Super Edge-Magic Labeling of Some Fan Graphs

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Abstract—For a graph $G(V, E)$ with $p$ vertices and $q$ edges, a bijective function $f$ from $V(G) \cup E(G)$ to $\{1, 2, ..., p+q\}$ is called a super edge-magic labeling of $G$ if $f(V(G)) = \{1, 2, ..., p\}$ and there exists a constant $k$ such that for any edge $uv$ of $G$, $f(u) + f(v) + f(uv) = k$. A graph $G$ is called super edge-magic if there exists a super edge-magic labeling of $G$. In this paper, we shows that the fan graph $F_n,2$ and $mF_n,2$ is a super edge-magic where $n$ is a positive integer, $m$ positive odd number and $m \geq 3$.

Index Terms—super edge-magic graphs, edge-magic labelings.

I. INTRODUCTION

The concept of super edge-magic labeling is motivated by edge-magic labeling. Let $G$ be a graph with $p$ vertices and $q$ edges. The edge-magic labeling of $G$ is a bijective function $f$ from $V(G) \cup E(G)$ to $\{1, 2, ..., p+q\}$ which there exists a constant $k$ such that for any edge $uv$ of $G$, $f(u) + f(v) + f(uv) = k$. In this case, $G$ is said to be edge-magic. In 1998, Enomoto et al. [2] defined a super edge-magic labeling of a graph $G$ as an edge-magic labeling $f$ of $G$ such that $f(V(G)) = \{1, 2, ..., p\}$ and $G$ is called super edge-magic if there exists a super edge-magic labeling of $G$. In 2001, Figueroa-Centero et al. [3] analysed a necessary and sufficient condition for a graph to be super edge-magic and proved that the fan $F_n \cong P_n + K_1$ is an edge-magic for every positive integer $n$ and $F_n$ is a super edge-magic if $n \leq 6$.

Lemma 1.1. [3] A graph $G$ with $p$ vertices and $q$ edges is a super edge-magic if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, ..., p\}$ such that the set $S = \{f(u) + f(v) \mid uv \in E(G)\}$ consists of $q$ consecutive integers. In this case, $f$ extends to a super edge-magic labeling of $G$ with constant $k = p + q + s$ where $s = \min(S)$ and $S = \{f(u) + f(v) \mid uv \in E(G)\} = \{k-(p+1), k-(p+2), ..., k-(p+q)\}$.

In 2008, Nguarah et al. [4] proved that the graph $K_2 + P_n$ is a super edge-magic if and only if $n \leq 2$. Later, Nguarah and Simanjuntak (2014)[5] shown that for any integers $m, n$ such that $m \geq 3$, the graph $K_m + P_n$ is super edge-magic if and only if $n \in \{1, 2\}$. In this paper, we proved that graph fan $F_{n,2}$ is a super edge-magic for any positive integer $n$ and $mF_{n,2}$ is a super edge-magic where $n$ is a positive integer, $m$ positive odd number and $m \geq 3$.

II. MAIN RESULTS

A fan graph $F_{n,2}$ is defined as the graph joint $K_n + P_2$ where $K_n$ is the empty graph of $n$ vertices and $P_2$ is the path of 2 vertices which is a graph with $n + 2$ vertices and $2n + 1$ edges as follow Fig 1.

Theorem II.1. For any positive integer $n$, the graph $F_{n,2}$ is super edge-magic with $k = 3n + 6$.

Proof: Let $n \in \mathbb{N}$, $V(F_{n,2}) = \{v_1, v_2, ..., v_{n+2}\}$ and $E(F_{n,2}) = \{v_1v_2, v_1v_3, ..., v_1v_{n+1}, v_1, v_{n+2}\} \cup \{v_{n+2}v_2, v_{n+2}v_3, ..., v_{n+2}v_{n+1}\}$.

Define $f : V(F_{n,2}) \rightarrow \{1, 2, ..., n+2\}$ by $f(v_i) = i$. Hence $f$ is a bijective function. Then $S = \{f(u) + f(v) \mid uv \in E(F_{n,2})\} = \{f(v_1) + f(v_i) \mid i \in \{2, 3, ..., n+2\}\}$ and $S = \{f(v_{n+2}) + f(v_j) \mid j \in \{2, 3, ..., n+1\}\} = \{3, 4, ..., 2n+3\}$ is the set of $2n+1$ consecutive integers which $\min(S) = 3$.

By Lemma I.1, $f$ extends to a super edge-magic labeling of $F_{n,2}$. Hence graph $F_{n,2}$ is a super edge-magic with $k = n + 2 + 2n + 1 + 3 = 3n + 6$.

Remark. From Theorem II.1, $F_{2,2}$ is a super edge-magic with $k = 12$ as follow Fig 2. and from observation, we have another super edge-magic labeling of $F_{2,2}$ as follow Fig 3.

Fig 1. $F_{n,2}$

Fig 2. Super Edge-Magic Labeling of $F_{2,2}$

Fig 3. Super Edge-Magic Labeling of $F_{2,2}$
For any positive integer \( m \), the disjoint union of \( m \) copies of \( F_{n,2} \) denoted by \( mF_{n,2} \) is a graph with \( m(n+2) \) vertices and \( m(2n+1) \) edges.

**Theorem II.2.** If \( m \geq 3 \) is an odd number and \( n \) is a positive integer. Then graph \( mF_{n,2} \) is super edge-magic with 
\[
k = 3 \left( \frac{mn + 3m + 1}{2} \right).
\]

**Proof:** Let \( m, n \in \mathbb{N} \) such that \( m \geq 3 \) is an odd number. Let \( V(mF_{n,2}) = V_1 \cup V_2 \cup \ldots \cup V_m \) and \( E(mF_{n,2}) = E_1 \cup E_2 \cup \ldots \cup E_m \) where \( V_i = \{v_1, v_2, \ldots, v_{n+1}\}, E_i = \{v_1v_2, v_1v_3, \ldots, v_{n+1}v_{n+2}, v_2v_{n+2}, v_3v_{n+2}, \ldots, v_{n+1}v_{n+2}\} \).

**Subcase 1.2:** \( \frac{m+1}{2} < r \leq m \). We have that
\[
f(v_{q+1}^{2m+2-2r}) = (q + 1)m - \frac{m + 2 - 2r - 2}{2} = k.
\]

**Case 2:** \( m(n+1) < k \leq m(n+2) \). Let \( s = k - m(n+1) \) hence \( 1 \leq s \leq m \).

**Subcase 2.1:** \( 1 \leq s \leq \frac{m+1}{2} \). Thus
\[
f(v_{q+1}^{m+1-2s}) = m(n+2) - \frac{m + m + 1 - 2s - 1}{2} = k.
\]

**Subcase 2.2:** \( \frac{m+1}{2} \leq s \leq m \). We have
\[
f(v_{q+1}^{m+1-2s}) = m(n+2) - \frac{2m + 1 - 2r - 1}{2} = k.
\]

Thus \( f \) is a bijective function. Let \( i \in \{1, 2, \ldots, m\} \).

Recall that \( E_i = \{v_1v_2, v_1v_3, \ldots, v_{n+1}v_{n+2}\} \cup \{v_2v_{n+2}, v_3v_{n+2}, \ldots, v_{n+1}v_{n+2}\} \cup \{v_1v_2, v_1v_3, \ldots, v_{n+1}v_{n+2}\} \). Let \( S_i = \{f(x) + f(y) | xy \in E_i \} \).

**Subcase 1.1:** \( 1 \leq r \leq \frac{m+1}{2} \). Thus
\[
f(v_{q+1}^{m+1-2r}) = (q + 1)m - \frac{m + m + 2 - 2r}{2} = k.
\]

**References**


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