Proceedings of the International MultiConference of Engineers and Computer Scientists 2017 Vol II,
IMECS 2017, March 15 - 17, 2017, Hong Kong

Super Edge-Magic Labeling of Some Fan Graphs

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***Abstract*—For a graph $G(V, E)$ with p vertices and q edges, a bijective function f from $V(G)\cup E(G)$ to $\{1, 2, ..., p+q\}$ is called a super edge-magic labeling of G if $f(V(G)) = \{1, 2, ..., p\}$ and there exists a constant k such that for any edge uv of G, $f(u)+f(v)+f(uv) = k$. A graph G is called super edge-magic if there exists a suber edge-magic labeling of G. In this paper, we shows that the fan graph $F_{n,2}$ and $mF_{n,2}$ is a super edge-magic where n is a positive integer, m positive odd number and $m \geq 3$.**

***Index Terms*—super edge-magic graphs, edge-magic labelings.**

I. Introduction

THE concept of super edge-magic labeling is motivated by edge-magic labeling. Let G be a graph with p vertices and q edges. The *edge-magic labeling* of G is a bijective function f from $V(G) \cup E(G)$ to $\{1, 2, ..., p + q\}$ which there exists a constant k such that for any edge uv of G, $f(u) + f(v) + f(uv) = k$. In this case, G is said to be *edge-magic*. In 1998, Enomoto et al. [2] defined a *super edge-magic labeling* of a graph G as an edge-magic labeling f of G such that $f(V(G)) = \{1, 2, ..., p\}$ and G is called *super edge-magic* if there exists a super edge-magic labeling of G. In 2001, Figueroa-Centero et al. [3] analysed a necessary and sufficient condition for a graph to be super edge-magic and proved that the fan $F_n \cong P_n + K_1$ is an edge-magic for every positive integer n and F_n is a super edge-magic if $n \leq 6$.

Lemma I.1. [3] *A graph G with p vertices and q edges is a super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, ..., p\}$ such that the set*

$$S = \{f(u) + f(v) \mid uv \in E(G)\}$$

consists of q consecutive integers. In this case, f extends to a super edge-magic labeling of G with constant $k = p+q+s$ where $s = \min(S)$ and

$$\begin{aligned} S &= \{f(u) + f(v) \mid uv \in E(G)\} \\ &= \{k - (p + 1), k - (p + 2), ..., k - (p + q)\}. \end{aligned}$$

In 2008, Ngurah et al. [4] proved that the graph $\overline{K_2} + P_n$ is a super edge-magic if anf only if $n \leq 2$. Later, Ngurah and Simanjuntak (2014)[5] shown that for any integers m, n such that $m \geq 3$, the graph $\overline{K_m} + P_n$ is super edge-magic if and only if $n \in \{1, 2\}$. In this paper, we proved that graph fan $F_{n,2}$ is a super edge-magic for any positive integer n and $mF_{n,2}$ is a super edge-magic where n is a positive integer, m positive odd number and $m \geq 3$.

Manuscript received Jan 19, 2017; revised Feb 11, 2017. This work was supported in part by the Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang.

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II. Main Results

A fan graph $F_{n,2}$ is defined as the graph joint $\overline{K_n} + P_2$ where $\overline{K_n}$ is the empty graph of n vertices and P_2 is the path of 2 vertices which is a graph with $n + 2$ vertices and $2n + 1$ edges as follow Fig 1.

Theorem II.1. *For any positive integer n, the graph $F_{n,2}$ is super edge-magic with $k = 3n + 6$.*

Proof: Let $n \in \mathbb{N}$, $V(F_{n,2}) = \{v_1, v_2, ..., v_{n+2}\}$ and $E(F_{n,2}) = \{v_1v_2, v_1v_3, ..., v_1v_{n+1}, v_1, v_{n+2}\} \cup \{v_{n+2}v_2, v_{n+2}v_3, ..., v_{n+2}v_n, v_{n+2}v_{n+1}\}$.

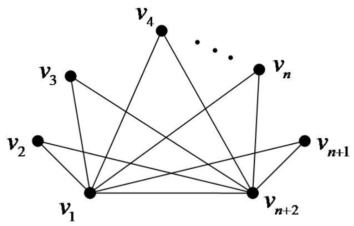

Fig 1. $F_{n,2}$

Define $f : V(F_{n+2}) \to \{1, 2, ..., n+2\}$ by $f(v_i) = i$. Hence f is a bijective function. Then

$$\begin{aligned} S =&\{f(u) + f(v) \mid uv \in E(F_{n,2})\} \\ =&\{f(v_1) + f(v_i) \mid i \in \{2, 3, ..., n + 2\}\} \\ &\cup \{f(v_{n+2}) + f(v_j) \mid j \in \{2, 3, ..., n + 1\}\} \\ =&\{3, 4, ..., 2n + 3\} \end{aligned}$$

is the set of $2n+1$ consecutive integers which $\min(S) = 3$. By Lemma I.1, f extends to a super edge-magic labeling of $F_{n,2}$. Hence graph $F_{n,2}$ is a super edge-magic with $k = n + 2 + 2n + 1 + 3 = 3n + 6$. ■

Remark. From Theorem II.1, $F_{2,2}$ is a super edge-magic with $k = 12$ as follow Fig 2. and from observation, we have another super edge-magic labeling of $F_{2,2}$ as follow Fig 3.

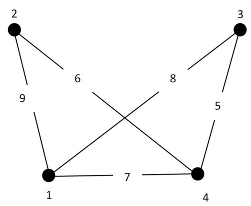

Fig 2. Super Edge-Magic Labeling of $F_{2,2}$

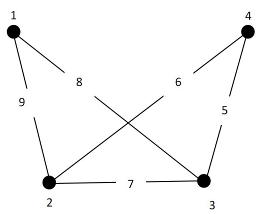

Fig 3. Super Edge-Magic Labeling of $F_{2,2}$

ISBN: 978-988-14047-7-0
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

For any positive integer m, the disjoint union of m copies of $F_{n,2}$ denoted by $mF_{n,2}$ is a graph with $m(n+2)$ vertices and $m(2n+1)$ edges.

Theorem II.2. *If $m \geq 3$ is an odd number and n is a positive integer. Then graph $mF_{n,2}$ is super edge-magic with $k = 3\left(mn + \dfrac{3m+1}{2}\right)$.*

Proof: *Let $m, n \in \mathbb{N}$ such that $m \geq 3$ is an odd number. Let $V(mF_{n,2}) = V_1 \cup V_2 \cup ... \cup V_m$ and $E(mF_{n,2}) = E_1 \cup E_2 \cup ... \cup E_m$ where $V_i = \{v_1^i, v_2^i, ..., v_{n+2}^i\}$, $E_i = \{v_1^i v_2^i, v_1^i v_3^i, ..., v_1^i v_{n+1}^i, v_1^i v_{n+2}^i, v_2^i v_{n+2}^i, v_3^i v_{n+2}^i, ..., v_n^i v_{n+2}^i, v_{n+1}^i v_{n+2}^i\}$.*

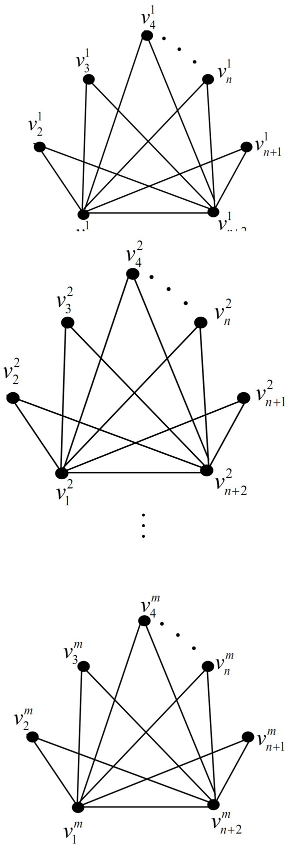

Fig 4. $mF_{n,2}$

Define $f : V(mF_{n,2}) \to \{1, 2, ..., m(n+2)\}$ as follows :

$$f(v_j^i) = \begin{cases} i & \text{if } j = 1, \\ jm + 1 - \frac{m+i}{2} & \text{if } 2 \leq j \leq n+1 \\ & \quad \text{and } i \text{ is odd}, \\ jm - \frac{i-2}{2} & \text{if } 2 \leq j \leq n+1 \\ & \quad \text{and } i \text{ is even}, \\ m(n+2) - \frac{i-1}{2} & \text{if } j = n+2 \text{ and } i \text{ is odd}, \\ m(n+2) - \frac{m+i-1}{2} & \text{if } j = n+2 \text{ and } i \text{ is even}. \end{cases}$$

We can see that $m(j-1) < f(v_j^i) \leq mj$ for all $i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n+2\}$ so f is injection. Next, let $k \in \{1, 2, ..., m(n+2)\}$. If $1 \leq k \leq m$, then $k = f(v_1^k)$.
Case 1 : $m+1 \leq k \leq m(n+1)$. Then there are $q, r \in \mathbb{N}$, $k = qm + r$ where $1 \leq q \leq n$ and $1 \leq r \leq m$.
Subcase 1.1 : $1 \leq r \leq \frac{m+1}{2}$. Thus

$$f(v_{q+1}^{m+2-2r}) = (q+1)m + 1 - \frac{m+m+2-2r}{2} = k.$$

Subcase 1.2 : $\frac{m+1}{2} < r \leq m$. We have that

$$f(v_{q+1}^{2m+2-2r}) = (q+1)m - \frac{2m+2-2r-2}{2} = k.$$

Case 2 : $m(n+1) < k \leq m(n+2)$. Let $s = k - m(n+1)$ hence $1 \leq s \leq m$.
Subcase 2.1 : $1 \leq s \leq \frac{m-1}{2}$. Thus

$$f(v_{n+2}^{m+1-2s}) = m(n+2) - \frac{m+m+1-2s-1}{2} = k.$$

Subcase 2.2 : $\frac{m+1}{2} \leq s \leq m$. We have

$$f(v_{n+2}^{2m+1-2s}) = m(n+2) - \frac{2m+1-2r-1}{2} = k.$$

Thus f is a bijective function. Let $i \in \{1, 2, .., m\}$. Recall that $E_i = \{v_1^i v_2^i, v_1^i v_3^i, ..., v_1^i v_{n+1}^i\} \cup \{v_1^i v_{n+2}^i\} \cup \{v_2^i v_{n+2}^i, v_3^i v_{n+2}^i, ..., v_n^i v_{n+2}^i, v_{n+1}^i v_{n+2}^i\}$. Let $S_i = \{f(x) + f(y) \mid xy \in E_i\}$. If i is even, then

$$\begin{aligned} S_i = & \left\{ jm + \frac{i+2}{2} \,\middle|\, j = 2, 3, ..., n+1 \right\} \\ & \cup \left\{ jm + m(n+1) + \frac{m+1-2i}{2} \,\middle|\, j = 2, 3, ..., n+1 \right\} \\ & \cup \left\{ m(n+1) + \frac{m+1+i}{2} \right\}. \end{aligned}$$

If i is odd, we have

$$\begin{aligned} S_i = & \left\{ jm + 1 - \frac{m-i}{2} \,\middle|\, j = 2, 3, ..., n+1 \right\} \\ & \cup \left\{ jm + m(n+1) + \frac{m+1-2i}{2} \,\middle|\, j = 2, 3, ..., n+1 \right\} \\ & \cup \left\{ m(n+2) + \frac{i+1}{2} \right\}. \end{aligned}$$

Therefore the set

$$\begin{aligned} S &= \{f(x) + f(y) \mid xy \in E(mF_{n,2})\} \\ &= S_1 \cup S_2 \cup ... \cup S_m \\ &= \left\{ \frac{3m+3}{2}, \frac{3m+5}{2}, ..., 2mn + \frac{5m+1}{2} \right\} \end{aligned}$$

is the set of $m(2n+1)$ consecutive integers and

$$\begin{aligned} k &= m(n+2) + m(2n+1) + \min(S) \\ &= m(n+2) + m(2n+1) + \frac{3m+3}{2} \\ &= 3\left(mn + \frac{3m+1}{2}\right). \end{aligned}$$

By Lemma I.1, f extends to a super edge-magic labeling of $mF_{n,2}$ with $k = 3\left(mn + \dfrac{3m+1}{2}\right)$. ∎

REFERENCES

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, third edition, Chapman & Hall, 1996.
[2] H. Enomoto, A.S. Llado, T. Nakamigawa and G. Ringel, "Super edge-magic graphs," *SUT J. Math.* V34, N2, pp. 105-109.
[3] R.M. Figuroa-Centeno, R. Ichishima and F.A. Muntaner-Batie, "The place of super edge-magic labeling among other classes of labelings," *Discrete Math.* V231, pp. 153-168.
[4] A.A.G. Ngurah, E.T. Baskoro and R. Simanjuntak, "On the super edge-magic deficiencies of graphs," *Australas. J. Combin.* V40, pp. 3-14.
[5] A.A.G. Ngurah and R. Simanjuntak, "Super edge-magic deficiency of joint-product graphs," *arXiv:1401.4522v2[math.CO] 26 Apr 2014.*