Some Results of the Bipolar Max-product Fuzzy Relational Equations

Chia-Cheng Liu, Yung-Yih Lur, Yan-Kuen Wu*

Abstract—In the literature, some studies have shown that determining the consistency of bipolar fuzzy relational equations is NP-complete. Namely, the solution procedure for solving a system of bipolar fuzzy relational equations with max-product composition contains a high computational complexity. In this study, some sufficient conditions for consistency of bipolar fuzzy relational equations are proposed to reduce the difficulty of detecting the case of an empty solution set. Numerical examples illustrate that the proposed properties are simple to use and do not require to generate the set of possible feasible solution pairs.

Index Terms—bipolar fuzzy relational equalities, max-product composition, NP-complete.

I. INTRODUCTION

In the literature, a system of fuzzy relational equations usually formulates in a matrix form as follows:

\[ x \circ A = b, \]

where \( x = (x_i)_{1 \times m}, A = [a_{ij}]_{m \times n} \) and \( b = (b_j)_{1 \times n} \) are all defined over \([0, 1]\). The operation “\( \circ \)” represents a well-defined algebraic composition for matrix multiplication.

Fuzzy relational equations have played an important role in the field of fuzzy set theory since the first study proposed by Sanchez [17] in 1976. After then, fuzzy relational equations or inequalities with different kinds of compositions have been proposed over the years [11], [12], [19]. The commonly seen max-min, \( \text{max}_{i \in I}(x_i \land a_{ij}) = b_j, \forall j \in J \), and max-product compositions, \( \text{max}_{i \in I}(a_{ij} x_i) = b_j, \forall j \in J \), are special cases of the max-triangular-norm (max-t-norm) composition. Di Nola et al. [3] indicated that the solution set of fuzzy relational equations with max-continuous t-norm composition can be completely determined by a unique maximum solution and a finite number of minimal solutions. However, finding all of the minimal solutions becomes much more difficult because it is NP-hard [1], [2], [10]. Many researchers continuously investigated relevant properties of minimal solution and proposed novel solution methods [9], [14], [15], [18], [20]. Furthermore, Lin et al. [8] presented that all systems of max-continuous u-norm fuzzy relational equations (e.g., max-product, max-continuous Archimedean t-norm and max-arithmetical mean) are essentially equivalent, because they all are equivalent to the covering problem.


Recently, Ferson et al. [4] considered a generation of the linear optimization problem subject to a system of bipolar fuzzy relational equations with max-min composition. They wanted to pursue the idea of taking into account antagonistic effects for this new optimization problem. For instance, consider a supplier who wants to optimize its public awareness and attributes a degree of appreciation to their products. Such a degree of appreciation can be denoted by a real number \( x_i \) in the unit interval \([0, 1]\) whose complement \( \tilde{x}_i = 1 - x_i \) in \([0, 1]\) stands for the degree of disappreciation. Generally, when the positive effect \( x_i \) increases, the negative effect \( \tilde{x}_i = 1 - x_i \) will fall. It is called the bipolar character. It is clear that the bipolar fuzzy relational equations contain the decision vector and its negation simultaneously. Motivated by Ferson et al. [4], Li and Liu [6] considered the linear optimization problem with bipolar max-Łukasiewicz equation constraints and transformed this problem into a 0-1 integer linear programming problem.

A system of bipolar fuzzy relational equations with max-product composition formulates in the matrix form as follows:

\[ x \circ A^+ \lor \tilde{x} \circ A^- = b \]

where \( x = (x_i)_{1 \times m}, \tilde{x} = (\tilde{x}_i)_{1 \times m}, A^+ = [a^+_{ij}]_{m \times n}, A^- = [a^-_{ij}]_{m \times n} \) and \( b = (b_j)_{1 \times n} \) are all defined over \([0, 1]\). The notation “\( \lor \)” denotes max operation and the operation “\( \circ \)” represents the max-product composition. \( \tilde{x}_i = 1 - x_i \) denotes the bipolar character.

If either \( A^+ \) or \( A^- \) is the zero matrix, the system (1) degenerates into unipolar max-product fuzzy relational equations as \( x \circ A^+ = b \) or \( \tilde{x} \circ A^- = b \), respectively. Essentially, solving the bipolar fuzzy relational equations with max-product composition is to find a set of solution vectors \( x = (x_i)_{i \in I} \) such that

\[ \text{max}_{i \in I} \{a^+_{ij} x_i, a^-_{ij} \tilde{x}_i\} = b_j, j \in J, \]

where index sets \( I = \{1, 2, \cdots, m\} \) and \( J = \{1, 2, \cdots, n\} \), respectively.

For investigating the solution set of bipolar max-min fuzzy relational equations to (1),

\[ \text{max}_{i \in I} \{\text{min}(a^+_{ij} x_i), \text{min}(a^-_{ij} \tilde{x}_i)\} = b_j, j \in J, \]

Chia-Cheng Liu and Yung-Yih Lur are with the Department of Industrial Management, Vanung University, Taoyuan, Taiwan, R.O.C., Email: lihht@vnu.edu.tw and yylur@vnu.edu.tw.

Yan-Kuen Wu is with the Department of Business Administration, Vanung University, Taoyuan, Taiwan, R.O.C., Email: ykw@vnu.edu.tw.
Freson et al. [4] first analyzed each single equation by a piecewise linear function. Based on the analyzed results which obtained from all of equations, then they structured the solution set of (3) by taking proper intersections and unions. They also figured out that the solution set of a system of bipolar fuzzy relational equations can be determined by a finite set of maximal and minimal solution pairs. However, Li and Liu [6] presented that determining the consistency of a system of bipolar fuzzy relational equations is NP-complete. That is to say, applying the proposed solution procedure by Freson et al. [4] for solving bipolar max-product fuzzy relational equations of (2) contains the high computation complexity. To improve the difficulty of solving the problem, this study proposes some properties for the bipolar fuzzy relational equations with max-product composition. Numerical examples illustrate that the proposed properties can be easily used to detect the case of an empty solution set of (2).

II. SOME RESULTS

For a system of fuzzy relational equations with continuous max-t-norm composition, a well-known property exists according to which its solution set, if non-empty, can be completely determined using a unique maximum solution and a finite number of minimal solutions. However, this structural property cannot extend to the solution set of bipolar fuzzy relational equations because the (2) contains the decision vector and its negation simultaneously. To investigate the property of the solution set, denoted by \(X(A^+, A^-, b)\), for the bipolar fuzzy relational equations with max-product composition in (2), some results are given as follows:

**Lemma 1.** If \(a_{ij}^+ < b_j \) and \(a_{ij}^- < b_j \) for all \(i \in I \) holds for some \(j \in J \) in (2), then the solution set \(X(A^+, A^-, b)\) is empty.

**Proof.** Due to \(0 \leq x_i \leq 1 \) for each \(i \in I \), if \(a_{ij}^+ < b_j \) and \(a_{ij}^- < b_j \) for all \(i \in I \) holds for some \(j \in J \) in (2), then this result leads to

\[
a_{ij}^- x_i < b_j \quad \text{and} \quad a_{ij}^- \bar{x}_i < b_j
\]

and no solution for \(x \in X(A^+, A^-, b)\) can satisfy the \(j\)th equation in (2). □

According to Lemma 1, we can conclude that if solution set \(X(A^+, A^-, b)\) of (2) is nonempty, for each \(j \in J \), \(a_{ij}^+ \geq b_j \) or \(a_{ij}^- \geq b_j \), for each \(i \in I \), that is \(b_j \leq \max_{i \in I} \{a_{ij}^+, a_{ij}^-\}\) must hold true.

Henceforth, this study assumes \(\frac{b_j}{a_{ij}^-} \to \infty \) and \(\frac{b_j}{a_{ij}^+} \to \infty \) for all \(i \in I, j \in J \) if \(a_{ij}^+ = 0 \) or \(a_{ij}^- = 0 \). We also define \(\frac{a_{ij}^-}{a_{ij}^+} \) or \(\frac{a_{ij}^+}{a_{ij}^-} \) if \(a_{ij}^+ = 0 \) and \(a_{ij}^- = 0 \).

**Lemma 2.** If \(x = (x_i)_{i \in I} \in X(A^+, A^-, b) \neq \emptyset\) is a feasible solution for (2), \(\max_{j \in J} \{1 - \frac{b_j}{a_{ij}^-}\} \leq x_i \leq \min_{j \in J} \{1 - \frac{b_j}{a_{ij}^+}\}, \forall i \in I \).

**Proof.** For any solution \(x = (x_i)_{i \in I} \in X(A^+, A^-, b) \neq \emptyset\),

\[
\max_{i \in I} \{a_{ij}^+ x_i, a_{ij}^- \bar{x}_i\} = b_j, j \in J,
\]

This implies that \(a_{ij}^+ x_i \leq b_j \) and \(a_{ij}^- \bar{x}_i \leq b_j \), for all \(i \in I \), for each \(j \in J \).

Consider the situation where \(a_{ij}^+ x_i \leq b_j \) to yield \(x_i \leq \frac{b_j}{a_{ij}^+} \), for each \(j \in J \).

The other situation is where \(a_{ij}^- \bar{x}_i = a_{ij}^- (1-x_i) \leq b_j \) to yield \(x_i \geq 1 - \frac{b_j}{a_{ij}^-} \), for each \(j \in J \).

Combining the results of these two situations with each variable \(x_i \in [0, 1], i \in I \), can yield

\[
\max_{j \in J} \{1 - \frac{b_j}{a_{ij}^-}\} \leq x_i \leq \min_{j \in J} \{1 - \frac{b_j}{a_{ij}^+}\}, \forall i \in I. \quad \square
\]

**Lemma 2** shows that if \(x = (x_i)_{i \in I} \in X(A^+, A^-, b) \neq \emptyset\) is a feasible solution for (2), the value of each variable \(x_i\) is bound between \(\max_{j \in J} \{1 - \frac{b_j}{a_{ij}^-}\} \) and \(\min_{j \in J} \{1 - \frac{b_j}{a_{ij}^+}\}, \forall i \in I \). They can be called the lower and upper bounds of variable \(x_i\), denoted using \(\bar{x}_i\) and \(\bar{x}_i\), respectively.

The lower bound \(\bar{x}_i = \max_{j \in J} \{1 - \frac{b_j}{a_{ij}^-}\}\) and the upper bound \(\bar{x}_i = \min_{j \in J} \{1 - \frac{b_j}{a_{ij}^+}\}\) of variable \(x_i\), \(i \in I\) can be easily computed, but they may not be solutions for (2).

**Remark.** Clearly, Lemma 2 can further deduce that if variable \(x_i\) exists the case \(\bar{x}_i > \bar{x}_i\), for some \(i \in I\), then the solution set of (2) is empty.

**Lemma 3.** If \(x = (x_i)_{i \in I} \in X(A^+, A^-, b) \neq \emptyset\) is a feasible solution for (2), \(\max_{i \in I} \{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\} \leq b_j \leq \max_{i \in I} \{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\}\) for all \(j \in J \).

**Proof.** According to Lemma 2, if \(x = (x_i)_{i \in I} \in X(A^+, A^-, b) \neq \emptyset\) is a feasible solution for (2), \(\bar{x}_i = \max_{j \in J} \{1 - \frac{b_j}{a_{ij}^-}\}\) for all \(j \in J \).

In addition, \(b_j \leq \max_{i \in I} \{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\}\) according to Lemma 1. Hence, \(b_j \leq \max_{i \in I} \{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\}\) for all \(j \in J \). □

**Lemma 3** shows that if the value of \(b_j\) is not in the range of \(\max_{i \in I} \{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\}\) to \(\max_{i \in I} \{\frac{a_{ij}^- a_{ij}^+}{a_{ij}^- + a_{ij}^+}\}\) for some \(j \in J\), the system of (2) is inconsistent. Hence, Lemma 3 can be used to check whether the solution set of (2) is empty or not.

**Example 1.** Consider the following matrix form of bipolar fuzzy relational equations with max-product composition.

We use this example to detect the case of an empty solution set by verifying Lemma 3.

\[
x \circ A^+ \vee \bar{x} \circ A^- = b
\]

where \(x = (x_1, x_2, \ldots, x_6), \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6), \bar{x}_i = 1 - x_i, i \in I = \{1, 2, \ldots, 6\},
\]

\[
A^+ = \begin{bmatrix}
0.45 & 0.11 & 0.18 & 0.05 & 0.16 & 0.23 \\
0.21 & 0.12 & 0.11 & 0.08 & 0.17 & 0.16 \\
0.32 & 0.32 & 0.15 & 0.18 & 0.37 & 0.20 \\
0.05 & 0.19 & 0.30 & 0.25 & 0.24 & 0.35 \\
0.96 & 0.32 & 0.25 & 0.21 & 0.37 & 0.36 \\
0.27 & 0.21 & 0.22 & 0.12 & 0.26 & 0.27
\end{bmatrix}
\]
Theorem 1 shows that for any solution \( x = (x_i)_{i \in \mathcal{I}} \in X(A^+, A^-, b) \), if \( x_i \) is a binding variable, \( x_i \) or \( \bar{x}_i \) is also binding there, that is \( J(x_i) \subseteq J(\bar{x}_i) \cup J(x_i) \).

**Definition 2.** Let \( \bar{x}_i = \max_{j \in \mathcal{J}} \{ 1 - \frac{b_{ij}}{a_{ij}}, 0 \} \) and \( x_i = \min_{j \in \mathcal{J}} \{ \frac{b_{ij}}{a_{ij}}, 1 \} \) be the corresponding lower bound and upper bound of variable \( x_i, i \in \mathcal{I} \) for (2). Two index sets define as follows:

\[
I_j := \{ i \in \mathcal{I} | a_{ij} = b_j, i \in \mathcal{I} \},
\]

and

\[
\bar{I}_j := \{ i \in \mathcal{I} | \bar{x}_{ij} = b_j, i \in \mathcal{I}, \forall j \in \mathcal{J} \}.
\]

For Definition 2, index sets \( I_j \) and \( \bar{I}_j \) denote that the possible variables of \( x \) may be selected as a binding variable in the \( j \)-th equation.

**Lemma 4.** If index sets with \( I_j = \bar{I}_j = \emptyset \) for some \( j \in \mathcal{J} \) exists, then the solution set \( X(A^+, A^-, b) \) of (2) is empty.

**Proof.** For any solution \( x = (x_i)_{i \in \mathcal{I}} \in X(A^+, A^-, b) \), all equations must be satisfied. Furthermore, Theorem 1 shows that if \( x_i \) is a binding variable, \( \bar{x}_i \) or \( x_i \) is also binding there. Hence, index sets with \( I_j = \bar{I}_j = \emptyset \) show that no any possible variables of \( x \) can be selected as a binding variable in the \( j \)-th equation. Hence the solution set \( X(A^+, A^-, b) \) of (2) is empty. \( \square \)

**Example 2.** Consider the following matrix form of bipolar fuzzy relational equations with max-product composition. We use this example to detect the case of an empty solution set by verifying Lemma 4.

\[
x \circ A^+ \lor \bar{x} \circ A^- = b
\]

where \( x = (x_1, x_2, \ldots, x_6), \bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_6), \bar{x}_i = 1 - x_i, i \in \mathcal{I} = \{ 1, 2, \ldots, 6 \},
\]

\[
A^- = \begin{bmatrix}
0.18 & 0.09 & 0.12 & 0.14 & 0.10 & 0.21 \\
0.37 & 0.29 & 0.10 & 0.01 & 0.09 & 0.49 \\
0.16 & 0.77 & 0.07 & 0.12 & 0.02 & 0.44 \\
0.24 & 0.11 & 0.20 & 0.04 & 0.13 & 0.47 \\
0.09 & 0.19 & 0.09 & 0.27 & 0.13 & 0.30 \\
0.01 & 0.17 & 0.08 & 0.17 & 0.06 & 0.22 \\
\end{bmatrix}
\]

\[
b = (0.32, 0.25, 0.10, 0.16, 0.24, 0.18).
\]
Clearly, Lemma 3 is satisfied by above results. However, we cannot verify the consistency of Example 2.

Computing the corresponding lower bound and upper bound of variable $x_i$ for Example 2 can obtain:

\[
\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}} = (0.711, 0.960, 0.649, 0.640, 0.649, 0.923)
\]

and

\[
\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}} = (0.333, 0.429, 0.609, 0.404, 0.407, 0.059).
\]

Above results show that each of the lower bound $\bar{x}_i$ is less than the upper bound $\bar{x}_i$, $\forall i \in \mathcal{I} = \{1, 2, \cdots, 6\}$. Namely, the above results do not violate Lemma 2. However, we also cannot verify the consistency of Example 2.

For Example 2, the corresponding index sets $I_j$ and $I'_j, j \in \mathcal{J} = \{1, 2, \cdots, 6\}$ can be yielded by Definition 2 as follows:

\[
I_1 = \emptyset, \quad I'_2 = \{1\}; \quad I_2 = \{3\}, \quad I'_2 = \emptyset;
\]

\[
I_3 = \emptyset, \quad I'_3 = \emptyset; \quad I_4 = \{1, 5, 6\}, \quad I'_4 = \{4\};
\]

\[
I_5 = \emptyset, \quad I'_5 = \{2, 3, 5, 6\}; \quad I_6 = \{2, 4\}, \quad I'_6 = \emptyset.
\]

Since $I_3 = I'_3 = \emptyset$, it leads that the solution set of Example 2 is empty by Lemma 4. □

III. CONCLUSION

In this study, we propose some properties to detect the case of an empty solution set of bipolar fuzzy relational equations with max-product composition. Numerical examples illustrate that for detecting the case of an empty solution set, the proposed properties is simple to use and does not require to generate the set of possible feasible solution pairs.

ACKNOWLEDGMENT

This work is supported under grants no. MOST 105-2410-H-238-001 and MOST 105-2115-M-238-001, Ministry of Science and Technology, Taiwan, R.O.C.

REFERENCES


