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On A-efficient Treatment-control Designs

Kazuhiro Ozawa*, Shinji Kuriki

***Abstract*—We consider a treatment-control design in the design of experiments. We give a construction of an A-efficient treatment-control design generated by a cyclic design. We show such efficient designs obtained by computer search.**

***Index Terms*—A-optimality, cyclic design, efficiency, treatment-control design.**

I. Introduction

In the design of experiments, we consider an experiment to compare v treatments (called test treatments) with a standard treatment (called a control) using b blocks, each of size k. Such a design is called a treatment-control design. The control is denoted by 0 and the test treatments are denoted by $1, 2, \cdots, v$. Under the usual additive linear model, for given v, b and k, the problem is to find a treatment-control design which minimizes

$$\sum_{i=1}^{v} \mathrm{Var}(\hat{\tau}_i - \hat{\tau}_0), \tag{1.1}$$

where $\hat{\tau}_i - \hat{\tau}_0$ is the BLUE of $\tau_i - \tau_0$, τ_i is the treatment effect of the test treatment i for $i = 1, 2, \cdots, v$ and τ_0 is the treatment effect of the control, assuming the treatment-control design is connected. If a treatment-control design minimizes (1.1), it is called an A-optimal treatment-control design. Throughout, we consider a treatment-control design with $k \leq v$ and we denote the class of all connected treatment-control designs by $\mathcal{D}(v, b, k)$.

Bechhofer and Tamhane [1] defined a balanced treatment incomplete block (BTIB) design, which is useful to find an A-optimal treatment-control design. A treatment-control design $d \in \mathcal{D}(v, b, k)$ is called a BTIB design if there exist nonnegative integers λ_c and λ such that

$$\lambda_{0i} = \lambda_c, \quad \text{for} \ \ i = 1, 2, \cdots, v,$$

and

$$\lambda_{ii'} = \lambda, \quad \text{for} \ \ i, i' = 1, 2, \cdots, v, \ \ i \neq i',$$

where λ_{ij} is the number of blocks containing the treatments i and j for $i, j = 0, 1, 2, \cdots, v, \ \ i \neq j$.

An A-optimal treatment-control design belongs to a subclass of BTIB designs. A BTIB design $d \in \mathcal{D}(v, b, k)$ is called a BTIB$(v, b, k; t, s)$ if

(i) d is binary in test treatments,

and

(ii) there are s blocks each of which contains exactly $t + 1$ replications of the control, while each of the remaining $b - s$ blocks contains exactly t replications of the control.

If $s = 0$, it is called an R-type design, while if $0 < s < b$, it is called an S-type design.

Let

$$\begin{aligned} g(x, z) \ = \ & vk(v-1)^2[bvk(k-1) \\ & -(bx+z)(vk-v+k) + (bx^2+2xz+z)]^{-1} \\ & +vk[k(bx+z) - (bx^2+2xz+z)]^{-1} \end{aligned}$$

and

$$\Lambda = \{0, 1, \cdots, \lfloor k/2 \rfloor - 1\} \times \{0, 1, \cdots, b\} - \{0, 0\}.$$

Majumdar and Notz [5] showed the following theorem.

Theorem 1. Let t and s be integers defined by

$$g(t, s) = \min_{(x,z) \in \Lambda} g(x, z).$$

For any treatment-control design $d \in \mathcal{D}(v, b, k)$,

$$\sum_{i=1}^{v} \mathrm{Var}(\hat{\tau}_i - \hat{\tau}_0) \geq g(t, s)\sigma^2 \tag{1.2}$$

holds. Furthermore, the equality of (1.2) holds if d is a BTIB$(v, b, k; t, s)$. Hence a BTIB$(v, b, k; t, s)$ is an A-optimal treatment-control design.

Das, Dey, Kageyama and Sinha [2] have given a list of A-efficient BTIB designs in the practically useful ranges of the parameters. In this paper, we give a list of A-efficient treatment-control (not always BTIB) designs and we compare the efficiencies of our designs with the efficiencies of their designs as treatment-control designs. The efficiency of a treatment-control design $d \in \mathcal{D}(v, b, k)$ is defined by

$$e = \frac{g(t, s)\sigma^2}{\sum_{i=1}^{v} \mathrm{Var}(\hat{\tau}_i - \hat{\tau}_0)}$$

(see Stufken [6]). Das, Dey, Kageyama and Sinha [2] constructed A-efficient BTIB designs from balanced incomplete block designs and partially balanced incomplete block designs, with

$$2 \leq k \leq 10, \quad r \leq 10, \quad k \leq v \leq b \leq 50$$

and

$$e \geq 0.950,$$

where r is the number of replications of each test treatment. A BTIB design with $e \geq 0.950$ is said to be highly efficient by them. The total number of designs listed by them is 155. Table I show the A-efficient BTIB designs with $k = 4, r \leq 10, 4 \leq v \leq b \leq 50$ in part of the list in Das, Dey, Kageyama and Sinha [2].

Kazuhiro Ozawa is with the Department of Nursing, Gifu College of Nursing, Gifu, 5016295 Japan (e-mail: ozawa@gifu-cn.ac.jp).

Shinji Kuriki is with Department of Mathematical Sciences, Graduate School of Engineering, Osaka Prefecture University, Osaka 599-8531, Japan (e-mail: kuriki@ms.osakafu-u.ac.jp).

TABLE I
THE LIST OF A-EFFICIENT BTIB DESIGNS WITH $k = 4, r \leq 10$, $4 \leq v \leq b \leq 50$ IN DAS, DEY, KAGEYAMA AND SINHA [2]

v	b	r	r_c	e
4	4	3	4	1
5	7	4	8	0.953
	10	6	10	1
6	10	5	10	1
7	7	3	7	1
8	26	10	24	0.993
9	12	4	12	1
10	25	8	20	0.986
	30	9	30	0.999
12	19	5	16	0.998
13	20	5	15	0.954
	26	6	26	0.995
15	35	7	35	0.991
16	28	6	16	0.969
	36	7	32	0.998
20	50	8	40	0.966
21	50	8	32	0.968

Here, among such designs with the same values of v and k, they do not list the designs that satisfy both the following conditions:

(i) small value of e and (ii) large number of b.

That is, if for the same values of v and k, there are two designs, say d_1 and d_2 having b_1 and b_2 blocks and A-efficiencies e_1 and e_2 respectively, such that $b_1 \geq b_2$, then d_1 is not included in the list if $e_1 \leq e_2$.

II. A-EFFICIENT TREATMENT-CONTROL DESIGNS

Let $d = (V, \mathcal{B})$ be a design, where V is a set of v treatments and $\mathcal{B}$ is a collection of k-subsets of V. We construct a treatment-control design d' adding exactly f replications of the control to each block of $\mathcal{B}$. Then $d' \in \mathcal{D}(v, b, k+f)$ and d' is an R-type design, where b is the number of blocks of $\mathcal{B}$. Here the treatments of d are considered as the test treatments of d'.

As the original design $d = (V, \mathcal{B})$, we use a cyclic design.

Let $V = \{0, 1, \cdots, v-1\}$ (mod v), the residues of modulo v. For a block $B = \{b_0, b_1, \cdots, b_{k-1}\} \in \mathcal{B}$, let

$$B = \{b_0, b_1, \cdots, b_{k-1}\} \pmod{v}.$$

Then

$$\{B + i \pmod{v} \mid i = 0, 1, \cdots, v-1\}$$

is called a cyclic class. If $\mathcal{B}$ is divided into some cyclic classes, then $(V, \mathcal{B})$ is called a cyclic design.

By John [4], in a cyclic design, the canonical efficiency factor is given by

$$e_i = \frac{k-1}{k} - \frac{1}{rk} \sum_{h=1}^{v-1} \lambda_h \cos\left(\frac{2\pi hi}{v}\right)$$

for $i = 1, 2, \cdots, v-1$, where r is the replications of each treatment and λ_h is the number of blocks containing the treatments 0 and h. It is expected that if the original design d is A-efficient, then the resulting design d' is also A-efficient.

We use A-efficient cyclic designs given by John [4], and we consider A-efficient treatment-control designs $\mathcal{D}(v, b, k)$ with

$$3 \leq k \leq 10, \quad r \leq 10, \quad k \leq v \leq 50, \quad b \leq 50$$

TABLE II
THE LIST OF A-EFFICIENT TREATMENT-CONTROL DESIGNS WITH $k = 4, r \leq 10, 4 \leq v \leq 50, b \leq 50$ BY USING A CYCLIC DESIGN

v	b	r	r_c	e	f
5	5	3	5	0.994	1
6	6	3	6	0.994	1
8	8	3	8	0.993	1
	16	6	16	0.997	1
	24	9	24	0.999	1
9	9	3	9	0.987	1
10	10	3	10	0.981	1
	20	6	20	0.996	1
11	11	3	11	0.974	1
	22	6	22	0.995	1
	33	9	33	0.997	1
12	12	3	12	0.971	1
13	13	3	13	0.967	1
14	14	3	14	0.960	1
	28	6	28	0.986	1
	42	9	42	0.989	1
15	15	3	15	0.954	1
	30	6	30	0.986	1
16	32	6	32	0.981	1
17	34	6	34	0.978	1
18	36	6	36	0.973	1
19	38	6	38	0.970	1
20	40	6	40	0.965	1
21	42	6	42	0.963	1
22	44	6	44	0.958	1
23	46	6	46	0.955	1
24	48	6	48	0.952	1

and

$$e \geq 0.950.$$

We use the same rule considered by Das, Dey, Kageyama and Sinha [2]. We obtained the 188 A-efficient treatment-control designs by using computer search. Table II show the A-efficient treatment-control designs with $k = 4, r \leq 10, 4 \leq v \leq 50, b \leq 50$ by using a cyclic design.

In Tables I and II, the number of blocks of the A-efficient treatment-control designs generated by a cyclic design are smaller than that of Das, Dey, Kageyama and Sinha [2] in the same v treatments.

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