An Improvement in the Artificial-free Technique along the Objective Direction for the Simplex Algorithm

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Abstract—The artificial-variable free technique along the objective direction for the simplex algorithm was proposed for solving a linear programming problem in 2014. It was designed to deal especially with unrestricted variables. Before starting the simplex algorithm, a single unrestricted variable is rewritten using two nonnegative variables causing the number of variables increasing. In this paper, we present that this technique can deal with nonnegative variables which the real world problems needed the variables to be nonnegative. Moreover, we propose some criteria for selecting a variable to transform the problem. The computational results show that the average number of iterations from the selected variable which has the maximum coefficient of the objective function outperform the original simplex algorithm.

Index Terms—artificial-free, linear programming, transformed problem, relaxed problem, objective direction.

I. INTRODUCTION

THE simplex algorithm [1] has been popularly used to solve linear programming problems and it is quite efficient in practice for small or medium size [2]. However, in 1972, a collection of linear programming problems which shows the worst case running time for the simplex algorithm using the original Dantzig's rule was given by Klee and Minty [3]. Then, Karmarkar [4] has proposed a new faster algorithm and many researchers have improved pivoting rule for the simplex algorithm [5]. However, finding the new initial feasible point which is closer the optimal point and solving without artificial variables were proposed [2], [6], [7], [9], [10], [11], [12].

The simplex algorithm starts at an initial basic feasible solution, $\mathbf{x} = \mathbf{0}$ (the origin point). If the origin point is not a basic feasible solution then the artificial variables will be added for finding the basic feasible solution. Big-M and Two-Phase method are well-known algorithms which are used to handle artificial variables. By adding artificial variables, the number of variables will be increased.

In 1997, Arsham [6], [7] proposed the algorithm without using artificial variables. However, in 1998, Enge and Huhn [8] proposed a counterexample, in which Arsham's algorithm declared the infeasibility of a feasible problem.

In 2000, the algorithm for solving a linear programming problem without adding artificial variables was proposed again by Pan [9]. The algorithm starts at the initial basis

which corresponds to a solution in primal and a solution in dual. If the primal solution and the dual solution are infeasible, then costs of objective function in primal will be perturbed to some positive number for dual feasibility and the dual simplex method can start. The computational results were shown to be superior for small problems and this algorithm starts from the origin point.

Arsham [10], [11] then presented the new solution algorithm without using artificial variables repeatedly in 2006. Before the algorithm starts, the right hand side values need to be nonnegative. The simplex algorithm can start without using artificial variable by relaxing the $\geq$ constraints. Then all relaxed constraints will be reinserted to the problem to guarantee the optimal solution. However, the computational result was not shown the effectiveness of the algorithm. In that year, Corley et al. [12] proposed the algorithm without introducing artificial variables for nonnegative right hand side problems. They solve the relaxed problem which consists of the original objective function subject to a single constraint which makes a largest cosine angle with the gradient vector of the objective function. At each subsequent iteration, the constraint which had the new maximum cosine angle among those constraints would be added, and the dual simplex method is applied. However, their research is restricted for a feasible and bounded linear programming problem and the computational experiment was not shown.

In 2014, Boonperm and Sinapiromsaran [13] presented the algorithm without using artificial variables which starts by fixing one of variables which has a nonzero coefficient of the objective function in term of another variable in the objective plane. Then constraints are split into three groups: the positive group, the negative group and the zero group. Moving on the objective direction, the optimal solution may be formed by some constraints in the positive group. So the algorithm starts by relaxing constraints from the negative and the zero groups. The simplex algorithm is applied to solve this relaxed problem, and additional constraints from the negative the zero coefficient groups will be introduced to the relaxed problem, and the dual simplex method is used to determine the new optimal solution. However, they lacked the computational result to show the effectiveness of the algorithm. Then, in that year, Boonperm and Sinapiromsaran [14] proposed the algorithm without using artificial variables by separating constraints into two groups: the acute constraint group and the non-acute constraint group. They start the algorithm by relaxing the non-acute constraint and solve the relaxed problem first. The results of the algorithm are superior than the original simplex algorithm. However, the algorithm deals with unrestricted variables which it

is rewritten using two nonnegative variables causing the number of variables increasing by a factor of two for each unrestricted variable.

From the artificial-free techniques above, in this paper, we improve the artificial-free technique along the objective direction [13] by suggestion the criteria to choose the variable which is used to map the problem. Moreover, the improved algorithm can deal with the nonnegative varaiables. The proposed algorithm starts by using the objective plane to split constraints into three groups by considering the sign of coefficient for each constraint. Then constraints from the nonpositive groups (negative and zero coefficient groups) are relaxed. The relaxed problem can identify a feasible point by our theorem. Then it will be transformed for starting the simplex algorithm without using artificial variables. Constraints from the nonpositive groups are added for checking the solution from the relaxed problem. Since artificial variables is not used, the number of variables by our algorithm is less than or equal to the number of variables by the simplex algorithm. The number of constraints which solved by our algorithm is less-than or equal to the number of constraints which solved by the simplex algorithm. This is an obvious advantage of our algorithm. Moreover, we suggest the criteria to select the variable for mapping. From the computational results, we found that iterations from the selected variable which has the maximum cost of the objective function outperform the simplex algorithm, and another criteria. The main concept of our algorithm and theorems for guarantee feasibility of a relaxed linear programming problem is shown in section 2. In section 3, after the relaxed problem is solved by the simplex algorithm, constraints from the negative group and the zero group will be added and analysed for finding the optimal solution. In section 4, the computational results show the effectiveness of the algorithm. In the last section, we conclude and discuss our new findings.

II. PRELIMINARIES

Consider a linear programming problem in the following form:

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c^T x} \\ \text{subject to} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned} \tag{1}$$

where $\mathbf{c}$ is a nonzero vector and $\mathbf{x}$ is an n-dimensional column vector, $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ is an m-dimensional column vector. Let

$$y = \mathbf{c^T x} = c_1 x_1 + c_2 x_2 + \cdots + c_i x_i + \cdots + c_n x_n. \tag{2}$$

If $c_i \neq 0$, we determine

$$x_i = \frac{y}{c_i} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{c_j}{c_i} x_j. \tag{3}$$

After replacing x_i into the problem (1), the problem in an equivalent form can be written as follows:

Maximize y

subject to $\quad \frac{a_{li}}{c_i} y + \sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{lj} - \frac{a_{li} c_j}{c_i}) x_j \quad \leq b_l, l = 1, 2, ..., m$

$$\frac{y}{c_i} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{c_j}{c_i} x_j \quad \geq 0$$

$$x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \quad \geq 0. \tag{4}$$

Since $x_i \geq 0$, $\frac{y}{c_i} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{c_j}{c_i} x_j \geq 0$ is added to be the $(m+1)^{th}$ constraint. Then the problem (4) can be rewritten as follows:

Maximize y

subject to $\quad y \leq b'_r - \sum_{\substack{j=1, \\ j \neq i}}^{n} a'_{rj} x_j, \quad r \in M_1$

$$y \geq b'_s - \sum_{\substack{j=1, \\ j \neq i}}^{n} a'_{sj} x_j, \quad s \in M_2$$

$$\sum_{\substack{j=1, \\ j \neq i}}^{n} a_{tj} x_j \leq b_t, \quad t \in M_3$$

$$x_1, ..., x_{i-1}, \quad x_{i+1}, ..., x_n \geq 0. \tag{5}$$

where M_1, M_2 and M_3 are the set of the constraint r, s, t if $\frac{a_{ri}}{c_i} > 0$, $\frac{a_{si}}{c_i} < 0$ and $\frac{a_{ti}}{c_i} = 0$ respectively, and $b'_l = \frac{c_i b_l}{a_{li}}$, $a'_{lj} = \frac{c_i a_{lj}}{a_{li}} - c_j$, $l \in \{M_1 \cup M_2\}$, $j = 1, 2, ..., n$ and $j \neq i$.

Consider the $(m+1)^{th}$ constraint, we let $a'_{m+1,j} = c_j$, $j = 1, ..., i-1, i+1, ..., n$ and $b'_{m+1} = 0$. If $c_i > 0$ then $m+1$ will be added to M_2. Otherwise, $m+1$ will be added to M_1. So $|M_1| = m_1, |M_2| = m_2, |M_3| = m_3$ and $m_1 + m_2 + m_3 = m + 1$.

Since we classify groups by the value of coefficients, M_1 is the group of positive coefficients, M_2 is the group of negative coefficients, and M_3 is the group of zero coefficients.
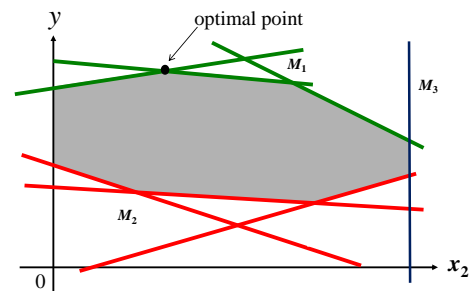

Fig. 1. Example of a feasible region after mapping in $\mathbb{R}^2$

Pleasantly, the problem (5) can be written in the matrix form as follows:

$$\begin{aligned} \text{Maximize} \quad & y \\ \text{subject to} \quad & \mathbf{1}_{m_1} y \leq \mathbf{b}'_{m_1} - \mathbf{A}'_{m_1} \mathbf{x}_{\bar{i}}, \\ & \mathbf{1}_{m_2} y \geq \mathbf{b}'_{m_3} - \mathbf{A}'_{m_2} \mathbf{x}_{\bar{i}}, \\ & \mathbf{A}_{m_3} \mathbf{x}_{\bar{i}} \leq \mathbf{b}_{m_3} \\ & \mathbf{x}_{\bar{i}} \geq \mathbf{0} \end{aligned} \tag{6}$$

where $\mathbf{1}_{m_1}$ is an m_1-dimensional column vector of 1, $\mathbf{1}_{m_2}$ is an m_2-dimensional column vector of 1, $\mathbf{b}'_{m_1}$, $\mathbf{b}'_{m_2}$ and $\mathbf{b}_{m_3}$ are the right hand side vector corresponding to the group M_1, M_2 and M_3, respectively. $\mathbf{A}'_{m_1}$, $\mathbf{A}'_{m_2}$ and $\mathbf{A}_{m_3}$ are submatrices corresponding to the group M_1, M_2 and M_3, repectively. $\mathbf{x}_{\bar{i}}^{\mathbf{T}} = [x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n]^{\mathbf{T}}$.

A. The relaxed problem

Consider the problem (6), it will be feasible when y is between constraints in the group of M_1 and constraints in group M_2, and $\mathbf{x}_{\bar{i}}$ satisfies constraints in group M_3 as figure 1. Since we want to maximize y, the optimal solution will be formed by some constraints from group M_1. So we will relax some constraints from group M_2 and M_3. Then, we will solve the problem with constraints from the group M_1 first. Therefore, the relaxed problem can be rewritten as follows:

$$\begin{aligned} \text{Maximize} \quad & y \\ \text{subject to} \quad & \mathbf{1}_{m_1} y \le \mathbf{b}'_{m_1} - \mathbf{A}'_{m_1} \mathbf{x}_{\bar{i}} \\ & \mathbf{x}_{\bar{i}} \ge \mathbf{0} \end{aligned} \tag{7}$$
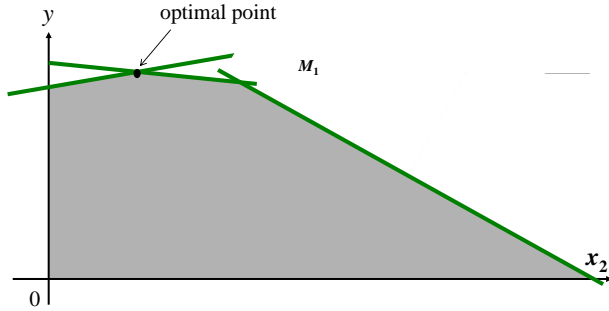
We will show that the problem (7) is always feasible.

Fig. 2. Example of a feasible region of the relaxed problem (7) in $\mathbb{R}^2$

Theorem 1. *If $M_1 \ne \emptyset$ and $b'_{min} = \min\limits_{r \in M_1} \{b'_r\}$. Then $(y_0, \mathbf{x_{\bar{i}0}^T}) = (\mathbf{b'_{min}}, \mathbf{0^T})$ is a feasible point of the problem (7).*

Proof: Suppose $M_1 \ne \emptyset$ and $b'_{min} = \min\limits_{r \in M_1} \{b'_r\}$. Choose $(y_0, \mathbf{x}_{\bar{i}0}^T) = (b'_{min}, \mathbf{0^T})$. We get $\mathbf{1}_{m_1} y_0 = \mathbf{1}_{m_1} b'_{min} \le \mathbf{b}'_{m_1}$. So $(y_0, \mathbf{x}_{\bar{i}0}^T) = (b'_{min}, \mathbf{0^T})$ is a feasible point of the problem (7). The problem (7) is always feasible. ∎

If $M_1 = \emptyset$ and $M_3 = \emptyset$ then the problem (6) can be rewritten as follows:

$$\begin{aligned} \text{Maximize} \quad & y \\ \text{subject to} \quad & \mathbf{1}_{m_2} y \ge \mathbf{b}'_{m_2} - \mathbf{A}'_{m_2} \mathbf{x}_{\bar{i}}, \\ & \mathbf{x}_{\bar{i}} \ge \mathbf{0} \end{aligned} \tag{8}$$

We can show that the problem (8) is unbounded.

Theorem 2. *If $M_1 = \emptyset$ and $M_3 = \emptyset$ then the problem (8) is unbounded.*

Proof: Assume $M_1 = \emptyset$ and $M_3 = \emptyset$. Let $X = \{(y, \mathbf{x}_{\bar{i}}^T) | \mathbf{1}_{m_2} y \ge \mathbf{b}_{m_2} - \mathbf{A}'_{m_2} \mathbf{x}_{\bar{i}}\}$ and $b'_{max} = \max\limits_{s \in M_2} \{b'_s\}$. Show that X is not empty.

Choose $(y_0, \mathbf{x}_{\bar{i}}^T) = (b'_{max}, \mathbf{0^T})$. We get $\mathbf{1}_{m_2} y_0 = \mathbf{1}_{m_2} b'_{max} \ge \mathbf{b}'_{m_2}$. So $(y_0, \mathbf{x}_{\bar{i}}^T) = (b'_{max}, \mathbf{0^T})$ is a feasible point of the problem (8). So X is not empty.

We will show that $\mathbf{d^T} = [1, 0, ..., 0]^T$ is a recession direction[1] where $\mathbf{d}$ is an n-dimensional column vector. For all $\alpha > 0$ and $(y_0, \mathbf{x}_{\bar{i}0}^T) + \alpha \mathbf{d^T} = (y_0 + \alpha, \mathbf{x}_{\bar{i}0}^T + \alpha \mathbf{0^T}) = (y_0 + \alpha, \mathbf{x}_{\bar{i}0}^T) \in X$. Since $\mathbf{1}_{m_2} y_0 \ge \mathbf{b}'_{m_2} - \mathbf{A}'_{m_2} \mathbf{x}_{\bar{i}0}$ and $\mathbf{1}_{m_2} \alpha > \mathbf{0}$,

$$\begin{aligned} \mathbf{1}_{m_2}(y_0 + \alpha) &= \mathbf{1}_{m_2} y_0 + \mathbf{1}_{m_2}\alpha \ge \mathbf{b}'_{m_2} - \mathbf{A}'_{m_2}(\mathbf{x}_{\bar{i}0} + \mathbf{0}) \\ &= \mathbf{b}'_{m_2} - \mathbf{A}'_{m_2} \mathbf{x}_{\bar{i}0}. \end{aligned}$$

Therefore $\mathbf{d}$ is a recession direction of X.

Consider the objective cost of the problem (8) is $\mathbf{c^T} = [1, 0, ..., 0]^T$, $(y_0 + \alpha, \mathbf{x}_{\bar{i}0}^T + \alpha \mathbf{0^T})\mathbf{c} = y_0 + \alpha$. Since $\alpha > 0$, $y_0 + \alpha \to \infty$ as $\alpha \to \infty$. Therefore the problem (8) is unbounded. ∎

B. The transformed problem

The relaxed problem (7) is always feasible. If $\mathbf{b}'_{m_1} \ge 0$ then $(y_0, \mathbf{x}_{\bar{i}0}^T) = \mathbf{0^T}$ is a feasible point and we can start the simplex algorithm by adding slack variables. Otherwise, $(y_0, \mathbf{x}_{\bar{i}0}^T) = (b'_{min}, \mathbf{0^T})$ is a feasible point. Then we transform the problem (7) using $y' = y - y_0$. The equivalent form is in the following:

$$\begin{aligned} \text{Maximize} \quad & y' + y_0 \\ \text{subject to} \quad & \mathbf{1}_{m_1} y' + \mathbf{A}'_{m_1} \mathbf{x}_{\bar{i}} \le \mathbf{b}'_{m_1} - \mathbf{1}_{m_1} y_0 \\ & \mathbf{x}_{\bar{i}} \ge \mathbf{0} \end{aligned} \tag{9}$$

Constraints of group M_2 will be transformed as $\mathbf{1}_{m_2} y' + \mathbf{A}_{m_2} \mathbf{x}_{\bar{i}} \ge \mathbf{b}'_{m_2} - \mathbf{1}_s y_0$. The variable y is not in constraints in group M_3. So constraints in group M_3 is not transformed. The transformed problem can be written in the following:

$$\begin{aligned} \text{Maximize} \quad & y' + y_0 \\ \text{subject to} \quad & \mathbf{1}_{m_1} y' + \mathbf{A}'_{m_1} \mathbf{x}_{\bar{i}} \le \mathbf{b}'_{m_1} - \mathbf{1}_{m_1} y_0 \\ & \mathbf{1}_{m_2} y' + \mathbf{A}'_{m_2} \mathbf{x}_{\bar{i}} \ge \mathbf{b}'_{m_2} - \mathbf{1}_{m_2} y_0 \\ & \mathbf{A}_{m_3} \mathbf{x}_{\bar{i}} \le \mathbf{b}_{m_3} \\ & \mathbf{x}_{\bar{i}} \ge \mathbf{0} \end{aligned} \tag{10}$$

If the transformed problem is infeasible or unbounded then the problem (6) will be infeasible or unbounded, respectively. If the optimal solution of the transformed problem $(y'^*, \mathbf{x}_{\bar{i}}^{*T})$ is found then the optimal solution $(y^*, \mathbf{x}_{\bar{i}}^{*T})$ of the problem (6) will be found by letting $y^* = y'^* + y_0$. Then optimal solution of the original problem is $\begin{bmatrix} x^*_{\bar{i}} \\ \mathbf{x}^*_{\bar{i}} \end{bmatrix} = \begin{bmatrix} \frac{y^*}{c_i} - \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} \frac{c_j}{c_i} x^*_j \\ \mathbf{x}^*_{\bar{i}} \end{bmatrix}$.

III. THE PROPOSED ALGORITHM

Consider the problem (9), we found that $\mathbf{b}'_{m_1} - \mathbf{1}_{m_1} y_0 \ge \mathbf{0}$. So the standard form can be written as follows:

$$\begin{aligned} \text{Maximize} \quad & y^+ - y^- + y_0 \\ \text{subject to} \quad & \mathbf{1}_{m_1} y^+ - \mathbf{1}_{m_1} y^- + \mathbf{A}'_{m_1} \mathbf{x}_{\bar{i}} + \mathbf{s} = \mathbf{b}'_{m_1} - \mathbf{1}_{m_1} y_0 \\ & y^+, y^-, \mathbf{x}_{\bar{i}}, \mathbf{s} \ge \mathbf{0} \end{aligned} \tag{11}$$

Let $\hat{\mathbf{b}}_{m_1} = \mathbf{b}'_{m_1} - \mathbf{1}_{m_1} y_0$. So the initial tableau of the relaxed problem (11) can be shown as follows:

	z	y^+	y^-	$\mathbf{x}_{\bar{i}}$	$\mathbf{s}$	RHS
z	1	-1	1	$\mathbf{0^T}$	$\mathbf{0^T}$	y_0
$\mathbf{s}$	0	$\mathbf{1}_{m_1}$	$-\mathbf{1}_{m_1}$	$\mathbf{A}'_{m_1}$	$\mathbf{I}_{m_1}$	$\hat{\mathbf{b}}_{m_1}$

where $\mathbf{I}_{m_1}$ is an $m_1 \times m_1$ identity matrix. Since the problem (11) is always feasible, the solution can be one of two cases: optimal or unbounded solution.

A. The optimal solution case

After we found the optimal solution of the problem (11), then we will check that it will be the optimal solution of the original problem by adding constraints from group M_2 and M_3 into the problem (11). For the transformed problem, constraints of group M_2 will be changed as $\mathbf{1}_{m_2}y' + \mathbf{A}'_{m_2}\mathbf{x}_{\bar{i}} \geq \mathbf{b}'_{m_2} - \mathbf{1}_{m_2}y_0$ or $-\mathbf{1}_{m_2}y' - \mathbf{A}'_{m_2}\mathbf{x}_{\bar{i}} \leq -\mathbf{b}_{m_2} + \mathbf{1}_{m_2}y_0$. The standard form for the transformed problem is $-\mathbf{1}_{m_2}y^+ + \mathbf{1}_{m_2}y^- - \mathbf{A}_{m_2}\mathbf{x}_{\bar{i}} + \mathbf{s}_{M_2} = -\mathbf{b}_{m_2} + \mathbf{1}_{m_2}y_0$ where $\mathbf{s}_{M_2}$ is a nonnegative column vector.

Let $\mathbf{B}_{m_1^*}$ and $\mathbf{N}_{m_1^*}$ are the optimal basis and the associated nonbasic matrix of the problem (11), respectively. The corresponding tableau of the relaxed problem is as follows:

	z	$\mathbf{x}_{\mathbf{B}_{m_1^*}}$	$\mathbf{x}_{\mathbf{N}_{m_1^*}}$	RHS
z	1	0	$\mathbf{Z}_{\mathbf{N}_{m_1^*}} - \mathbf{c}^{\mathbf{T}}_{\mathbf{N}_{m_1^*}}$	$z_{m_1^*} + y_0$
$\mathbf{x}_{\mathbf{B}_{m_1^*}}$	0	$\mathbf{I}_{m_1}$	$\mathbf{B}_{m_1^*}^{-1}\mathbf{N}_{m_1^*}$	$\mathbf{B}_{m_1^*}^{-1}\hat{\mathbf{b}}_{m_1}$

where $\mathbf{Z}_{\mathbf{N}_{m_1^*}}, = \mathbf{c}^{\mathbf{T}}_{\mathbf{B}_{m_1^*}}\mathbf{B}_{m_1^*}^{-1}\mathbf{N}_{m_1^*}, z_{m_1^*} = \mathbf{c}^{\mathbf{T}}_{\mathbf{B}_{m_1^*}}\mathbf{B}_{m_1^*}^{-1}\hat{\mathbf{b}}_{m_1}$, $\mathbf{c}^{\mathbf{T}}_{\mathbf{B}_{m_1^*}}$ and $\mathbf{c}^{\mathbf{T}}_{\mathbf{N}_{m_1^*}}$ are costs of objective function vectors of the problem (11) which rearranged by basic and nonbasic columns.

Let $\hat{\mathbf{A}} = \begin{bmatrix} -\mathbf{1}_{m_2} & \mathbf{1}_{m_2} & -\mathbf{A}'_{m_2} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{m_3} \end{bmatrix}$ be the combined coefficient matrix of group M_2 and M_3. $\mathbf{I}_{23} = \begin{bmatrix} \mathbf{I}_{m_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m_3} \end{bmatrix}$ where $\mathbf{I}_{m_2}$ is an $m_2 \times m_2$ identity matrix and $\mathbf{I}_{m_3}$ is an $m_3 \times m_3$ identity matrix. $\mathbf{b}_{23} = \begin{bmatrix} -\mathbf{b}'_{m_2} - \mathbf{1}_{m_2}y_0 \\ \mathbf{b}_{m_3} \end{bmatrix}$ So the additional constraints from the group M_2 and M_3 is rewritten as follows:

$$\hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}\mathbf{x}_{\mathbf{B}_{m_1^*}} + \hat{\mathbf{A}}_{\mathbf{N}_{m_1^*}}\mathbf{x}_{\mathbf{N}_{m_1^*}} + \mathbf{I}_{23}\mathbf{s}_{23} = \mathbf{b}_{23} \qquad (12)$$

where $\hat{\mathbf{A}} = [\hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}, \hat{\mathbf{A}}_{\mathbf{N}_{m_1^*}}]$ are rearranged by basic and nonbasic columns. After adding constraints (12) into tableau, we get

	z	$\mathbf{x}_{\mathbf{B}_{m_1^*}}$	$\mathbf{x}_{\mathbf{N}_{m_1^*}}$	$\mathbf{s}_{23}$	RHS
z	1	0	$\mathbf{Z}_{\mathbf{N}_{m_1^*}} - \mathbf{c}^{\mathbf{T}}_{\mathbf{N}_{m_1^*}}$	0	$z_{m_1^*} + y_0$
$\mathbf{x}_{\mathbf{B}_{m_1^*}}$	0	$\mathbf{I}_{m_1}$	$\mathbf{B}_{m_1^*}^{-1}\mathbf{N}_{m_1^*}$	0	$\mathbf{B}_{m_1^*}^{-1}\hat{\mathbf{b}}_{m_1}$
$\mathbf{s}_{23}$	0	$\hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}$	$\hat{\mathbf{A}}_{\mathbf{N}_{m_1^*}}$	$\mathbf{I}_{23}$	$\mathbf{b}_{23}$

We can eliminate $\hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}$ by multiplying the second row by $\hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}$ and subtracting from the third row gives the following tableau:

	z	$\mathbf{x}_{\mathbf{B}_{m_1^*}}$	$\mathbf{x}_{\mathbf{N}_{m_1^*}}$	$\mathbf{s}_{23}$	RHS
z	1	0	$\mathbf{Z}_{\mathbf{N}_{m_1^*}} - \mathbf{c}^{\mathbf{T}}_{\mathbf{N}_{m_1^*}}$	0	$z_{m_1^*} + y_0$
$\mathbf{x}_{\hat{\mathbf{B}}_{m_1^*}}$	0	$\mathbf{I}_{m_1}$	$\mathbf{B}_{m_1^*}^{-1}\mathbf{N}_{m_1^*}$	0	$\mathbf{B}_{m_1^*}^{-1}\hat{\mathbf{b}}_{m_1}$
$\mathbf{s}_{23}$	0	0	$\hat{\mathbf{A}}'_{\mathbf{N}_{m_1^*}}$	$\mathbf{I}_{23}$	$\hat{\mathbf{b}}_{23}$

where $\hat{\mathbf{b}}_{23} = \mathbf{b}_{23} - \hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}\mathbf{B}_{m_1^*}^{-1}\hat{\mathbf{b}}_{m_1}$ and $\hat{\mathbf{A}}'_{\mathbf{N}_{m_1^*}} = \hat{\mathbf{A}}_{\mathbf{N}_{m_1^*}} - \hat{\mathbf{A}}_{\mathbf{B}_{m_1^*}}\mathbf{B}_{m_1^*}^{-1}\mathbf{N}_{m_1^*}$. We can obtain the optimal solution by considering the sign of the right hand side in row $\mathbf{s}_{23}$. If $\hat{\mathbf{b}}_{23} \geq \mathbf{0}$, then the current solution is optimal. Otherwise, perform the dual simplex method to find the solution. Then we can conclude that if we find the optimal solution by the dual simplex method using Dantzig's rule, the value of the right hand side in the optimal tableau is the optimal solution of the

transformed problem. Otherwise, if the dual is unbounded, we can conclude that the original problem is infeasible.

B. The unbounded case

Let $\mathbf{B}_{m_1}$ be the basis and $\mathbf{N}_{m_1}$ be the associated nonbasic matrix of the problem (11). The corresponding tableau of the relaxed problem is as follows:

	z	$\mathbf{x}_{\mathbf{B}_{m_1}}$	$\mathbf{x}_{\mathbf{N}_{m_1}}$	RHS
z	1	0	$\mathbf{Z}_{\mathbf{N}_{m_1}} - \mathbf{c}^{\mathbf{T}}_{\mathbf{N}_{m_1}}$	$z_{m_1} + y_0$
$\mathbf{x}_{\mathbf{B}_{m_1}}$	0	$\mathbf{I}_{m_1}$	$\mathbf{B}_{m_1}^{-1}\mathbf{N}_{m_1}$	$\mathbf{B}_{m_1}^{-1}\hat{\mathbf{b}}_{m_1}$

where $\mathbf{Z}_{\mathbf{N}_{m_1}} = \mathbf{c}^{\mathbf{T}}_{\mathbf{B}_{m_1}}\mathbf{B}_{m_1}^{-1}\mathbf{N}_{m_1}$, $z_{m_1} = \mathbf{c}^{\mathbf{T}}_{\mathbf{B}_{m_1}}\mathbf{B}_{m_1}^{-1}\hat{\mathbf{b}}_{m_1}$. Let R be an index set of nonbasic variables and $z_j = \mathbf{c}^{\mathbf{T}}_{\mathbf{B}_{m_1}}\mathbf{B}_{m_1}^{-1}\mathbf{N}_{m_1:j}, j \in R$. If the relaxed problem is unbounded, it means that there is $z_j - c_j < 0$ and $\mathbf{B}_{m_1}^{-1}\mathbf{N}_{m_1:j} \leq \mathbf{0}$. We will find the solution of the transformed problem by adding all constraints from group M_2 and M_3 into the current relaxed tableau.

Similarly, we can use the equation (12) like the optimal solution case for adding to the current tableau when $\mathbf{B}_{m_1^*}$ and $\mathbf{N}_{m_1^*}$ are replaced by $\mathbf{B}_{m_1}$ and $\mathbf{N}_{m_1}$. After elimination, if $\hat{\mathbf{b}}_{23} \geq \mathbf{0}$, then the current solution is primal feasible then the primal simplex will be applied. Otherwise, both primal and dual solutions are infeasible at the current iteration because of $z_j - c_j < 0$. In [9], his method perturbs $z_j - c_j < 0$ to a positive value to obtain the dual feasible and then perform the dual simplex. After the optimal solution is found, the original $z_j - c_j$ will be restored and the primal simplex is used. However, if the dual problem is unbound, then the original problem is infeasible.

C. The special case

If $M_1 = \emptyset$ and $M_2 = \emptyset$ then the problem (6) remains in the following

$$\begin{aligned} \text{Maximize} \quad & y \\ \text{subject to} \quad & \mathbf{A}_{m_3}\mathbf{x}_{\bar{i}} \leq \mathbf{b}_{m_3} \\ & \mathbf{x}_{\bar{i}} \geq \mathbf{0} \end{aligned} \qquad (13)$$

From the problem (13), the solution can be one of two cases: unbounded or infeasible. Because of the variable y is not in constraints, if there is $\mathbf{x}_{\bar{i}0}$ that $\mathbf{A}_{m_3}\mathbf{x}_{\bar{i}0} \leq \mathbf{b}_{m_3}$ then the problem is feasible and y can increase to infinity. So the problem will be unbounded. Otherwise, the problem is infeasible. In this case, we will start with the first constraint and relax the remaining constraints as in the following:

$$\begin{aligned} \text{Maximize} \quad & y \\ \text{subject to} \quad & \mathbf{a}^{\mathbf{T}}_{1:}\mathbf{x}_{\bar{i}} \leq b_1 \\ & \mathbf{x}_{\bar{i}} \geq \mathbf{0} \end{aligned} \qquad (14)$$

where $\mathbf{a}_{1:}$ is the coefficient vector of the first constraint.

For fixing $j \in \{1, 2, ..., n\}$, if $a_{1j} \neq 0$ where $j \neq i$ then $y = 0, x_j = \frac{b_1}{a_{1j}}$ and $x_l = 0$ is a feasible point of the problem (14) where $l = 1, 2, ..., n, l \neq j \neq i$. Then the transformed problem is

$$\begin{aligned} \text{Maximize} \quad & y \\ \text{subject to} \quad & \frac{a_{11}}{a_{1j}}x_1 + \cdots + x'_j + \cdots + \frac{a_{1n}}{a_{1j}}x_n + s = 0 \\ & x_1, ..., x'_j, ..., x_n, \quad s \geq 0 \end{aligned} \qquad (15)$$

where $x'_j = x_j - \frac{b_1}{a_{1j}}$. This problem is unbounded. Then we will add the remaining constraints and use the unbounded case for checking the solution.

D. Summary of the algorithm

The algorithm is summarized in the following:

Step 1: Map the original problem with $x_i = y - (c_1 x_1 + c_{i-1} x_{i-1} + \cdots + c_{i+1} x_{i+1} + \cdots + c_n x_n)/c_i$ where $c_i \neq 0$ to the problem (4) and split constraints into 3 groups.

Step 1.1: If $M_1 \neq \emptyset$, relax constraints from group M_2 and M_3. If $b'_r \geq 0$ for all $r \in M_1$, start the simplex algorithm at the origin point. Otherwise, transform the problem using $y' = y - b'_{min}$ where $b'_{min} = \min_{r \in M_1} \{b'_r\}$ and start the simplex algorithm.

If the optimal solution is found, go to Step 2 (the optimal solution case.). Otherwise, go to Step 3.

Step 1.2: Otherwise, if $M_3 = \emptyset$, the problem is unbounded. Then stop.

Otherwise, relax constraints from the group M_3 and transform the problem using $y' = y - b'_{max}$ where $b'_{max} = \max_{s \in M_2} \{b'_s\}$. This relaxed problem is unbounded. Go to Step 3 (the unbounded case.).

Step 2: Test the optimal solution with all constraints from group M_2 and M_3.

Step 2.1: If it satisfies all constraints, it is the optimal solution. Then stop.

Step 2.2: Otherwise, perform the dual simplex at the unsatisfied constraints until the optimal solution is found. Then stop.

Step 3: Add all constraints from group M_2 and M_3.

Step 3.1: If the current solution satisfies all constraints, perform the primal simplex algorithm until the optimal solution is obtained. Then stop.

Step 3.2: Otherwise, perturb the negative reduced cost to a positive value for dual feasibility, then perform the dual simplex until the optimal solution is found. Restore the original reduced cost and perform the primal simplex until the optimal solution is found. Then stop.

E. The proposed criteria

In this paper, we suggest three criteria to choose the mapping variable as follows:

1. Choose the variable which has the smallest m_1.
2. Choose the variable which has the largest m_1.
3. Choose the variable which has the largest coefficient of the objective function.

IV. COMPUTATIONAL RESULTS

In this section, we tested the algorithm based on simulated linear programming problems. The randomly generated linear programming test problems

- are maximization problems;
- have a vector $\mathbf{c}$ with $c_i \in [-9, 9], i = 1, 2, ..., n$;

TABLE I
COMPARISON THE AVERAGE NUMBER OF ITERATIONS BETWEEN THREE CRITERIA AND TWO-PHASE METHOD

m	n	The average number of iterations			
		RP-Min	RP-Max	C-Max	Two-Phase
50	5	20.32	**19.83**	20.21	45.32
100	5	21.19	21.16	**20.11**	78.52
150	5	25.07	24.55	**22.04**	110.88
200	5	24.73	25.54	**22.40**	144.70
250	5	27.09	26.08	**24.91**	174.53
100	10	74.18	66.74	**65.94**	111.32
200	10	87.66	76.39	**73.89**	191.80
300	10	93.36	83.16	**73.84**	270.79
400	10	99.40	82.54	**74.38**	351.02
500	10	94.87	88.73	**77.14**	425.27
200	20	319.86	282.38	**272.92**	283.07
400	20	370.59	298.67	**282.45**	494.46
600	20	397.14	348.87	**292.04**	693.51
800	20	404.12	359.22	**295.24**	876.95
1000	20	405.71	348.63	**295.71**	1063.09
300	30	742.76	656.23	624.88	**503.17**
600	30	858.64	759.43	**697.54**	867.75
900	30	930.86	816.53	**725.81**	1195.50
1200	30	1000.85	860.37	**719.15**	1542.60
1500	30	1061.17	851.59	**731.21**	1869.10

- have a matrix $\mathbf{A}$ with $a_{ij} \in [-9, 9], i = 1, 2, ..., m, j = 1, 2, ..., n$;
- have a vector $\mathbf{x}$ with $x_i \in [0, 9], i = 1, 2, ..., n$;

Then we derive a vector $\mathbf{b}$ with $b_i = \mathbf{A}_{i:}\mathbf{x}$ where $i \in 1, 2, ..., n$ and $b_j = \mathbf{A}_{j:}\mathbf{x} + 1$ where $j \in n + 1, n + 2, ..., m$.

The different sizes of the number of variables (n) and the number of constraints (m) were tested with our method with three criteria and Two-Phase method where $m > n$, $n \in \{5, 10, 20, 30\}$ and m increases by 10, 20, 30, 40 and 50 times of the number of variables. For each size of the tested problems, the average number of iterations of 100 different problems were compared and shown in table I.
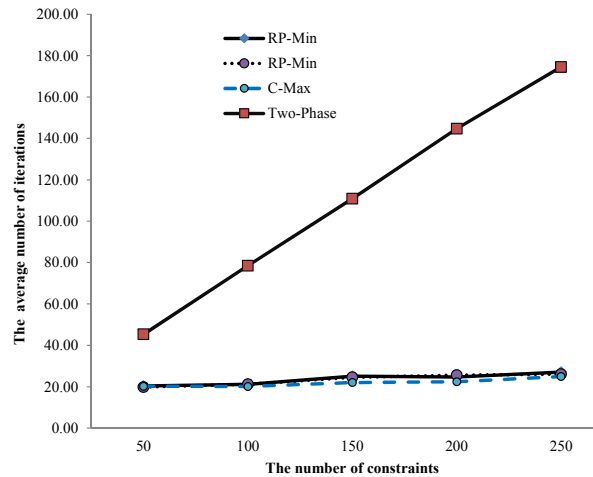

Fig. 3.   The average number of iterations for five variables.

In table I, the boldface numbers identify that the smallest average number of iterations. The result in table I is plotted as in Fig.3.-6. RP-Min, RP-Max and C-Max identify three criteria: the smallest m_1, the largest m_1 and the largest coefficient of the objective function, respectively.

From the computational results, we found that the average number of iterations of the proposed algorithm with three criteria are less than two-phase method except the problem with 30 variables and 300 constraints. The average number
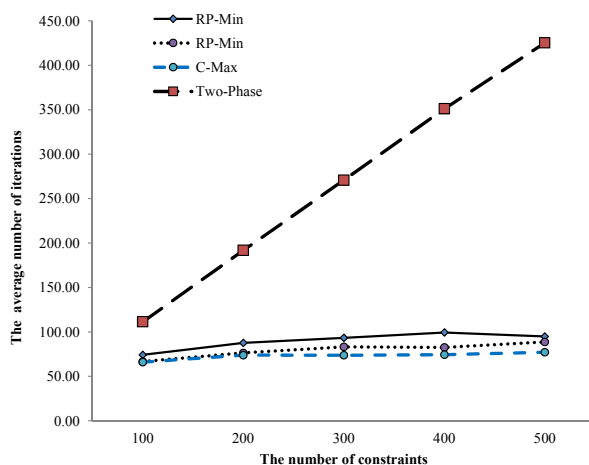
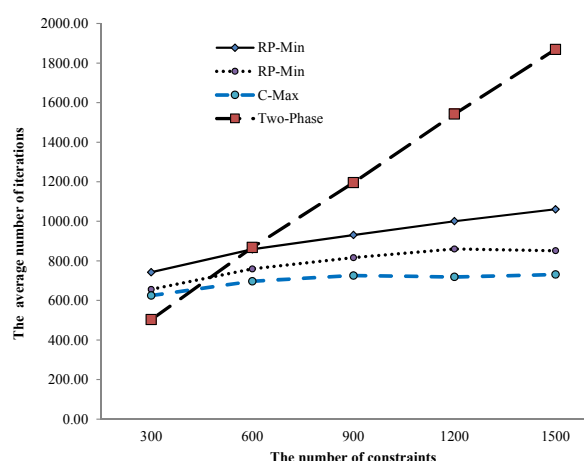Fig. 4.   The average number of iterations for ten variables.

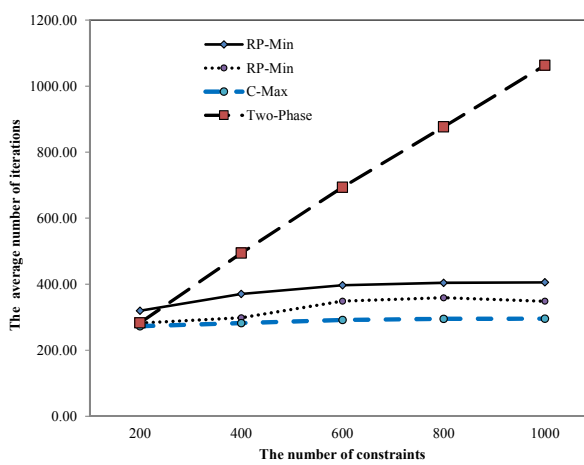Fig. 6.   The average number of iterations for 30 variables.

Fig. 5.   The average number of iterations for 20 variables.

of iterations of the selected variable which has the largest coefficient of the objective function outperform two-phase method and another criteria.

V. Conclusions

In this paper, we proposed the artificial-free technique to improve the simplex algorithm. The objective plane can split constraints into three groups by considering the sign of coefficient for each constraint. Then constraints from the nonpositive groups (negative and zero coefficient groups) are relaxed. The relaxed problem can identify a feasible point by our theorem. Then it will be transformed for starting the simplex algorithm without using artificial variables. Constraints from the nonpositive groups are added for checking the solution from the relaxed problem. Since artificial variables is not used, the number of variables by our algorithm is less than or equal to the number of variables by the simplex algorithm. The number of constraints which solved by our algorithm is less-than or equal to the number of constraints which solved by the simplex algorithm. If all constraints contain in the group M_1, then the number of constraints which solved by our algorithm and the simplex algorithm is equal. At each iteration, our algorithm solves partial tableau from the simplex tableau. So the dimension of parameters which used by our algorithm is less than or equal to the

dimension of parameters which the simplex algorithm used. This is one of the advantage of our algorithm. If the group M_1 and M_3 is empty, then we can conclude that the original problem is unbounded without using the simplex algorithm. This is an obvious advantage of our algorithm. Moreover, we suggest the criteria to select the variable for mapping. From the computational results, we found that iterations from the selected variable which has the maximum cost of the objective function outperform the simplex algorithm, and another criteria.

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