_{L^2}^2.
\]

**Proof:** Multiplication of the equation (1)-(2) by \( -\Delta f \), using integrations by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla f(t)\|_{L^2}^2 \leq \frac{\beta n}{2} \|\nabla f(t)\|_{L^2}^2 + n \int \int \nabla f \cdot \nabla v_f
\]
\[
- \sigma \int \int \Delta f \Delta_v f = \alpha \|\Delta f\|_{L^2}^2.
\]

Analogously, multiplication of the equation (1)-(2) by \( -\Delta_v f \), we get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla f(t)\|_{L^2}^2 \leq \frac{\beta n}{2} \|\nabla f(t)\|_{L^2}^2 - n \int \int \nabla f \cdot \nabla v_f
\]
\[
- \sigma \|\Delta f(t)\|_{L^2}^2 - \alpha \int \int \Delta f \Delta_v f.
\]

Moreover, it yields
\[
\frac{1}{2} \frac{d}{dt} \|\nabla f(t)\|_{L^2}^2 \leq \frac{\beta n}{2} \|\nabla f(t)\|_{L^2}^2
\]
\[
- (\alpha + \sigma) \int \int \Delta f \Delta_v f - \min\{\alpha, \sigma\} \|\Delta f(t)\|_{L^2}^2.
\]

Note that
\[
\int \int \Delta f \Delta_v f = \int \int \left( \sum \partial_x \partial_y f \right)^2.
\]

Hence, we can get
\[
\frac{1}{2} \frac{d}{dt} \|\nabla f(t)\|_{L^2}^2 \leq \frac{\beta n}{2} \|\nabla f(t)\|_{L^2}^2
\]
\[
- \min\{\alpha, \sigma\} \|\Delta f(t)\|_{L^2}^2.
\]

### III. Proof of Theorem 1

For a start, we consider the small time behavior of solutions.

**Lemma 3** For every \( f_0 \in L^2(R^n) \), let the equation (1)-(3) satisfy (4), any solution satisfies
\[
sup_{0 \leq t \leq 1} \|f(t)\|_{L^2(R^n)} \leq C, \quad \|\nabla f(t)\|_{L^2(R^n)} \leq C t^{-\frac{1}{2}}.
\]

**Proof:** Now, by [11] we can utilize a linear expansion in \( t \) to prove the assertion. Let \( T \in (0, 1] \) be given. Consider \( t \in (0, T) \) and define
\[
y(t) = \|f(t)\|_{L^2}^2 + k t \|\nabla f(t)\|_{L^2}^2,
\]
where \( k < \min\{2\alpha, 2\sigma, 2\theta\} \).

We differentiate this quantity, and use Lemmas 1-2 to find
\[
y'(t) = \frac{d}{dt} \|f(t)\|_{L^2}^2 + k \|\nabla f(t)\|_{L^2}^2 + k t \frac{d}{dt} \|\nabla f(t)\|_{L^2}^2
\]
\[
\leq n \beta \|f(t)\|_{L^2}^2 - 2 \min\{\alpha, \theta\} \|\nabla f(t)\|_{L^2}^2
\]
\[
- 2 \min\{\sigma, \theta\} \|\nabla f(t)\|_{L^2}^2 + 2 k t \frac{\beta n}{2} \|\nabla f(t)\|_{L^2}^2
\]
\[
+ k \|\nabla f(t)\|_{L^2}^2 - 2 k t \min\{\alpha, \sigma\} \|\Delta f(t)\|_{L^2}^2
\]
\[
\leq n \beta \|f(t)\|_{L^2}^2 + k n \beta t \|\nabla f(t)\|_{L^2}^2
\]
\[
\leq C y(t).
\]

A straightforward application of Gronwall’s inequality then implies
\[
\|f(t)\|_{L^2}^2 + k t \|\nabla f(t)\|_{L^2}^2 \leq C T \|f_0\|_{L^2}^2.
\]
The assertion follows directly by applying the above inequality.

Next, we consider the large time behavior of solutions. For a start, we should collect a few standard inequalities for the reader’s convenience.

**Lemma 4** There is a constant \( C = C_\Omega \) such that
\[
\int_{\Omega} \int |f(x, v)|^2 dx dv \leq C \int_{\Omega} \int |\nabla f|^2 dx dv.
\]

**Proof:** The proof is similar to that of the classical Poincaré inequality, and is only sketched. Without loss of generality, \( \Omega \) is a rectangle \( \Pi^n_{i=1} (a_i, b_i) \). First, the result is
\[
C(b_1 - a_1) \int_{\Omega} \int |f(x, v)| \partial_{x_1} f(x, v) dx dv.
\]
With use of the Hőlder inequality, it follows that
\[
\int_{\mathbb{R}^n} \int_{\Omega} |f(x, v)|^2 dx dv \leq C \| \partial_x f(x, v) \|_{L^2} \| f(x, v) \|_{L^2}.
\]

**Lemma 5** For every \( f_0 \in H^1(\mathbb{R}^{2n}) \), let the equation (1)-(3) satisfy (4). If \( v \in \mathbb{R}^n \) and \( x \in \Omega \) (\( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \)), \( f \) obeys the prescribed boundary conditions on \( \partial \Omega \)
\[
f(t, x, v) = f(t, y, v), \quad \forall \ x, y \in \partial \Omega, \quad v \in \mathbb{R}^n, \quad t \geq 0.
\]
Setting \( \lambda = \min\{\alpha C_\Omega^{-1}, \theta C_\Omega^{-1}\} - n\beta \geq 0 \), any solution of (1.1) satisfies
\[
\| f(t) \|_{L^2} \leq \| f(t_0) \|_{L^2} e^{-\frac{t}{2\lambda}},
\]
where \( C_\Omega \) is defined by Lemma 4. Moreover, If \( \lambda > 0 \), then one has \( f \to 0 \) as \( t \to \infty \).

**Proof:** Since Lemma 1, we have
\[
\frac{1}{2} \frac{d}{dt} \| f(t) \|_{L^2}^2 \leq \frac{n\beta}{2} \| f \|_{L^2}^2 - \min\{\alpha, \theta\} \| \nabla_x f \|_{L^2}^2.
\]
where \( k_1 = -\left(\frac{\alpha}{2} - \sigma + \frac{\theta}{2}\right) \geq 0 \). Using Lemma 4, it is clear that
\[
\frac{d}{dt} \| f(t) \|_{L^2} + \lambda \| f \|_{L^2} \leq 0.
\]
The assertion follows directly by applying the Gronwall inequality.

Now, we begin with the proof of Theorem 1.

**Proof of Theorem 1** Collecting Lemmas 3 and 5, we obtain the desired results. The proof of Theorem 1 is complete.

**REFERENCES**


