A Dynamic Geometry Approach to the Fagnano’s Problem

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Abstract—The Fagnano’s problem is a famous optimization problem in plane geometry, whose solution is given by the inscribed orthic triangle of a given acute triangle. In this paper, we discuss how to solve this problem by the principle of reflection via a dynamic geometry approach. Compared with the method of solutions based on Calculus or Euclidean Geometry only, this approach can be easily adapted to study similar problems in other settings. We will also introduce a very useful formula for finding the perimeter of the orthic triangle.

Index Terms—Fagnano problem, minimal perimeter, orthic triangle, periodic billiard path, dynamic geometry.

I. INTRODUCTION

In 1775, the Italian mathematician called Giovanni Fagnano posed an interesting optimization problem as follows [1-3]:

“For a given acute triangle \( \Delta ABC \), determine the inscribed triangle with minimal perimeter.”

Fagnano used the technique of calculus to solve the problem by himself. He concluded that the inscribed triangle with minimal perimeter is the orthic triangle of \( \Delta ABC \). By definition, the orthic triangle of \( \Delta ABC \) is the triangle formed by joining the endpoints of the altitudes drawn from the vertices of \( \Delta ABC \) (see Fig. 1).

![Fig. 1. \( \Delta DEF \) is the orthic triangle of \( \Delta ABC \)](image)

There are some important and interesting results concerned with the orthic triangle, which are stated below.

These results can be proved by elementary plane geometry and they will not be discussed in this paper (see [2, 3, 5]). For convenience, we will use the same notations as above to describe the results.

Theorem 1.1. The altitudes \( \angle CFB, \angle AED, \angle BFC \) of \( \Delta DEF \) bisect the angles \( \angle DFE, \angle FED, \angle EDF \) of \( \Delta DEF \), respectively.

Theorem 1.2. The following angles in \( \Delta ABC \) are equal.

\[(i) \angle ACB = \angle AFD = \angle BFE
(ii) \angle ABC = \angle ADF = \angle CDE
(iii) \angle CAB = \angle BEF = \angle CED
(iv) \angle CAE = \angle CFD = \angle CFE = \angle CBD
(v) \angle BCF = \angle BDE = \angle BDF = \angle BAE
(vi) \angle ABD = \angle AEF = \angle AED = \angle ACF\]

Besides using calculus, there are other methods for solving the Fagnano’s problem, which can be found in [2, 3, 6, 8, 9, 11, 13]. In the next section, we will introduce a simple approach to solve this problem by the principle of reflection, via the use of the dynamic geometry software GeoGebra as a tool. Then, we will describe a useful formula for finding the length of the perimeter of the inscribed orthic triangle \( \Delta DEF \). This approach can be easily adapted to study similar problems in other settings (e.g. \( \Delta ABC \) is right or obtuse triangle). We expect this approach will be found useful for reference by researchers working on the related areas, as well as lecturers or teachers who are interested to introduce the Fagnano’s problem for illustration or discussion purposes in their own classes.

II. HOW TO SOLVE THE FAGNANO’S PROBLEM

Before we discuss how to solve the Fagnano’s problem, let us introduce a very useful technique for solving the shortest path problem below [4].

“If \( A, B \) are two points lying on the same side of a line, find a point \( O \) on the line such that the path \( A-O-B \) is the shortest.”

Fig. 2 shows that we can first use the given line as the axis of reflection to locate the image of reflection of \( B \), say \( B_1 \), on the opposite side, and then determine the position of the required point \( O \), which is the intersection between \( AB_1 \) and the given line. According to the principle of reflection, \( OB = OB_1 \), so \( AO + OB = AO + OB_1 = AB_1 \). Since \( AB_1 \) is a line segment, the path obtained is the shortest possible. In fact, if one choose an arbitrary point \( P \) not at the position \( O \), the path \( A-P-B \) will be longer than \( A-O-B \) since \( AP + PB = AP + PB_1 > AB_1 \), due to the well-known “triangular inequality”
By the principle of reflection, we have \( \angle \alpha = \angle \beta \) (see Fig. 2). On the other hand, \( \angle \beta = \angle \gamma \), as they are vertically opposite angles. Hence, \( \angle \alpha = \angle \gamma \). We can summarize this important result as follows [9].

**Theorem 2.1.** The shortest path joining two given points on one side of a line, and meeting this line, is a broken line whose parts make equal angles with the given line.

We now illustrate how to solve the Fagnano's problem by using the dynamic geometry software GeoGebra as an aids. First, we draw an acute triangle \( \Delta ABC \) with altitudes \( AE, BD \) and \( CF \), and a point \( P \) on \( AB \) as shown in Fig. 4.

By the principle of reflection, we have \( \angle CMN = \angle BMP \) and \( \angle CNM = \angle ANP \). However, in general, \( \angle APN \) may not equal to \( \angle BPM \). Now, by moving \( P \) along \( AB \) and using the angle measurement tool built-in GeoGebra, we can observe that \( \angle APN = \angle BPM, \angle CMN = \angle BMP \) and \( \angle CNM = \angle ANP \) occur when \( P, M, N \) coincide with the points \( F, E \) and \( D \), respectively (see Fig. 6). According to Theorem 2.1, the shortest path from \( N \) to \( M \) via \( P \) is \( N-P-M \). Similarly, the shortest path from \( P \) to \( N \) via \( M \) is \( P-M-N \), and the shortest path from \( M \) to \( P \) via \( N \) is \( M-P-N \). This implies that the inscribed triangle of \( \Delta ABC \) with minimal perimeter is the orthic triangle \( \Delta DFE \), which is the solution of the Fagnano's problem. In fact, it is a unique solution since there is only one inscribed orthic triangle in \( \Delta ABC \).
We can summarize the conclusion by the following theorem.

**Theorem 2.2.** Of all triangles inscribed in the given acute triangle \( \triangle ABC \), the orthic triangle \( \triangle DEF \) has the minimal perimeter.

### III. FINDING THE PERIMETER OF THE ORTHIC TRIANGLE

Let \( BC = a \), \( AC = b \), \( AB = c \), and \( \angle CAB = \theta_1 \), \( \angle ACB = \theta_2 \), \( \angle ABC = \theta_3 \). We now derive a formula for finding the perimeter of the inscribed orthic triangle \( \triangle DEF \).

We can refer to the diagram below (Fig. 7).

![Fig. 7. Several equal angles in \( \triangle ABC \)](image)

By Theorem 1.2, \( \theta_1 = \alpha_1 \), \( \theta_1 = \alpha_2 \), \( \theta_1 = \beta_1 \), \( \theta_2 = \gamma_1 = \gamma_2 \). Since \( \triangle ADF \sim \triangle ABC \), so \( DF/BC = AF/AC \). Hence,

\[
DF = a \times \frac{b \sin(\pi/2 - \theta_1)}{b} = a \cos \theta_1
\]

Also, \( \triangle BFE \sim \triangle BCA \), so \( FE/AC = BE/AB \). Hence,

\[
EF = b \times \frac{c \sin(\pi/2 - \theta_1)}{c} = b \cos \theta_3
\]

Similarly, \( \triangle CED \sim \triangle CAB \), so \( DE/AB = CE/AC \). Hence,

\[
DE = c \times \frac{b \sin(\pi/2 - \theta_1)}{b} = c \cos \theta_2
\]

Thus, we have the following result.

**Theorem 3.1.** The perimeter of the orthic triangle \( \triangle DEF \) inscribed in \( \triangle ABC \) can be computed by the formula:

\[
\delta_{\triangle DEF} = a \cos \theta_1 + b \cos \theta_3 + c \cos \theta_2
\]

where \( \delta_{\triangle DEF} \) denotes the perimeter of \( \triangle DEF \).

**Example.** Find the perimeters of the sequence of orthic triangles in \( \triangle ABC \) if the latter is an equilateral triangle with unit side length.

Solution. Let \( \triangle i \) denotes the \( i \)-th orthic triangle in \( \triangle ABC \). By Theorem 3.1, we have

\[
\delta_{\triangle i} = 3 \cos \frac{\pi}{3} = \frac{3}{2}
\]

\[
\delta_{\triangle 1} = 3 \times 1 \cos \frac{\pi}{3} = \frac{3}{4}
\]

\[
\delta_{\triangle 2} = \frac{1}{4} \cos \frac{\pi}{3} = \frac{3}{8}
\]

In general, we have: \( \delta_{\triangle i} = 3 \times 1 \cos \frac{\pi}{3} = \frac{3}{2} \).

### IV. CONCLUDING REMARKS

In this paper, we have discussed how to adopt a dynamic geometry approach to solve the Fagnano’s problem. We have also derived a useful formula for finding the perimeter of the inscribed orthic triangle of an acute triangle. Some recent researches on the study of periodic billiard trajectories in polygons are actually originated from the Fagnano’s minimal perimeter problem, whose solution (i.e. the inscribed orthic triangle) is now often called a closed 3-periodic billiard trajectory or orbit. If the given triangle is a right triangle instead of an acute triangle, then we can use similar dynamic geometry approach to explore and conclude that every right triangle has a periodic billiard trajectory, although the latter is not closed. For instance, Fig. 8 illustrate an example of an open periodic billiard trajectory in a right triangle. In fact, there are some open problems related to periodic billiard trajectories in triangles or \( n \)-sided (\( n > 3 \)) polygons. Readers may refer to [7, 11, 12] for getting some ideas on the recent results and open problems in this interesting research area.
Fig. 8. A periodic billiard trajectory in a right triangle

REFERENCES