Proceedings of the International MultiConference of Engineers and Computer Scientists 2018 Vol I
IMECS 2018, March 14-16, 2018, Hong Kong

Base Hypergraphs and Orbits of CNF Formulas

Stefan Porschen *

Abstract— **The existence problem of specific base hypergraphs of formulas on basis of the fibre perspective on the propositional satisfiability problem is addressed. Further, the orbit structure of the set of fibre-transversals imposed by the flipping operation is investigated. Moreover some complexity results are proven. Methodogically, the concept of linear and exact linear hypergraphs, respectively, formulas is exploited.**

Keywords: satisfiability, hypergraph, fibre-transversal, orbit

1 Introduction

A fundamental open question in mathematics is the NP versus P problem which is attacked within the theory of NP-completeness [8]. The genuine and one of the most important NP-complete [6] problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas. More precisely, SAT is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover, SAT plays an essential role in the theory of designing exact algorithms, and it offers various applications due to the fact that instances of numerous computational problems can be encoded as equivalent instances of CNF-SAT via reduction [10]. The reason for this is the high expressiveness of the CNF language. In industrial applications most often the modelling CNF formulas are of a specific structure. And therefore it would be desirable to have fast algorithms for such instances. Also from a theoretical point of view one is interested in classes for which SAT can be solved in polynomial time. There are known several classes, for which SAT can be tested efficiently, such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested and co-nested formulas etc. [2, 4, 5, 7, 11, 12, 13, 9, 19, 21]. It turns out that a useful tool in revealing the structure of CNF-SAT is provided by linear CNF formulas. Note that the complexity of various satisfiability problems on linear formula classes is well studied confer, e.g. [16, 18]. On basis of LCNF-SAT we discuss in this paper the complexity of SAT restricted to classes defined through the flipping operation on CNF formulas. Further, we exploit the fibre view on clause sets and investigate the structure and existence questions of specific base hypergraphs of formulas. Also the orbit structure of fibre-transversals is investigated with respect to the flipping operation on formulas. Specifically, we show that the orbit space of the class of diagonal fibre-transversals generally is non-trivial for diagonal base hypergraphs.

*Mathematics Group, Department 4, HTW Berlin, D-10313 Berlin, Germany, Email: porschen@htw-berlin.de.

2 Preliminaries

A Boolean or propositional variable x taking values from $\{0, 1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$ also called the *flipped* or *complemented* variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of different literals and it is represented as a set $c = \{l_1, \ldots, l_k\}$. A clause containing no negative literal is called *positive*. A clause containing only negated variables is called *negative*. A *unit* clause contains exactly one literal. A conjunctive normal form formula C, for short formula, is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. A formula can also be empty which is denoted as $\varnothing$. Let CNF be the collection of all formulas. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Let CNF_+ (CNF_-) denote that part of CNF containing only positive (negative) clauses. A formula $C \in \mathrm{CNF}$ is called *linear* if each pair $c_i, c_j \in C$, $i \neq j$, satisfies $|V(c_i) \cap V(c_j)| \leq 1$. By LCNF the class of linear formulas is denoted. In an *exact* linear formula the variable sets of distinct clauses have exactly one member in common. For a finite set M, let 2^M denote its powerset. Given a finite group G, let $\mathrm{Gn}(G)$ denote a set of generators of G. Given $C \in \mathrm{CNF}$, SAT asks whether there is a truth assignment $t : V(C) \to \{0, 1\}$ such that there is no $c \in C$ all literals of which are set to 0. If such an assignment exists it is called a *model* of C, and $M(C)$ denotes the collection of all models of C. Let SAT $\subseteq$ CNF denote the collection of all clause sets for which there is a model, and UNSAT := CNF $\setminus$ SAT. Clauses containing a complemented pair of literals are always satisfied. Hence, it is assumed throughout that clauses only contain literals over different variables.

3 Base-Hypergraphs of Formulas

The hyperedge set $B(C)$ of the base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ assigned to a formula $C \in \mathrm{CNF}$ is defined as $B(C) := \{V(c) : c \in C\} \in \mathrm{CNF}_+$. As intro-

ISBN: 978-988-14047-8-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

duced in [14] the collection of all clauses c such that $V(c) = b$, for a fixed $b \in B(C)$, is the fibre C_b of C over b. Conversely, a hypergraph $\mathcal{H} = (V, B)$ can be regarded as base hypergraph if its vertex set V is a finite non-empty set of Boolean variables such that for every $x \in V$ there is a $b \in B$ containing x meaning $B \neq \emptyset$. By $W_b := \{c : V(c) = b\}$ denote the collection of all possible clauses over a fixed $b \in B$. By definition, a hypergraph $\mathcal{H} = (V, B)$ is *linear* if $|b \cap b'| \leq 1$, for all distinct $b, b' \in B$, and $\mathcal{H}$ is *exact linear* if the symbol $\leq$ above is replaced with $=$. Recall that a hypergraph $\mathcal{H} = (V, B)$ is called *loopless* if $|b| \geq 2$, for all $b \in B$ [3]. Observe that the base hypergraph $\mathcal{H}(C)$ is (exact) linear if the formula C is (exact) linear. Moreover $\mathcal{H}(C)$ is loopless if C is free of unit clauses. The set of all clauses over $\mathcal{H}$ is $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$. A *$\mathcal{H}$-based* formula is a subset $C \subseteq K_{\mathcal{H}}$ such that $C_b := C \cap W_b \neq \emptyset$, for every $b \in B$. For $\mathcal{H}$-based $C \subsetneq K_{\mathcal{H}}$ let $\bar{C} := K_{\mathcal{H}} \setminus C$ be its complement formula. If C satisfies $\bar{C}_b := W_b \setminus C_b \neq \varnothing$, for all $b \in B$, then $\bar{C}$ also is $\mathcal{H}$-based with non-empty fibres $\bar{C}_b$, for every $b \in B$. A *fibre-transversal* of $K_{\mathcal{H}}$ is a $\mathcal{H}$-based formula $F \subset K_{\mathcal{H}}$ such that $|F \cap W_b| = 1$, for every $b \in B$, this clause is denoted as $F(b)$. By $\mathcal{F}(K_{\mathcal{H}})$ denote the set of all fibre-transversals of $K_{\mathcal{H}}$. Observe that, given a linear base hypergraph $\mathcal{H}$ then every fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ is linear. A *compatible* fibre-transversal is defined by the property that $\bigcup_{b \in B} F(b) \in W_V$. $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ is the set of all compatible fibre-transversals of $K_{\mathcal{H}}$. A *diagonal* fibre-transversal is defined through the property that for each $F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ one has $F \cap F' \neq \emptyset$. Finally, let $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ be the collection of all diagonal fibre-transversals of $K_{\mathcal{H}}$. We call a base hypergraph $\mathcal{H}$ diagonal [15] if and only if $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \emptyset$. As for the total clause set $K_{\mathcal{H}}$ we can define fibre-transversals for a $\mathcal{H}$-based formula $C \subset K_{\mathcal{H}}$ as follows. A fibre-transversal F of C contains exactly one clause of each fibre C_b of C. The collection of all fibre-transversals of C is denoted as $\mathcal{F}(C)$. We also have compatible and diagonal fibre-transversals of C via $\mathcal{F}_{\text{comp}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, and $\mathcal{F}_{\text{diag}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$.

4 The Flipping Operation

For a fixed finite and non-empty set of propositional variables V let $\text{CNF} := \text{CNF}(V)$ denote the set of all CNF formulas with $V(C) \subseteq V$. Let c^X be the clause obtained from c by complementing all variables in $X \cap V(c)$, where X is an arbitrary subset of V, for short we set $c^\gamma := c^{V(c)}$, and further $c^\emptyset := c$. This *flipping operation* $\varphi(c, X) := c^X$ on clauses induces via the identification of c with $\{c\}$ an action on arbitrary members of CNF: For $C = \{c_1, \ldots, c_m\} \in \text{CNF}$ and $X \in 2^V$, we define $\varphi : \text{CNF} \times 2^V \to \text{CNF}$, through $\varphi(C, X) := \{\varphi(c_1, X), \ldots, \varphi(c_m, X)\} =: C^X \in \text{CNF}$. Again set $C^\gamma := C^{V(C)}$ in case that all variables in C are flipped, and $C^\emptyset := C$. Thus formally we obtain the *G_V-action* of the abelian group $G_V := (2^V, \oplus)$ with neutral element $\emptyset$ on CNF providing CNF as a *G_V-space*. Indeed, first flipping C by $X \in G_V$ then by $Y \in G_V$ obviously yields $\varphi(\varphi(C, X), Y) = (C^X)^Y = C^{X \oplus Y} = \varphi(C, X \oplus Y)$ where $\varphi(\varnothing, X) := \varnothing \in \text{CNF}$, for every $X \in G_V$. Observe that as the group is commutative the operation is the same regarded as a left action or as a right action. In case $V(C) \neq V$ the relevant subgroup of G_V is $G_{V(C)} = (2^{V(C)}, \oplus)$. By $\mathcal{O}(C) := \{C^X : X \in G_{V(C)}\} = \{C^X : X \in G_V\}$ denote the *(G_V-)orbit* of C in CNF yielding the classes of an equivalence relation on CNF whose quotient space CNF/G_V therefore is usually called the *orbit space*. Let $G_{V(C)}(C) := \{X \in G_{V(C)} : C^X = C\}$ denote the *isotropy group* also called *stabilizer* of $C \in \text{CNF}$. More generally $G_V(\mathcal{C}) := \{X \in G_V : C \in \mathcal{C} \Rightarrow C^X \in \mathcal{C}\}$ denotes the isotropy group of the class $\mathcal{C} \subseteq \text{CNF}$. Observe that also the latter concept can easily be verified to be a subgroup of G_V. Indeed, for $X, Y \in G_V(\mathcal{C})$ and arbitrary $C \in \mathcal{C}$ let $C^X =: C' \in \mathcal{C}$ then we have $C^{X \oplus Y^{-1}} = C^{X \oplus Y} = C'^Y \in \mathcal{C}$ hence $X \oplus Y^{-1} \in G_V(\mathcal{C})$. The mapping $f_X : \text{CNF} \to \text{CNF}$ defined by $f_X(C) := C^X$ is *G_V-equivariant*, by definition meaning that $f_X(C^Y) = [f_X(C)]^Y$, for every $Y \in G_V$ and every $C \in \text{CNF}$. Clearly, $V(C') = V(C)$ for $C' \in \mathcal{O}(C)$. So $C^X = C$ and $C' = C^Y \in \mathcal{O}(C)$ implies $C'^X = (C^Y)^X = (C^X)^Y = C'$ and vice versa. Therefore $G_{V(C)}(C') = G_{V(C)}(C)$ for all $C' \in \mathcal{O}(C)$. Also as usual a *fixed point* of an operation, cf. e.g. [20], is the unique member of an 1-point *invariant subspace*, so by definition its isotropy group equals the whole group.

Lemma 1 *[15] $\varnothing \neq C \in \text{CNF}$ is a fixed point of the G_V-action if and only if $C_b = W_b$, for all $b \in B(C)$.*

Given $C \in \text{CNF}$, we set $A(C) := \{c \in C : c^\gamma \notin C\}$ and $S(C) := \{c \in C : c^\gamma \in C\}$. This yields subclasses of CNF, namely $\mathcal{A} := \{C \in \text{CNF} : C = A(C)\}$ denoting the set of *anti-symmetric* formulas and $\mathcal{S} := \{C \in \text{CNF} : C = C^\gamma\}$ which is the set of *symmetric* formulas as introduced in [17]. Observe that $A(C)$ can be the empty formula as is the case for $C = W_b \in \mathcal{S}$, $b \subseteq V$. If $C \in \text{LCNF}$ we also have $S(C) = \varnothing$ therefore $\{\varnothing\} = \mathcal{S} \cap \mathcal{A}$. A specific class of symmetric formulas is given by those $C \in \mathcal{S}$ such that every clause $c \in C$ either belongs to CNF_+ or to CNF_-. Let $\mathcal{S}_\pm$ denote the collection of such formulas yielding the key to the complexity of $\mathcal{S}$-SAT as is shown next. We also prove that the complexity does not decrease if the input is restricted to anti-symmetric instances.

Theorem 1 *The computational complexity of* SAT *restricted to $\mathcal{C}$ is NP-complete, for $\mathcal{C} \in \{\mathcal{S}_\pm, \mathcal{S}, \mathcal{A}\}$.*

Proof. Clearly, $\mathcal{C}$-SAT belongs to NP, for $\mathcal{C} \in \{\mathcal{S}_\pm, \mathcal{S}, \mathcal{A}\}$. Regarding $\mathcal{S}_\pm$ first recall that the NP-complete hypergraph bicolorability problem for given hypergraph $\mathcal{H} = (V, B)$ asks whether a 2-coloring $t : V \to \{0, 1\}$ of the vertex set exists such that no hyperedge appears monochromatic. Next B can be interpreted as a set

ISBN: 978-988-14047-8-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

of positive clauses $B \in \mathrm{CNF}_+$ if its vertices are assigned to Boolean variables. Clearly one has $B^\gamma \in \mathrm{CNF}_-$ and $C := B \cup B^\gamma \in \mathcal{S}_\pm$. Thus the decision of the bicolorability problem for $\mathcal{H}$ means a decision of SAT for C. Hence a polynomial-time reduction is provided establishing that SAT restricted to $\mathcal{S}_\pm$ is NP-complete. The assertion for $\mathcal{S}$ therefore holds true via the inclusion $\mathcal{S}_\pm \subset \mathcal{S}$. Regarding the additional statement, first recall that SAT remains NP-complete for the class of linear formulas that are free of unit clauses according to [18]. Next assume there is $C \in \mathrm{LCNF}$ free of unit clauses such that $c, c^\gamma \in C$ implying $|V(c) \cap V(c^\gamma)| \geq 2$ and contradicting the linearity, hence such a formula must be a member of $\mathcal{A}$. In consequence also the computational complexity of SAT restricted to $\mathcal{A}$ is NP-complete. □

Note that both classes $\mathcal{A}, \mathcal{S}$ are stable subspaces of CNF under the flipping operation [15], meaning that $G_V(\mathcal{C}) = G_V$, for $\mathcal{C} \in \{\mathcal{A}, \mathcal{S}\}$. However, the subclasses $\mathrm{CNF}_-, \mathrm{CNF}_+ \subseteq \mathcal{A}$ are non-invariant subspaces. Also $C \in \mathcal{S}_\pm$ then $\mathcal{O}(C) \not\subseteq \mathcal{S}_\pm$. Moreover one has:

Lemma 2 *Given $C \in \mathcal{S}_\pm$ then a set of generators of $G_V(C)$ can be computed in polynomial time.*

For a given base-hypergraph $\mathcal{H}$ let $\mathcal{C}(\mathcal{H}) := \{C \in \mathcal{C} : \mathcal{H}(C) = \mathcal{H}\}$, for $\mathcal{C} \in \{\mathcal{A}, \mathcal{S}\}$.

Theorem 2 *For $\mathcal{H} = (V, B)$ one has. (1) There is an G_V-equivariant bijection $\sigma : \mathcal{A}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$. (2) Given $C \in \mathcal{A}(\mathcal{H})$ with $G := G_V(C)$ then $\mathrm{Gn}(G_V(\sigma(C))) = \mathrm{Gn}(G) \cup \mathrm{Gn}(G_B)$, where $G_B := G_V(B \cup B^\gamma)$. Moreover for input $\mathrm{Gn}(G)$, $\mathrm{Gn}(G_V(\sigma(C)))$ can be computed in polynomial time.*

5 Orbits of Fibre-Transversals

This section is devoted to study to some extent the orbit structure with respect to the flipping operation within the set of fibre-transversals over specific base hypergraphs. To that end, first let $\mathcal{H} = (V, B)$ be a fixed but arbitrary base hypergraph with total clause set $K_\mathcal{H}$. Before, several results which are proven elsewhere and turn out to be useful are recalled here. A first one characterizes the satisfiability of a formula C in terms of compatible fibre-transversals in its based complement formula $\bar{C} = K_\mathcal{H} \setminus C$.

Theorem 3 *[14] For $\mathcal{H} = (V, B)$, let $C \subset K_\mathcal{H}$ be a $\mathcal{H}$-based formula such that $\bar{C}$ is $\mathcal{H}$-based, too. Then C is satisfiable if and only if $\bar{C}$ admits a compatible fibre-transversal F. Moreover, the union $\bigcup c$ of all clauses $c \in F^\gamma$ is a member of $M(C)$.*

Lemma 3 *[17] (i) $\mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$ is bijective to W_V. (ii) $F \in \mathcal{F}_{\mathrm{diag}}(K_\mathcal{H}) \Leftrightarrow F \in \mathrm{UNSAT}$, $F \in \mathcal{F}(K_\mathcal{H})$.*

Fibre-transversals yield stable classes with respect to the flipping operation.

Lemma 4 *For $\mathcal{H} = (V, B)$ each of the classes $\mathcal{F}(K_\mathcal{H})$, $\mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$, $\mathcal{F}_{\mathrm{diag}}(K_\mathcal{H})$ and $\mathcal{F}(K_\mathcal{H}) \setminus (\mathcal{F}_{\mathrm{comp}}(K_\mathcal{H}) \cup \mathcal{F}_{\mathrm{diag}}(K_\mathcal{H}))$ is invariant under G_V-action.*

Proof. For $F \in \mathcal{F}(K_\mathcal{H})$ and $X \subseteq V$ we have by definition $F^X = \{c^X : c \in F\} = \{(F(b))^X : b \in B\}$. Hence it is $(F^X)(b) = (F(b))^X \in W_b$, thus $1 = |F \cap W_b| = |F^X \cap W_b|$, for every $b \in B$, implying $F^X \in \mathcal{F}(K_\mathcal{H})$ proving the first claim. Next assume $F \in \mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$ then $\bigcup_{b \in B}(F^X)(b) = \bigcup_{b \in B}(F(b))^X = (\bigcup_{b \in B} F(b))^X \in W_V$, for every $X \in G_V$ hence $F^X \in \mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$. Next, take an arbitrary $F \in \mathcal{F}_{\mathrm{diag}}(K_\mathcal{H}) \neq \varnothing$ otherwise the claim holds for sure. As $F'^X \in \mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$ for every $X \subseteq V$ and every $F' \in \mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$ there exists $b \in B$ depending on F', X such that $F(b) = (F'^X)(b) = F'(b))^X$ equivalent with $(F(b)^X = (F^X)(b) = F'(b)$. Therefore $F^X \in \mathcal{F}_{\mathrm{diag}}(K_\mathcal{H})$, for every $X \in G_V$. The last claim directly follows from the first three results. □

Observe that every fibre-transversal $F \in \mathcal{F}(K_\mathcal{H})$ belongs to $\mathcal{A}$. Therefore as a direct consequence from Lemma 1 one has that there are no fixed points within sets of fibre-transversals at all, except for the empty base hypergraph. One even obtains stronger results as follows.

Theorem 4 *[15] Let $\mathcal{H} = (V, B)$. (i) $F^X \neq F$ for every $F \in \mathcal{F}(K_\mathcal{H})$ and every $\emptyset \neq X \in G_V$. (ii) $\mathcal{F}_{\mathrm{comp}}(K_\mathcal{H}) = \mathcal{O}(F)$, for any fixed $F \in \mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})$.*

Observe that the statement (ii) above means that the G_V-action is *transitive* [20] restricted to the space of compatible fibre-transversals. Further, statement (i) implies that every fibre-transversal over a base hypergraph has the trivial isotropy group E. Regarding the space of diagonal fibre-transversals of a base hypergraph $\mathcal{H} = (V, B)$, let the integer $\delta(\mathcal{H}) \geq 0$ denote the cardinality of the orbit space $\mathcal{F}_{\mathrm{diag}}(K_\mathcal{H})/G_V$. Clearly $\delta(\mathcal{H})$ essentially depends on the structure of B. Further denote the number of orbits in the space of all fibre-transversals of $\mathcal{H}$ by $\omega(\mathcal{H})$ and set $\beta(\mathcal{H}) := \sum_{b \in B} |b| - |V| \geq 0$.

Corollary 1 *Given a base hypergraph $\mathcal{H} = (V, B)$, then we have $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})} \geq 1$, $|\mathcal{F}(K_\mathcal{H})| = \omega(\mathcal{H})2^{|V|}$, and $|\mathcal{F}_{\mathrm{diag}}(K_\mathcal{H})| = \delta(\mathcal{H})2^{|V|}$.*

Proof. $r := |B|$ and $n := |V|$ are positive integers. According to Lemma 4 and Theorem 4 (i), for every $F \in \mathcal{F}(K_\mathcal{H})$ we have $|\mathcal{O}(F)| = 2^n$, therefore $|\mathcal{F}_{\mathrm{diag}}(K_\mathcal{H})| = \delta(\mathcal{H})2^n$, and $|\mathcal{F}(K_\mathcal{H})| = \omega(\mathcal{H})2^n$. Next one has $|\mathcal{F}(K_\mathcal{H})| \geq |\mathcal{F}_{\mathrm{comp}}(K_\mathcal{H})| = 2^n$ according to Lemma 3 (i), hence $\omega(\mathcal{H}) \geq 1$. Finally one obtains

$$|\mathcal{F}(K_\mathcal{H})| = \prod_{i=1}^{r} |W_{b_i}| = 2^{\sum_{i=1}^{r} |b_i|} = 2^{\beta(\mathcal{H})}2^n$$

hence $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$. $\square$

The question whether $\delta(\mathcal{H})$ can be larger than one is addressed later. A connection to the monotonicity index as introduced in [15] appears as follows.

Theorem 5 *Let $\mathcal{H}$ then $\mu(F) = 0$, for every $F \in \mathcal{F}(K_\mathcal{H}) \setminus \mathcal{F}_{\rm diag}(K_\mathcal{H})$.*

Proof. Recall that for $C \in$ CNF the value $\mu(C) := \min\{\min\{|C'_+|, |C'_-|\} : C' \in \mathcal{O}(C)\}$ is the monotonicity index of C, where $C_+ \in \text{CNF}_+$, respectively, $C_- \in \text{CNF}_-$ is the collection of all positive, respectively negative clauses in C. As shown in [15] we have $C \in$ SAT if and only if $\mu(C) = 0$. So, the claim follows directly from Lemma 3 (ii). $\square$

Let $\mathcal{H} = (V, B)$ be a base hypergraph, then have $|\mathcal{F}_{\rm comp}(K_\mathcal{H})| = |W_V|$ due to Lemma 3 (i). Whether $\mathcal{F}_{\rm diag}(K_\mathcal{H}) \neq \emptyset$ depends on the structure of the base hypergraph $\mathcal{H}$. Consider, e.g. a loopless and exact linear base hypergraph $\mathcal{H} = (V, B)$. Recalling that according to [18] each exact linear formula without unit clauses is satisfiable it follows that $\mathcal{F}_{\rm diag}(K_\mathcal{H}) = \emptyset$. However, choose an unsatisfiable linear formula [18] then its base hypergraph admits diagonal fibre-transversals. On that basis we can address the question whether the second assertion of Theorem 4 (ii) is valid for diagonal fibre-transversals also. As the result below tells us, the answer in general is no, implying that the orbit structure of the set of diagonal fibre-transversal can be more complex.

Theorem 6 *There exist loopless and linear diagonal base hypergraphs $\mathcal{H} = (V, B)$ such that the orbit space $\mathcal{F}_{\rm diag}(K_\mathcal{H})/G_V$ contains more than one element.*

Proof. Let $F_0 \in$ UNSAT be a linear formula free of unit clauses which exists according to [18], and let $\mathcal{H}_0 = (V_0, B_0) := \mathcal{H}(F_0)$ be its base hypergraph. Hence $F_0 \in \mathcal{F}_{\rm diag}(K_{\mathcal{H}_0}) \neq \emptyset$ meaning that $\mathcal{H}_0$ is a diagonal base hypergraph that is loopless and linear. Let b_i with $|b_i| \geq 2$, $V_0 \cap b_i = \emptyset$, $i = 1, 2$ and $b_1 \cap b_2 = \{x\}$. We set $\mathcal{H} = (V, B)$ where $V := V_0 \cup b_1 \cup b_2$, $B := B_0 \cup \{b_1, b_2\}$. Now we define distinct fibre-transversals $F, F' \in \mathcal{F}_{\rm diag}(K_\mathcal{H})$ by continuing F_0 over W_{b_1}, W_{b_2} as follows. Let $F := F_0 \cup \{b_1, b_2\}$ and $F' := F_0 \cup \{b_1, c_2\}$, where $c_2 = b_2^{\{x\}} \in W_{b_2}$. Hence, $b_2 \oplus c_2 = \{x, \bar{x}\}$, and $\mathcal{H} = \mathcal{H}(F) = \mathcal{H}(F')$. Since F_0 is linear, by construction it is ensured that F, F' are fibre-transversals over $K_\mathcal{H}$ which therefore is loopless and linear. Since $F_0 \in$ UNSAT the same obviously holds for F, F' and according to Lemma 3 (ii) indeed we obtain that F, F' are diagonal. Suppose there is $X \in G_V$ such that $F^X = F'$ specifically implying $b_1^X = b_1$ and $b_2^X = c_2$ as both formulas are fibre-transversals. The first equation means $x \notin X$, the second implies $x \in X$ yielding a contradiction. Thus it is verified that $\mathcal{O}(F) \neq \mathcal{O}(F')$ finishing the proof. $\square$

However unsatisfiable formulas $C \subset K_\mathcal{H}$ can exist also in case that $\mathcal{H}$ is not diagonal, as the next result tells us, repeated here for convenience.

Lemma 5 *[17] Let $\mathcal{H} = (V, B)$ be an exact linear base hypergraph such that there is a vertex $x \in V$ occurring in each $b \in B$. Let $C \subset K_\mathcal{H}$ be a $\mathcal{H}$-based formula such that $\bar{C}$ also is $\mathcal{H}$-based. Then we have: $C \in$ UNSAT if and only if $|\{b \in B : \forall c \in \bar{C}_b, x \in c\}| > 0$ and $|\{b \in B : \forall c \in \bar{C}_b, \bar{x} \in c\}| > 0$.*

So $C \in$ UNSAT generally does not mean $\mathcal{F}_{\rm diag}(C) \neq \emptyset$. A diagonal base hypergraph $\mathcal{H} = (V, B)$ is called strictly diagonal if for every $C \subset K_\mathcal{H}$ with $B(C) = B = B(\bar{C})$ one has the equivalence $C \in \text{UNSAT} \Leftrightarrow \mathcal{F}_{\rm diag}(C) \neq \emptyset$ [15]. The implication from right to left here is always valid according to Theorem 3. So, the existence of strictly diagonal base hypergraphs needs to be proven. As a first step we next show that the class of all strictly diagonal base hypergraphs does not coincide with the class of all diagonal base hypergraphs.

Theorem 7 *There exist loopless and linear diagonal base hypergraphs that are not strictly diagonal.*

Proof. To proceed semi-constructively, we start with a loopless exact linear base hypergraph $\mathcal{H}_1 = (V_1, B_1)$ such that according to Lemma 5 there is a $\mathcal{H}_1$-based unsatisfiable formula $C_1 \subset K_{\mathcal{H}_1}$ such that $\bar{C}_1$ also is $\mathcal{H}_1$-based. Next take any unsatisfiable $F' \in$ LCNF free of unit clauses and such that $V_2 := V(F')$ and V_1 are disjoint. Finally setting $\mathcal{H}_2 := \mathcal{H}(F')$ then specifically $F' \in \mathcal{F}_{\rm diag}(K_{\mathcal{H}_2})$ is ensured. Moreover $B_2 := B(F')$ and B_1 are disjoint also, therefore $\mathcal{H} = (V, B) := \mathcal{H}_1 \cup \mathcal{H}_2$ is a disjoint union. Now we claim that $\mathcal{H}$ is diagonal but not strictly diagonal establishing the theorem. To verify the claim, first observe that $\mathcal{F}_{\rm diag}(K_\mathcal{H}) \neq \emptyset$ thus $\mathcal{H}$ is a diagonal base hypergraph. Indeed, let $F \in \mathcal{F}(K_{\mathcal{H}_1})$ then $F \cup F' \in \mathcal{F}_{\rm diag}(K_\mathcal{H})$, because $F \cup F' \in \mathcal{F}(K_\mathcal{H})$ and, as $F' \in$ UNSAT, also $F \cup F' \in$ UNSAT using Lemma 3 (ii). Next, let $C_2 \in \mathcal{F}(K_{\mathcal{H}_2}) \setminus \mathcal{F}_{\rm diag}(K_{\mathcal{H}_2})$ hence $C_2 \in$ SAT. Then for the $\mathcal{H}$-based formula $C := C_1 \cup C_2 \in$ UNSAT holds because $C_1 \in$ UNSAT; further by construction $B(C) = B = B(\bar{C})$. Moreover, we have $\mathcal{F}_{\rm diag}(C) = \emptyset$ because $\mathcal{F}_{\rm diag}(K_{\mathcal{H}_1}) = \emptyset$ and $\mathcal{F}_{\rm diag}(C_2) = \emptyset$ proving that $\mathcal{H}$ is diagonal but not strictly diagonal. $\square$

We call a diagonal base hypergraph *simple* iff the orbit space of its diagonal fibre-transversals is trivial.

Lemma 6 *$\mathcal{H}$ is simple if and only if there is a G_v-equivariant bijection between $\mathcal{F}_{\rm comp}(K_\mathcal{H})$ and $\mathcal{F}_{\rm diag}(K_\mathcal{H})$.*

One has the following characterization.

Theorem 8 *[15] Let $\mathcal{H} = (V, B)$ be strictly diagonal. Then a fibre-transversal $F \in \mathcal{F}(K_\mathcal{H})$ is compatible if and*

ISBN: 978-988-14047-8-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

only if it satisfies $F \cap F' \neq \emptyset$ for every diagonal fibre-transversal $F' \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$.

We also have the following connection to minimal unsatisfiable formulas [1]. Recall that a formula is minimal unsatisfiable if and only if the removal of an arbitrary clause yields a satisfiable subformula. Similarly, call a diagonal base hypergraph *(inclusion-)minimal* if it is does not contain another diagonal base hypergraph.

Theorem 9 *Let $\mathcal{H} = (V, B)$ be a strictly minimal diagonal base hypergraph. A formula $C \subset K_{\mathcal{H}}$ with $B(C) = B$ is minimal unsatisfiable if and only if it is a diagonal fibre-transversal of $K_{\mathcal{H}}$.*

PROOF. $\mathcal{H}$ has at least two hyperedges otherwise it cannot be diagonal. Let $C \subset K_{\mathcal{H}}$ with $B(C) = B$ be a minimal unsatisfiable formula. Clearly, then C cannot contain a complete fibre of $K_{\mathcal{H}}$. It follows that $B(C) = B = B(\bar{C})$, therefore, since $\mathcal{H}$ is strictly diagonal we know that $\mathcal{F}_{\text{diag}}(C) \neq \emptyset$. In other words C contains a subformula $F \subseteq C$ that is a diagonal fibre-transversal of $K_{\mathcal{H}}$. So we conclude $C = F$. Otherwise there is a clause $c \in C \setminus F$ which can be removed yielding an unsatisfiable formula. Conversely, assume that F is a diagonal fibre-transversal. Then removing any clause c from it yields a fibre-transversal of the total clause set over the hypergraph $\mathcal{H}_c := (V, B \setminus \{V(c)\})$. Assume that $F \setminus \{c\}$ is unsatisfiable then, according to Lemma 3 (ii), it is a diagonal fibre-transversal of $K_{\mathcal{H}_c}$. So we obtain a contradiction because $\mathcal{H}$ is assumed to be minimal diagonal. Therefore F is minimal unsatisfiable. □

6 Open Problems and Concluding Remarks

It would be interesting to investigate the existence of non-trivial isotropy groups of arbitrary as well as structured formulas with respect to the flipping action. Also, the investigation of the orbit structure of $\mathcal{A}$ and $\mathcal{S}$ for $\mathcal{H}$-based formulas, as well as the detection of other invariant subspaces of CNF is devoted to future research. Further the study of loopless simple hypergraphs has to be continued also in connection to strictly diagonal base hypergraphs. Finally, the existence problem for loopless strictly diagonal hypergraphs is open. One might hope to find a loopless strictly diagonal hypergraph in case that there exists only compatible and diagonal fibre-transversals. Indeed, assume that there is a loopless simple diagonal base hypergraph $\mathcal{H}$ such that $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) \cup \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. Take an arbitrary $C \subset K_{\mathcal{H}}$ which is unsatisfiable and such that $C, \bar{C}$ are $\mathcal{H}$-based, then $\mathcal{F}(C) \neq \emptyset \neq \mathcal{F}(\bar{C})$. Moreover then one has $\mathcal{F}_{\text{comp}}(\bar{C}) = \emptyset$ relying on Theorem 3. Since $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) \neq \emptyset$ it follows $\mathcal{F}(\bar{C}) \subseteq \mathcal{F}(K_{\mathcal{H}}) \setminus \mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) = \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ implying $\bar{C} \in$ UNSAT. Now assume to the contrary that $\mathcal{F}_{\text{diag}}(C) = \emptyset$. Since $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \emptyset$ it follows $\mathcal{F}(C) \subseteq \mathcal{F}(K_{\mathcal{H}}) \setminus \mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. Therefore $\bar{C} \in$ SAT again according to Theorem 3 yielding a contradiction and verifying that $\mathcal{H}$ is strictly diagonal. However, a base hypergraph as required above exists if and only if there is a simple $\mathcal{H} = (V, B)$ such that $\beta(\mathcal{H}) = \sum_{b \in B} |b| - |V| = 1$, because then according to Corollary 1 it follows $\omega(\mathcal{H}) = 2$, and $\delta(\mathcal{H}) = 1$. Hence $\mathcal{F}(K_{\mathcal{H}})$ can only contain compatible and diagonal fibre-transversals at all. But such a situation cannot occur as corresponding unsatisfiable formulas do not exist such that $|C| \leq \frac{|V(C)|+1}{2}$ ensuring that $\mathcal{H}(C)$ is loopless and admits the properties as required. Although the previous argumentation may hint towards a direction for detecting a loopless strictly diagonal base hypergraph as desired.

References

[1] Aharoni, R., Linial, N., "Minimal non two-colorable hypergraphs and minimal unsatisfiable formulas," *J. Combin. Theory A* pp. 196-204, 43/1986.

[2] Aspvall, B., Plass, M.R., Tarjan, R.E., "A linear-time algorithm for testing the truth of certain quantified Boolean formulas," *Information Process. Lett.* pp. 121-123, 8/1979.

[3] Berge, C., *Hypergraphs*, North-Holland, Amsterdam, 1989.

[4] Boros, E., Crama, Y., Hammer, P.L., "Polynomial-time inference of all valid implications for Horn and related formulae," *Annals of Math. Artif. Intellig.* pp. 21-32, 1/1990.

[5] Boros, E., Crama, Y., Hammer, P.L., Sun, X., "Recognition of q-Horn formulae in linear time," *Discrete Appl. Math.* pp. 1-13, 55/1994

[6] Cook, S.A., "The Complexity of Theorem Proving Procedures," *3rd ACM Symposium on Theory of Computing* pp. 151-158, 1971.

[7] Franco, J., VanGelder, A., "A perspective on certain polynomial-time solvable classes of satisfiability," *Discrete Appl. Math.* pp. 177-214, 125/2003.

[8] Garey M.R., Johnson, D.S., *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman and Company, San Francisco, 1979.

[9] Minoux, M., "LTUR: A Simplified Linear-Time Unit Resolution Algorithm for Horn Formulae and Computer Implementation," *Information Process. Lett.* pp. 1-12, 29/1988.

[10] Karp, R.M., "Reducibility Among Combinatorial Problems," in: *Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y.*, Plenum, New York, pp. 85-103, 1972.

[11] Knuth, D.E., “Nested satisfiability,” *Acta Informatica* pp. 1-6, 28/1990.

[12] Kratochvil, J., Krivanek, M., “Satisfiability of co-nested formulas,” *Acta Informatica* pp. 397-403, 30/1993.

[13] Lewis, H.R., “Renaming a Set of Clauses as a Horn Set,” *J. ACM* pp. 134-135, 25/1978.

[14] Porschen, S., “A CNF Formula Hierarchy over the Hypercube,” Proc. AI 2007, *LNAI* pp. 234-243, 4830/2007.

[15] Porschen, S., “Structural Aspects of Propositional SAT,” Proc. IMECS 2016, Hong Kong, pp. 126-131.

[16] Porschen, S., Schmidt, T., Speckenmeyer, E., Wotzlaw, A., “XSAT and NAE-SAT of linear CNF classes,” *Discrete Appl. Math.* pp. 1-14, 167/2014.

[17] Porschen, S., Speckenmeyer, E., “A CNF class generalizing Exact Linear Formulas,” Proc. SAT 2008, *LNCS* pp. 231-245, 2008.

[18] Porschen, S., Speckenmeyer, E., Zhao, X., “Linear CNF formulas and satisfiability,” *Discrete Appl. Math.* pp. 1046-1068, 157/2009.

[19] Schlipf, J., Annexstein, F.S., Franco, J., Swaminathan, R.P., “On finding solutions for extended Horn formulas,” *Information Process. Lett.* pp. 133-137, 54/1995.

[20] Spannier, E.H., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[21] Tovey, C.A., “A Simplified NP-Complete Satisfiability Problem”, *Discrete Appl. Math.* pp. 85-89, 8/1984.

ISBN: 978-988-14047-8-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)