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An Algorithm for Solving Triple Hierarchical Pseudomonotone Variational Inequalities

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Abstract—**In this paper, we introduce and analyze a hybrid steepest-descent extragradient algorithm for solving triple hierarchical pseudomonotone variational inequalities in a real Hilbert space. The proposed algorithm is based on Korpelevich's extragradient method, Mann's iteration method, hybrid steepest-descent method and Halpern's iteration method. Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is derived. Our results improve and extend the corresponding results in the earlier and recent literature.**

Keywords: Triple hierarchical variational inequalities; Hybrid steepest-descent extragradient approach; Pseudomonotonicity; Lipschitz continuity; Global convergence.

1 Introduction

Throughout this paper, we let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. Recall that a mapping $S : C \to H$ is called

(i) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall\ x, y \in C$;

(ii) *pseudomonotone* if $\langle Ay, x - y\rangle \geq 0$ implies $\langle Ax, x - y\rangle \geq 0$, $\forall\ x, y \in C$;

(iii) *η-strongly monotone* if there exists a constant $\eta > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \forall\ x, y \in C$;

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(vi) *α-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall\ x, y \in C$;

(v) *L-Lipschitz continuous* (or *L-Lipschitzian*) if there exists a constant $L \geq 0$ such that $\|Sx - Sy\| \leq L\|x - y\|, \forall\ x, y \in C$. In particular, if $L = 1$ then S is called a *nonexpansive mapping*; if $L \in [0, 1)$ then S is called a *contraction.*

Let $\mathcal{A} : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (in short, VIP) is to find $x \in C$ such that

$$\langle \mathcal{A}x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The solution set of VIP (1) is denoted by $\mathrm{VI}(C, \mathcal{A})$.

The VIP (1) was first discussed by Lions [29]. There are many applications of VIP (1) in various fields; see e.g., [20, 32, 35, 41]. In 1976, Korpelevich [28] proposed an iterative algorithm, which is known as the extragradient method, for solving the VIP (1) in Euclidean space $\mathbf{R}^n$. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [5, 6, 9, 10, 11, 12, 14, 16, 17, 21, 31, 39, 43] and references therein, to name but a few.

In 2001, Yamada [38] introduced a hybrid steepest-descent method for solving the VIP (1) with $C = \mathrm{Fix}(T)$. The problem of finding a point in $\mathrm{VI}(\mathrm{Fix}(T), \mathcal{A})$ is called a hierarchical VIP or a hierarchical fixed point problem. Yamada's hybrid steepest-descent method has received great attention given by many authors, who improved it in various ways; see e.g., [8, 15, 36, 42] and references therein.

On the other hand, let $A : C \to H$ and $B : H \to H$ be two mappings. Consider the following bilevel variational inequality problem (BVIP):

BVIP. Finding $x^* \in \mathrm{VI}(C, B)$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathrm{VI}(C, B), \tag{2}$$

where $\mathrm{VI}(C, B)$ denotes the set of solutions of the VIP: Finding $y^* \in C$ such that

$$\langle By^*, y - y^*\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

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In particular, whenever $H = \mathbf{R}^n$, the BVIP was recently studied by Anh, Kim and Muu [1].

Bilevel variational inequalities are special classes of quasivariational inequalities (see [2, 4, 18, 37]) and of equilibrium with equilibrium constraints considered in [30, 25]. However it covers some classes of mathematical programs with equilibrium constraints (see [30]), bilevel minimization problems (see [34]), variational inequalities (see [19, 27, 40, 44]) and complementarity problems. In 2012, Anh, Kim and Muu [1] introduced an extragradient iterative algorithm for solving the above bilevel variational inequality.

It is well known that the Tikhonov regularization method is an important approach to the BVIP. Recently, in [22, 25, 26] the Tikhonov method with generalized regularization operators and bifunctions is extended to pseudomonotone variational inequalities and equilibrium problems, respectively. However in this case, the regularized subproblems, may fail to be strongly monotone, even pseudomonotone, since the sum of a strongly monotone operator and a pseudomonotone operator, in general, is not pseudomonotone. In our opinion, the existing methods that require some monotonicity properties cannot be applied to solve the regularized subvariational inequalities. Therefore the extragradient-type algorithm is an efficient approach for directly solving bilevel pseudomonotone variational inequalities.

Furthermore, we recall the variational inequality for a monotone operator $A_1 : H \to H$ over the fixed point set of a nonexpansive mapping $T : H \to H$ as follows: Find $\bar{x} \in \mathrm{VI}(\mathrm{Fix}(T), A_1)$, where

$$\begin{aligned}&\mathrm{VI}(\mathrm{Fix}(T), A_1)\\ &\quad := \quad \{\bar{x} \in \mathrm{Fix}(T) : \langle A_1\bar{x}, y - \bar{x}\rangle \geq 0, \forall y \in \mathrm{Fix}(T)\},\end{aligned}$$

and $\mathrm{Fix}(T) := \{x \in H : Tx = x\} \neq \emptyset$. In [23, 24], Iiduka introduced the following monotone variational inequality with variational inequality constraint over the fixed point set of a nonexpansive mapping:

Problem I (see [23, Problem 3.1]). Assume that

(i) $T : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$;

(ii) $A_1 : H \to H$ is α-inverse strongly monotone;

(iii) $A_2 : H \to H$ is β-strongly monotone and L-Lipschitz continuous;

(iv) $\mathrm{VI}(\mathrm{Fix}(T), A_1) \neq \emptyset$.
Then the objective is to find $x^* \in \mathrm{VI}(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2)$, where,

$$\begin{aligned}&\mathrm{VI}(\mathrm{VI}(\mathrm{Fix}(T), A_1), A_2)\\ &\quad := \quad \{x^* \in \mathrm{VI}(\mathrm{Fix}(T), A_1) : \langle A_2x^*, v - x^*\rangle \geq 0,\\ &\qquad\quad \forall\, v \in \mathrm{VI}(\mathrm{Fix}(T), A_1)\}.\end{aligned}$$

Since this problem has a triple structure in contrast with bilevel programming problems (see [30, 33]) or hierarchical constrained optimization problems or hierarchical fixed point problem, it is referred to as a triple-hierarchical constrained optimization problem (THCOP). More precisely, it is referred as a triple hierarchical variational inequality problem (THVIP); see Ceng, Ansari and Yao [8]. Very recently, some authors continued the study of Iiduka's THVIP (i.e., Problem I) and its variant and extension; see e.g., [3, 7, 8, 13, 15, 42].

Motivated and inspired by the above facts, we introduce and analyze a hybrid steepest-descent extragradient algorithm for solving triple hierarchical pseudomonotone variational inequalities, which will be defined in Section 2. The proposed algorithm is based on Korpelevich's extragradient method (see [28]), Mann's iteration method, hybrid steepest-descent method (see [38]) and Halpern's iteration method. Under mild conditions, the strong convergence of the iteration sequences generated by the algorithm is derived. Our results improve and extend the corresponding results announced by some others, e.g., Iiduka [23, Theorem 4.1] and Anh, Kim amd Muu [1, Theorem 3.1].

2 Main Results

Throughout this section, we always assume the following:

- $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$;
- $T : H \to H$ is a nonexpansive mapping;
- $A, B : H \to H$ satisfy the hypotheses (H1)-(H4):

  (H1) B is pseudomonotone on H and A is β-strongly monotone on H;

  (H2) A is L_1-Lipschitz continuous on H;

  (H3) B is L_2-Lipschitz continuous on H;

  (H4) $\mathrm{VI}(\mathrm{Fix}(T), B) \neq \emptyset$;
- μ and τ are two positive numbers such that $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < \tau \leq 1$ with $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

In this paper, we will consider the following triple hierarchical pseudomonotone variational inequality problem (THPVIP) defined over the fixed point set of a nonexpansive mapping $T : H \to H$.

Problem 1 *Finding* $x^* \in \mathrm{VI}(\mathrm{Fix}(T), B)$ *such that*

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathrm{VI}(\mathrm{Fix}(T), B), \tag{4}$$

where $\mathrm{Fix(T)} := \{x \in H : Tx = x\}$, *and* $\mathrm{VI}(\mathrm{Fix}(T), B)$ *denotes the set of solutions of the VIP: Finding* $y^* \in \mathrm{Fix}(T)$ *such that*

$$\langle By^*, y - y^*\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{5}$$

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For solving Problem 1, we propose the following algorithm:

Algorithm 2 *Initialization. Choose* $u \in H$, $x_0 \in H$, $k = 0$, $0 < \lambda \leq \frac{2\beta}{L_1^2}$, *positive sequences* $\{\delta_k\}, \{\lambda_k\}, \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ *and* $\{\bar{\epsilon}_k\}$ *such that*

$$\begin{cases} \lim_{k\to\infty} \delta_k = 0, \ \sum_{k=0}^{\infty} \bar{\epsilon}_k < \infty, \\ \alpha_k + \beta_k + \gamma_k = 1 \ \forall k \geq 0, \ \sum_{k=0}^{\infty} \alpha_k = \infty, \ \lim_{k\to\infty} \alpha_k = 0, \\ \lim_{k\to\infty} \beta_k = \xi \in (0, \frac{1}{2}], \ \lim_{k\to\infty} \lambda_k = 0, \ \lambda_k \leq \frac{1}{L_2} \quad \forall k \geq 0. \end{cases}$$

- **Step 1.** *Compute*
$$\begin{cases} v_k := Tx_k - \lambda_k \mu F(Tx_k), \\ y_k := P_{\mathrm{Fix}(T)}(v_k - \lambda_k B v_k), \\ z_k := P_{\mathrm{Fix}(T)}(v_k - \lambda_k B y_k). \end{cases}$$
- **Step 2.** *Inner loop* $j = 0, 1, \ldots$. *Compute*
$$\begin{cases} x_{k,0} := z_k - \lambda A z_k, \\ y_{k,j} := P_{\mathrm{Fix}(T)}(x_{k,j} - \delta_j B x_{k,j}), \\ x_{k,j+1} := \alpha_j x_{k,0} + \beta_j x_{k,j} \\ \qquad + \gamma_j P_{\mathrm{Fix}(T)}(x_{k,j} - \delta_j B y_{k,j}). \\ \mathit{If}\ \|x_{k,j+1} - P_{\mathrm{VI}(\mathrm{Fix}(T),B)} x_{k,0}\| \leq \bar{\epsilon}_k \ \textit{then set} \\ h_k := x_{k,j+1} \ \textit{and go to Step } 3. \\ \textit{Otherwise, increase } j \text{ by } 1 \textit{ and repeat} \\ \textit{the inner loop Step } 2. \end{cases}$$
- **Step 3.** *Set* $x_{k+1} := \alpha_k u + \beta_k x_k + \gamma_k h_k$. *Then increase* k *by* 1 *and go to Step 1.*

First of all, we have the following lemma.

Lemma 1 *Suppose that the hypotheses (H1)-(H4) hold. Then the sequence* $\{x_{k,j}\}$ *generated by Algorithm 2 converges strongly to the point* $P_{\mathrm{VI}(\mathrm{Fix}(T),B)}(z_k - \lambda A z_k)$ *as* $j \to \infty$. *Consequently, we have*

$$\|h_k - P_{\mathrm{VI}(\mathrm{Fix}(T),B)}(z_k - \lambda A z_k)\| \leq \bar{\epsilon}_k \quad \forall k \geq 0.$$

In the sequel we always suppose that the inner loop in the Algorithm 2 terminates after a finite number of steps. This assumption, by Lemma 1, is satisfied when B is monotone on $\mathrm{Fix}(T)$.

Moreover, the following lemmas are needed to derive the strong convergence of the iteration sequences generated by our algorithm 2.

Lemma 2 *Let sequences* $\{v_k\}, \{y_k\}$ *and* $\{z_k\}$ *be generated by Algorithm 2,* B *be* L_2*-Lipschitzian and pseudomonotone on* H*, and* $p \in \mathrm{VI}(\mathrm{Fix}(T), B)$. *Then, we have*

$$\begin{aligned} \|z_k - p\|^2 \ \leq \ & \|v_k - p\|^2 - (1 - \lambda_k L_2)\|v_k - y_k\|^2 \\ & -(1 - \lambda_k L_2)\|y_k - z_k\|^2. \end{aligned} \tag{6}$$

Lemma 3 *Suppose that the hypotheses (H1)-(H4) hold. Then the sequence* $\{x_k\}$ *generated by Algorithm 2 is bounded.*

Lemma 4 *Suppose that the hypotheses (H1)-(H4) hold. Assume that the sequences* $\{v_k\}$ *and* $\{z_k\}$ *are generated by Algorithm 2. Then, we have*

$$\begin{aligned} & \|z_{k+1} - z_k\| \\ \leq \ & (1 + \lambda_{k+1} L_2)\|v_{k+1} - v_k\| + \lambda_k \|B y_k\| \\ & + \lambda_{k+1}(\|B v_{k+1}\| + \|B y_{k+1}\| + \|B v_k\|). \end{aligned} \tag{7}$$

Moreover,

$$\lim_{k\to\infty} \|z_{k+1} - z_k\| = \lim_{k\to\infty} \|v_{k+1} - v_k\| = 0.$$

Lemma 5 *Suppose that the hypotheses (H1)-(H4) hold. Then for any solution* x^* *of Problem 1 (THPVIP) we have*

$$\begin{aligned} & \|x_{k+1} - x^*\|^2 \\ \leq \ & \alpha_k \|u - x^*\|^2 + \beta_k \|x_k - x^*\|^2 + \gamma_k \|v_k - x^*\|^2 \\ & + 2\gamma_k \bar{\epsilon}_k \|z_k - x^*\| + \gamma_k \bar{\epsilon}_k^2 \\ & - \gamma_k (1 - \lambda_k L_2)(\|v_k - y_k\|^2 + \|y_k - z_k\|^2). \end{aligned} \tag{8}$$

Moreover, if $\lim_{k\to\infty} \|x_k - Tx_k\| = 0$, *then*

$$\begin{aligned} & \lim_{k\to\infty} \|P_{\mathrm{VI}(\mathrm{Fix}(T),B)}(z_k - \lambda_k A z_k) - z_k\| \\ = \ & \lim_{k\to\infty} \|P_{\mathrm{VI}(\mathrm{Fix}(T),B)}(v_k - \lambda_k A v_k) - v_k\| = 0. \end{aligned}$$

By using the above lemmas, we can obtain the following result.

Theorem 3 *Suppose that the hypotheses (H1)-(H4) hold. If* $\lim_{k\to\infty} \|x_k - Tx_k\| = 0$, *then the two sequences* $\{x_k\}$ *and* $\{z_k\}$ *in Algorithm 2 with* $0 < \lambda < \frac{2\beta}{L_1^2}$ *converge strongly to the same point* x^* *which is a solution of Problem 1 (THPVIP).*

3 Concluding Remarks

Theorem 3 extends, improves, supplements and develops Iiduka [23, Theorem 4.1] and Anh, Kim and Muu [1, Theorem 3.1] in the following aspects.

- The problem of finding a solution x^* of Problem 1 (THPVIP) is very different from the problem of finding a solution x^* in Problem 3.1 (THVIP) of Iiduka [23], because our THPVIP generalizes the Iiduka's THVIP from the inverse-strongly monotone mapping A_1 to the pseudomonotone and Lipschitzian mapping B.

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- In the meantime, the problem of finding a solution x^* of Problem 1 (THPVIP) is very different from the problem of finding a solution x^* of BVIP, because our THPVIP generalizes Anh, Kim and Muu's BVIP in [1, Theorem 3.1] from the space $\mathbf{R}^n$ to the general Hilbert space H, and extends Anh, Kim and Muu's BVIP in [1, Theorem 3.1] to the setting of the THPVIP defined over the fixed point set of a nonexpansive mapping $T : H \to H$.

- The Algorithm 2.1 in [1] is extended to develop Algorithm 2 by virtue of Yamada's hybrid steepest-descent method [38]. The Algorithm 2 is more advantageous and more flexible than Algorithm 2.1 in [1] because it involves solving the BVIP (with $C = \mathrm{Fix}(T)$) and the fixed point problem of a nonexpansive mapping.

- Our Algorithm 2 is very different from Algorithm 2.1 in [1], because our Algorithm 2 is based on Korpelevich's extragradient method, Mann's iteration method, hybrid steepest-descent method and Halpern's iteration method.

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