Proceedings of the International MultiConference of Engineers and Computer Scientists 2017 Vol II,
IMECS 2017, March 15 - 17, 2017, Hong Kong

Small and Large Time Behavior of the Quantum Fokker-Planck System

Jie-qiong Shen, and Bin Li

Abstract—**This paper is concerned with the Quantum Fokker-Planck equation, which is a kinetic model for quantum mechanical (charged) particles-transport under the influence of diffusive effects. In this work, we study small and large time behavior of solutions, respectively. Though this setting is fairly introductory, our method of proof, which uses a priori estimates, can be extended to prove analogous results for transport equations, and nonlinear equations even when other tools, such as semigroup methods or the use of explicit fundamental solutions, are unavailable.**

Index Terms—**Quantum Fokker-Planck system, small time behavior, large time behavior.**

I. Introduction

THE Quantum Fokker-Planck (called QFP in the sequel for simplicity) equation is a parabolic partial differential equation. From the physical point of view, the QFP equation is a kinetic model for quantum mechanical (charged) particles-transport under the influence of diffusive effects, for example, in the description of quantum Brownian motion [8], quantum optics [9], and semiconductor device simulations [10].

In this paper, we are interested in the small and large time behavior of solutions of the following version of the QFP system, namely

$$f_t + v\cdot\nabla_x f - x\cdot\nabla_v f = \tag{1}$$

$$\beta\nabla_v\cdot(vf) + \sigma\Delta_v f + 2\gamma\nabla_x\cdot(\nabla_v f) + \alpha\Delta_x f, \tag{2}$$

$$f(x,v,0) = f_0(x,v), \tag{3}$$

where the Wigner function $f = f(t,x,v)$ is a probabilistic quasi-distribution function of particles at time $t \geq 0$, located at $x \in R^n$ with velocity $v \in R^n$. $\beta \geq 0$ is a friction parameter and the parameters $\sigma > 0$, $\alpha \geq 0$, $\gamma \geq 0$ constitute the phase-space diffusion matrix of the system, which are necessary in order to make the model consistent with quantum physics, that is,

$$\alpha\sigma \geq \gamma^2 + \frac{\beta^2}{16}, \tag{4}$$

which guarantees that the system is quantum mechanically correct, see [1], [2] for detail.

In last years, mathematical studies of QFP type equations mainly focused on the Cauchy problem, such as [1], [2], [3], [4], [5], [6], where different aspects regarding the derivation of the model were taken into consideration. In the present work we shall be mainly interested in the asymptotic behavior as $t \ll 1$ or $t \to \infty$ of solutions to (1)-(3). There is only a few results for this problem. For instance, if $f_0 \in L^1(R^{2n}) \cap L^2(R^{2n})$ Sparber et al. [7] proved that the solution of (1)-(3) converged exponentially towards the steady state by the use of explicit fundamental solutions and entropy-methods. If $f_0 \in L^2(R^{2n}, (1 + A(x,v))^m)$ with $A(x,v) = \frac{1}{4}(|x|^2 + 2x\cdot v + 3|v|^3)$ for some $m > 0$, Arnold et al. [12] proved that the (1)-(3) with $\beta = 2, \sigma = \alpha = 1, \gamma = 0$ admitted a unique stationary solution that converged towards the steady state with an exponential rate.

However, if f_0 only belongs to $L^2(R^{2n})$, the cited work can not be applied. In order to do so, we must introduce new tools, which uses a priori estimates, can be extended to prove analogous results for problems with time-dependent coefficients, transport equations, and nonlinear equations even when other tools, such as semigroup methods or the use of explicit fundamental solutions, are unavailable.

We are in a position to describe one of the main results:

Theorem 1 For every $f_0 \in L^2(R^{2n})$, let the equation (1)-(3) satisfy (4). If $\frac{\alpha+\sigma}{2} \geq \gamma\sqrt{1 + (\frac{\sigma-\alpha}{2\gamma})^2}$:

- For any $t \in (0,1]$, any solution of (1)-(3) satisfies

$$\sup_{0\leq t\leq 1}\|f(t)\|_{L^2(R^{2n})} \leq C,\ \|\nabla_x f(t)\|_{L^2(R^{2n})} \leq Ct^{-\frac{1}{2}};$$

- If $v \in R^n$ and $x \in \Omega$ ($\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$), f obeys the prescribed boundary conditions on $\partial\Omega$

$$f(t,x,v) = f(t,y,v),\ \forall\ x,y \in \partial\Omega,\ v \in R^n,\ t \geq 0.$$

Setting $\lambda = \min\{\alpha C_\Omega^{-1}, \theta C_\Omega^{-1}\} - n\beta \geq 0$, any solution of (1)-(3) satisfies

$$\|f(t)\|_{L^2} \leq \|f(t)_0\|_{L^2} e^{-\frac{1}{2}\lambda t},$$

where C_Ω is defined by Lemma 4. Moreover, If $\lambda > 0$, then one has $f \to 0$ as $t \to \infty$.

II. Preliminary

In the ensuing, we shall analyze the basic properties of the solutions of (1)-(3).

Lemma 1 Let the coefficients of the equation (1)-(3) satisfy (4).

- If $\gamma = 0$, then one has

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}^2 \leq \frac{n\beta}{2}\|f\|_{L^2}^2 - \alpha\|\nabla_x f\|_{L^2}^2 - \sigma\|\nabla_v f\|_{L^2}^2.$$

- If $\gamma > 0$, then one has

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}^2 \leq \frac{n\beta}{2}\|f\|_{L^2}^2 - \theta\|\nabla_x f\|_{L^2}^2 - \theta\|\nabla_v f\|_{L^2}^2,$$

where $\theta = \frac{\alpha+\sigma}{2} - \gamma\sqrt{1 + (\frac{\sigma-\alpha}{2\gamma})^2}$.

Proof: The first assertion follows directly by applying the integration by parts. Next we shall prove the second

Manuscript received September 14, 2016; revised October 17, 2016.

J. Shen and B. Li are with the Department of Mathematics, Jincheng College of Sichuan University, Chengdu 611731, P.R.CHINA.
E-mail: Jieqiongshen123@163.com; blimath@163.com.

assertion. Multiplication of the equation (1)-(3) by f, we have

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2} = \sum_{i=1}^{n}\int\int\{-v_i f_{x_i} f + x_i f_{v_i} f + \beta(v_i f)_{v_i} f dxdv + \sigma f_{v_i v_i} f + 2\gamma f_{x_i v_i} + \alpha f_{x_i x_i} f\}dxdv.$$

Using the integration by parts, it is clear that

$$\begin{aligned} -\int\int v_i f_{x_i} f dxdv &= \int\int (v_i f)_{x_i} f dxdv \\ &= \int\int v_i f_{x_i} f dxdv, \end{aligned}$$

hence $\int\int v_i f_{x_i} f dxdv = 0$.
Analogy, we have $\int\int x_i f_{v_i} f dxdv = 0$ and

$$\begin{aligned} \int\int \beta(v_i f)_{v_i} f dxdv &= \int\int \beta f^2 + \beta v_i f_{v_i} f dxdv \\ &= \int\int \beta f^2 - \beta(v_i f)_{v_i} f dxdv, \end{aligned}$$

thus

$$\int\int \beta(v_i f)_{v_i} f dxdv = \frac{\beta}{2}\int\int f^2 dxdv.$$

By [1] and the interpolation inequality

$$\int\int f_{x_i v_i} f dxdv \le \frac{\epsilon}{2}\|f_{x_i}\|_{L^2}^2 + \frac{1}{2\epsilon}\|f_{v_i}\|_{L^2}^2,$$

we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}^2 &\le \frac{n\beta}{2}\|f\|_{L^2}^2 - \sigma\|\nabla_v f\|_{L^2}^2 + \epsilon\gamma\|\nabla_x f\|_{L^2}^2 \\ &\quad + \frac{\gamma}{\epsilon}\|\nabla_v f\|_{L^2}^2 - \alpha\|\nabla_x f\|_{L^2}^2 \\ &\le \frac{n\beta}{2}\|f\|_{L^2}^2 + (\frac{\gamma}{\epsilon} - \sigma)\|\nabla_v f\|_{L^2}^2 \\ &\quad + (\epsilon\gamma - \alpha)\|\nabla_x f\|_{L^2}^2. \end{aligned}$$

Note that if $\epsilon = \frac{\alpha-\sigma}{2\gamma} + \sqrt{1 + (\frac{\sigma-\alpha}{2\gamma})^2}$ then one has

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}^2 \le \frac{n\beta}{2}\|f\|_{L^2}^2 - \theta\|\nabla_x f\|_{L^2}^2 - \theta\|\nabla_v f\|_{L^2}^2,$$

where $\theta = \frac{\alpha+\sigma}{2} - \gamma\sqrt{1 + (\frac{\sigma-\alpha}{2\gamma})^2}$.

Lemma 2 Let the coefficients of the equation (1)-(3) satisfy (4), then one has

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla_{x,v} f(t)\|_{L^2}^2 &\le \frac{\beta n}{2}\|\nabla_{x,v} f(t)\|_{L^2}^2 \\ &\quad - \min\{\alpha,\sigma\}\|\Delta_{x,v} f(t)\|_{L^2}^2. \end{aligned}$$

Proof: Multiplication of the equation (1)-(2) by $-\Delta_x f$, using integrations by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla_x f(t)\|_{L^2}^2 &\le \frac{\beta n}{2}\|\nabla_x f(t)\|_{L^2}^2 + n\int\int \nabla_x f \cdot \nabla_v f \\ &\quad -\sigma\int\int \Delta_x f \Delta_v f - \alpha\|\Delta_x f\|_{L^2}^2. \end{aligned}$$

Analogously, multiplication of the equation (1)-(2) by $-\Delta_v f$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla_v f(t)\|_{L^2}^2 &\le \frac{\beta n}{2}\|\nabla_v f(t)\|_{L^2}^2 - n\int\int \nabla_x f \cdot \nabla_v f \\ &\quad -\sigma\|\Delta_v f(t)\|_{L^2}^2 - \alpha\int\int \Delta_x f \Delta_v f. \end{aligned}$$

Moreover, it yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla_{x,v} f(t)\|_{L^2}^2 \le \frac{\beta n}{2}\|\nabla_{x,v} f(t)\|_{L^2}^2 - (\alpha+\sigma)\int\int \Delta_x f \Delta_v f - \min\{\alpha,\sigma\}\|\Delta_{x,v} f(t)\|_{L^2}^2.$$

Note that

$$\int\int \Delta_x f \Delta_v f = \int\int |\sum \partial_{x_i}\partial_{v_j} f|^2.$$

Hence, we can get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla_{x,v} f(t)\|_{L^2}^2 &\le \frac{\beta n}{2}\|\nabla_{x,v} f(t)\|_{L^2}^2 \\ &\quad - \min\{\alpha,\sigma\}\|\Delta_{x,v} f(t)\|_{L^2}^2. \end{aligned}$$

III. Proof of Theorem 1

For a start, we consider the small time behavior of solutions.

Lemma 3 For every $f_0 \in L^2(R^{2n})$, let the equation (1)-(3) satisfy (4), any solution satisfies

$$\sup_{0\le t\le 1}\|f(t)\|_{L^2(R^{2n})} \le C, \ \|\nabla_v f(t)\|_{L^2(R^{2n})} \le Ct^{-\frac{1}{2}}.$$

Proof: Now, by [11] we can utilize a linear expansion in t to prove the assertion. Let $T \in (0,1]$ be given. Consider $t \in (0,T]$ and define

$$y(t) = \|f(t)\|_{L^2}^2 + kt\|\nabla_{x,v} f\|_{L^2}^2,$$

where $k < \min\{2\alpha, 2\sigma, 2\theta\}$.

We differentiate this quantity, and use Lemmas 1-2 to find

$$\begin{aligned} y'(t) &= \frac{d}{dt}\|f(t)\|_{L^2}^2 + k\|\nabla_{x,v} f\|_{L^2}^2 + kt\frac{d}{dt}\|\nabla_{x,v} f(t)\|_{L^2}^2 \\ &\le n\beta\|f(t)\|_{L^2}^2 - 2\min\{\alpha,\theta\}\|\nabla_x f\|_{L^2}^2 \\ &\quad -2\min\{\sigma,\theta\}\|\nabla_v f\|_{L^2}^2 + 2kt\frac{\beta n}{2}\|\nabla_{x,v} f(t)\|_{L^2}^2 \\ &\quad +k\|\nabla_{x,v} f\|_{L^2}^2 - 2kt\min\{\alpha,\sigma\}\|\Delta_{x,v} f(t)\|_{L^2}^2 \\ &\le n\beta\|f(t)\|_{L^2}^2 + kn\beta t\|\nabla_{x,v} f\|_{L^2}^2 \\ &\le Cy(t). \end{aligned}$$

A straightforward application of Gronwall's inequality then implies

$$\|f(t)\|_{L^2}^2 + kt\|\nabla_{x,v} f\|_{L^2}^2 \le C_T\|f_0\|_{L^2}^2$$

The assertion follows directly by applying the above inequality.

Next, we consider the large time behavior of solutions. For a start, we should collect a few standard inequalities for the reader's convenience.

Lemma 4 There is a constant $C = C_\Omega$ such that

$$\int_{R^n}\int_\Omega |f(x,v)|^2 dxdv \le C_\Omega \int_{R^n}\int_\Omega |\nabla_x f|^2 dxdv.$$

Proof: The proof is similar to that of the classical Poincaré inequality, and is only sketched. Without loss of generality, Ω is a rectangle $\Pi_{i=1}^n(a_i,b_i)$. First, the result is

$$\int_{R^n}\int_\Omega |f(x,v)|^2 dxdv \le C(b_1-a_1)\int_{R^n}\int_\Omega |f(x,v)||\partial_{x_1} f(x,v)|dxdv.$$

ISBN: 978-988-14047-7-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

With use of the Hölder inequality, it follows that

$$\int_{R^n}\int_{\Omega}|f(x,v)|^2dxdv \le C\|\partial_{x_1}f(x,v)\|_{L^2}\|f(x,v)\|_{L^2}.$$

Lemma 5 For every $f_0 \in H^1(R^{2n})$, let the equation (1)-(3) satisfy (4). If $v \in R^n$ and $x \in \Omega$ ($\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$), f obeys the prescribed boundary conditions on $\partial\Omega$

$$f(t,x,v) = f(t,y,v),\ \forall\ x,y \in \partial\Omega,\ v \in R^n,\ t \ge 0.$$

Setting $\lambda = \min\{\alpha C_\Omega^{-1}, \theta C_\Omega^{-1}\} - n\beta \ge 0$, any solution of (1.1)satisfies

$$\|f(t)\|_{L^2} \le \|f(t)_0\|_{L^2}e^{-\frac{1}{2}\lambda t},$$

where C_Ω is defined by Lemma 4. Moreover, If $\lambda > 0$, then one has $f \to 0$ as $t \to \infty$.

Proof: Since Lemma 1, we have

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}^2 \le \frac{n\beta}{2}\|f\|_{L^2}^2 - \min\{\alpha,\theta\}\|\nabla_x f\|_{L^2}^2.$$

where $k_1 = -(\frac{\beta n}{2} - \sigma + \frac{n^2}{4\sigma}) \ge 0$. Using Lemma 4, it is clear that

$$\frac{d}{dt}\|f(t)\|_{L^2}^2 + \lambda\|f\|_{L^2}^2 \le 0.$$

The assertion follows directly by applying the Gronwall inequality.

Now, we begin with the proof of Theorem 1.

Proof of Theorem 1 Collecting Lemmas 3 and 5, we obtain the desired results. The proof of Theorem 1 is complete.

REFERENCES

[1] Arnold A., Dhamo E., Manzini C., "The Wigner-Poisson-Fokker-Planck system: global-in-time solution and dispersive effects", *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 2007, 24(4): 645-676.
[2] Arnold A., Carrillo JA., Dhamo E., "On the periodic Wigner-Poisson-Fokker-Planck system", *J. Math. Anal. Appl.*, 2002, 275: 263-276.
[3] Cañizo JA., López JL., Nieto J., "Global L^1 theory and regularity for the 3D nonlinear Wigner-Poisson-Fokker-Planck system", *J. Diff. Equ.*, 2004, 198: 356-373.
[4] Li B., Shen J., "The Wigner-Poisson-Fokker-Planck with exchange potential in weighted L^2 space", *Italian journal of pure and applied mathematics*, 2015, 35(2): 277-292.
[5] Li B., Shen J., "The Initial Boundary Value Problem of the Generalized Wigner System", *Engineering Letters*, 2015, 23(2): 110-114.
[6] Li B., Shen J., "The Wigner-Fokker-Planck system with Hartree-type nonlinearity", *Mathematica applicata*, 2015, 28(4): 836-845.
[7] Sparber C., Carrillo JA., Dolbeault J., Markowich P., "On the long time behavior of the Quantum Fokker-Planck Equation", *Monatsh. Math.*, 2004, 141: 237-257.
[8] Coffey WT., Kalmykov YP., Titov SV., et al., "Wigner function approach to the quantum Brownian motion of a particle in a potential", *Physical Chemistry Chemical Physics*, 2007, 9(26): 3361-3382.
[9] El M.k, Lambropoulos P., "Connection between approximate Fokker-Planck equations and the Wigner function applied to the micromaser", *Quantum Semiclass. Opt.*, 1996, 8: 23-37.
[10] Donarini A., Novotný T., Jauho AP., "Simple models suffice for the single-dot quantum shuttle", *New J. of Physics*, 2005, 7: 237-262.
[11] Pankavich S., Michalowski N., "A short proof of increased parabolic regularity", *Electronic Journal of Differential Equations*, 2015, 2015(205): 1-9.
[12] Arnold A., Gamba IM., Gualdani MP., et al., "The Wigner-Fokker-Planck equation: Stationary states and large time behavior", *Math. Mod. Methods Appl. Sci.*, 2012, 22(11): 1250034-1250065.

ISBN: 978-988-14047-7-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)