Classes of Ordinary Differential Equations Obtained for the Probability Functions of Kumaraswamy Distribution

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***Abstract*— In this paper, differential calculus was used to obtain the ordinary differential equations (ODE) of the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Kumaraswamy distribution. The parameters and support that define the distribution inevitably determine the nature, existence, uniqueness and solution of the ODEs. The method can be extended to other probability distributions, functions and can serve an alternative to estimation and approximation. Computer codes and programs can be developed and used for the implementation.**

***Index Terms*— Differentiation, quantile function, survival function, approximation, hazard function, Kumaraswamy.**

I. Introduction

CALCULUS is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The method of maximum likelihood is an example.

Differential equations often arise from the understanding and modeling of real life problems or some observed physical phenomena. Approximations of probability functions are one of the major areas of application of calculus and ordinary differential equations in mathematical statistics. The approximations are helpful in the recovery of the probability functions of complex distributions [1-10].

Apart from mode estimation, parameter estimation and approximation, probability density function (PDF) of probability distributions can be expressed as ODE whose solution is the PDF. Some of which are available. They include: beta distribution [11], Lomax distribution [12], beta prime distribution [13], Laplace distribution [14] and raised cosine distribution [15].

The aim of this research is to develop homogenous ordinary differential equations for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Kumaraswamy distribution. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the necessary conditions for the existence of the ODEs. Similar results for other distributions have been proposed, see [16-28] for details.

Kumaraswamy Distribution is one of the interval bounded support probability distributions and introduced by Kumaraswamy [29]. It is one of the most studied probability distribution as evidenced by the many research materials available. Some of the advantages of the distribution over the beta distribution were highlighted in [30] and [31]. A short note of the distribution was written by Nadarajah [32]. The boundary properties and inference of the distribution were discussed extensively in Okagbue [33] and Wang et al., [34] respectively.

Some aspects of the distribution investigated by authors include: generalized order statistics [35], improved point estimation [36], Bayesian estimation of the parameters under censored samples [37-38], conditional estimation [39], analysis of the distribution based on record data [40] and statistical moments for the generalized distribution [41].

Its flexibility, ease of computation and tractability properties of the distribution are motivations for the numerous modifications and generalizations of the distribution. Some of which are listed as follows; exponentiated Kumarswamy distribution [42], Kumaraswamy Weibull distribution [43], bivariate Kumaraswamy distribution [44], Kumaraswamy generalized gamma distribution [45], Kumaraswamy Lindley distribution [46]. Also available are; Kumaraswamy-generalized Lomax distribution [47], Kumaraswamy Pareto distribution [48], Kumaraswamy-geometric distribution [49], Kumaraswamy Birnbaum–Saunders distribution [50], Kumaraswamy linear exponential distribution [51], Kumaraswamy-generalized exponentiated Pareto distribution [52], generalized Kumaraswamy exponential distribution [53], Kumaraswamy power series distribution [54] and [55]. Also available are; Kumaraswamy GP distribution [56], Exponentiated Kumaraswamy-Dagum distribution [57] and [58], Kumaraswamy Quasi Lindley distribution [59], transmuted Kumaraswamy distribution [60], exp-kumaraswamy distributions [61], the weighted kumaraswamy distribution [62], Kumaraswamy skew-normal distribution [63] Kumaraswamy modified inverse

Manuscript received December 9, 2017; revised January 15, 2018. This work was sponsored by Covenant University, Ota, Nigeria.

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Weibull distribution [64]. Also available are; Kumaraswamy odd log-logistic distribution [65], Kumaraswamy binomial distribution [66], Kumaraswamy-power distribution [67], beta generated kumaraswamy-G family of distributions [68], Kumaraswamy Marshall-Olkin family of distributions [69], Kumaraswamy GEV distribution [70], Kumaraswamy skew-t distribution [71]. Also available are; Kumaraswamy-Burr III distribution [72], Kumaraswamy-transmuted exponentiated modified Weibull distribution [73], inverted Kumaraswamy distribution [74], exponentiated Kumaraswamy-power function distribution [75], Kumaraswamy complementary Weibull geometric distribution [76] and others. The ordinary differential calculus was used to obtain the results.

II. Probability Density Function

The probability density function of the Kumaraswamy distribution is given as;

$$f(x) = abx^{a-1}(1-x^a)^{b-1} \quad x \in (0,1) \tag{1}$$

To obtain the first order ordinary differential equation for the probability density function of the Kumaraswamy distribution, differentiate equation (1), to obtain;

$$f'(x) = \left\{ \frac{(a-1)x^{a-2}}{x^{a-1}} - \frac{a(b-1)x^{a-1}(1-x^a)^{b-1}}{(1-x^a)^{b-1}} \right\} f(x) \tag{2}$$

$$f'(x) = \left\{ \frac{(a-1)}{x} - \frac{a(b-1)x^{a-1}}{(1-x^a)} \right\} f(x) \tag{3}$$

The necessary condition for the existence of equation is that $a,b > 0, 0 < x < 1.$

The ordinary differential equations can be obtained for particular values of a and b.

Special cases;

1. When a = 1, b = 1, the equation (3) becomes;

$$f'(x) = 0 \tag{4}$$

2. When a = 1 and b = 2, 3, 4, … , n.

When a = 1, b = 2, substitute in equations 3, to obtain;

$$f'(x) = -\frac{f(x)}{1-x} \tag{5}$$

When a = 1, b = 3, substitute in equations 3, to obtain;

$$f'(x) = -\frac{2f(x)}{1-x} \tag{6}$$

When a = 1, b = 4, substitute in equations 3, to obtain;

$$f'(x) = -\frac{3f(x)}{1-x} \tag{7}$$

When a = 1, b = n, substitute in equations 3, to obtain;

$$f'(x) = -\frac{(n-1)f(x)}{1-x} \tag{8}$$

3. When b = 1 and a = 2, 3, 4, … , n.

When b = 1, a = 2, substitute in equations 3, to obtain;

$$f'(x) = \frac{f(x)}{x} \tag{9}$$

When b = 1, a = 3, substitute in equations 3, to obtain;

$$f'(x) = \frac{2f(x)}{x} \tag{10}$$

When b = 1, a = 4, substitute in equations 3, to obtain;

$$f'(x) = \frac{3f(x)}{x} \tag{11}$$

When b = 1, a = n, substitute in equations 3, to obtain;

$$f'(x) = \frac{(n-1)f(x)}{x} \tag{12}$$

To obtain a simplified ordinary differential equation that is independent of the powers of the parameters, differentiate equation (3);

$$f''(x) = \left\{ \begin{array}{l} -\dfrac{(a-1)}{x^2} - \dfrac{(b-1)(ax^{a-1})^2}{(1-x^a)^2} \\ -\dfrac{a(a-1)(b-1)x^{a-2}}{(1-x^a)} \end{array} \right\} f(x) \tag{13}$$

$$\left\{ \frac{(a-1)}{x} - \frac{a(b-1)x^{a-1}}{(1-x^a)} \right\} f'(x)$$

The necessary condition for the existence of equation is that $a,b > 0, 0 < x < 1.$

The following equations obtained from equation (3) are needed to simplify equation (13);

$$\frac{f'(x)}{f(x)} = \frac{a-1}{x} - \frac{a(b-1)x^{a-1}}{1-x^a} \tag{14}$$

$$\frac{a(b-1)x^{a-1}}{1-x^a} = \frac{a-1}{x} - \frac{f'(x)}{f(x)} \tag{15}$$

$$\left(\frac{a(b-1)x^{a-1}}{1-x^a} \right)^2 = \left(\frac{a-1}{x} - \frac{f'(x)}{f(x)} \right)^2 \tag{16}$$

$$\frac{(b-1)(ax^{a-1})^2}{(1-x^a)^2} = \frac{1}{b-1}\left(\frac{a-1}{x} - \frac{f'(x)}{f(x)} \right)^2 \tag{17}$$

$$\frac{a(a-1)(b-1)x^{a-1}}{1-x^a} = a-1\left(\frac{a-1}{x} - \frac{f'(x)}{f(x)} \right) \tag{18}$$

$$\frac{a(a-1)(b-1)x^{a-2}}{1-x^a} = \frac{a-1}{x}\left(\frac{a-1}{x} - \frac{f'(x)}{f(x)} \right) \tag{19}$$

Substitute equations (14), (17) and (19) into equation (13) to obtain;

$$f''(x) = \frac{f'^2(x)}{f(x)} - \left\{ \begin{array}{l} \dfrac{(a-1)}{x^2} + \dfrac{1}{b-1}\left(\dfrac{a-1}{x} - \dfrac{f'(x)}{f(x)} \right)^2 \\ + \dfrac{a-1}{x}\left(\dfrac{a-1}{x} - \dfrac{f'(x)}{f(x)} \right) \end{array} \right\} f(x) \tag{20}$$

$$f(0.1) = ab(0.1)^{a-1}(1-(0.1)^a)^{b-1} \tag{21}$$

$$f'(0.1)=ab(0.1)^{a-1}(1-(0.1)^a)^{b-1}\left\{10(a-1)-\frac{10a(b-1)(0.1)^a}{(1-(0.1)^a)}\right\} \quad (22)$$

The necessary condition for the existence of equation is that $a>0, 0<x<1, b>1$.

III. Quantile Function

The Quantile function of the Kumaraswamy distribution is given as;

$$Q(p)=(1-(1-p)^{\frac{1}{b}})^{\frac{1}{a}} \quad p\in(0,1) \quad (23)$$

To obtain the first order ordinary differential equation for the Quantile function of the Kumaraswamy distribution, differentiate equation (23), to obtain;

$$Q'(p)=\frac{1}{ab}(1-p)^{\frac{1}{b}-1}(1-(1-p)^{\frac{1}{b}})^{\frac{1}{a}-1} \quad (24)$$

$$Q'(p)=\frac{(1-p)^{\frac{1}{b}}(1-(1-p)^{\frac{1}{b}})^{\frac{1}{a}}}{ab(1-p)(1-(1-p)^{\frac{1}{b}})} \quad (25)$$

The necessary condition for the existence of equation is that $a,b>0, 0<p<1.$

Substitute equation (23) into equation (25);

$$Q'(p)=\frac{(1-p)^{\frac{1}{b}}Q(p)}{ab(1-p)(1-(1-p)^{\frac{1}{b}})} \quad (26)$$

The following equations obtained from equation (23) are needed to simplify equation (26);

$$Q^a(p)=1-(1-p)^{\frac{1}{b}} \quad (27)$$

$$(1-p)^{\frac{1}{b}}=1-Q^a(p) \quad (28)$$

Substitute equations (27) and (28) into equation (26);

$$Q'(p)=\frac{(1-Q^a(p))Q(p)}{ab(1-p)Q^a(p)} \quad (29)$$

The necessary condition for the existence of equation (29) is that: $a,b>0$ and $1-p\neq 0$.

$$ab(1-p)Q'(p)=(1-Q^a(p))Q^{1-a}(p) \quad (30)$$

The first order ordinary differential equation is given by;

$$ab(1-p)Q'(p)-Q^{1-a}(p)+Q(p)=0 \quad (31)$$

$$Q(0)=0 \quad (32)$$

The first ordinary differential equations for the Quantile function of the Kumaraswamy distribution can be obtained for particular values of a and b.

When a = 1, equation (31) becomes;

$$b(1-p)Q'(p)+Q(p)-1=0 \quad (33)$$

When a = 2, equation (31) becomes;

$$2b(1-p)Q(p)Q'(p)+Q^2(p)-1=0 \quad (34)$$

When a = 3, equation (31) becomes;

$$3b(1-p)Q^2(p)Q'(p)+Q^3(p)-1=0 \quad (35)$$

IV. Survival Function

The Survival function of the Kumaraswamy distribution is given as;

$$S(t)=(1-t^a)^b \quad (36)$$

To obtain the first order ordinary differential equation for the survival function of the Kumaraswamy distribution, differentiate equation (36), to obtain;

$$S'(t)=-abt^{a-1}(1-t^a)^{b-1} \quad (37)$$

$$S'(t)=-\frac{abt^a(1-t^a)^b}{t(1-t^a)} \quad (38)$$

The necessary condition for the existence of equation is that $a,b>0, 0<t<1.$

Substitute equation (36) into equation (38);

$$S'(t)=-\frac{abt^aS(t)}{t(1-t^a)} \quad (39)$$

The following equations obtained from equation (36) are needed to simplify equation (39);

$$S^{\frac{1}{b}}(t)=1-t^a \quad (40)$$

$$t^a=1-S^{\frac{1}{b}}(t) \quad (41)$$

Substitute equations (40) and (41) into equation (39);

$$S'(t)=-\frac{ab(1-S^{\frac{1}{b}}(t))S(t)}{tS^{\frac{1}{b}}(t)} \quad (42)$$

$$tS'(t)=-ab(1-S^{\frac{1}{b}}(t))S^{1-\frac{1}{b}}(t) \quad (43)$$

$$tS'(t)+ab(S^{1-\frac{1}{b}}(t)-S(t))=0 \quad (44)$$

$$S(0.1)=(1-(0.1)^a)^b \quad (45)$$

V. Inverse Survival Function

The inverse survival function of the Kumaraswamy distribution is given as;

$$Q(p)=(1-p^{\frac{1}{b}})^{\frac{1}{a}} \quad (46)$$

To obtain the first order ordinary differential equation for the inverse survival function of the Kumaraswamy distribution, differentiate equation (46), to obtain;

$$Q'(p)=-\frac{p^{\frac{1}{b}-1}(1-p^{\frac{1}{b}})^{\frac{1}{a}-1}}{ab} \quad (47)$$

$$Q'(p)=-\frac{p^{\frac{1}{b}}(1-p^{\frac{1}{b}})^{\frac{1}{a}}}{pab(1-p^{\frac{1}{b}})} \quad (48)$$

The necessary condition for the existence of equation is that $a,b>0, 0<p<1.$

Substitute equation (46) into equation (49);

$$Q'(p)=-\frac{p^{\frac{1}{b}}Q(p)}{pab(1-p^{\frac{1}{b}})} \quad (49)$$

The following equations obtained from equation (46) are needed to simplify equation (49);

$$Q^{a}(p)=(1-p^{\frac{1}{b}}) \quad (50)$$

$$p^{\frac{1}{b}}=1-Q^{a}(p) \quad (51)$$

Substitute equations (50) and (51) into equation (49);

$$Q'(p)=-\frac{(1-Q^{a}(p))Q(p)}{abpQ^{a}(p)} \quad (52)$$

The necessary condition for the existence of equation (52) is that: $a,b>0$ and $p\neq 0$

$$abpQ'(p)=-(1-Q^{a}(p))Q^{1-a}(p) \quad (53)$$

The first order ordinary differential equation is given by;

$$abpQ'(p)+Q^{1-a}(p)-Q(p)=0 \quad (54)$$

$$Q(0.1)=(1-0.1^{\frac{1}{b}})^{\frac{1}{a}} \quad (55)$$

The first ordinary differential equations for the inverse survival function of the Kumaraswamy distribution can be obtained for particular values of a and b.

When a = 1, equation (54) becomes;

$$bpQ'(p)-Q(p)+1=0 \quad (56)$$

When a = 2, equation (54) becomes;

$$2bpQ(p)Q'(p)-Q^{2}(p)+1=0 \quad (57)$$

When a = 3, equation (54) becomes;

$$3bpQ^{2}(p)Q'(p)-Q^{3}(p)+1=0 \quad (58)$$

VI. Hazard Function

The Hazard function of the Kumaraswamy distribution is given as;

$$h(t)=\frac{abt^{a-1}(1-t^{a})^{b-1}}{1-(1-t^{a})^{b}} \quad (59)$$

To obtain the first order ordinary differential equation for the Hazard function of the Kumaraswamy distribution, differentiate equation (59), to obtain;

$$h'(t)=\left\{\begin{array}{l}\frac{(a-1)t^{a-1}}{t^{a-2}}-\frac{a(b-1)t^{a-1}(1-t^{a})^{b-2}}{(1-t^{a})^{b-1}}\\-\frac{abt^{a-1}(1-t^{a})^{b-1}(1-(1-t^{a})^{b})^{-2}}{(1-(1-t^{a})^{b})^{-1}}\end{array}\right\}h(t) \quad (60)$$

$$h'(t)=\left\{\frac{(a-1)}{t}-\frac{a(b-1)t^{a-1}}{(1-t^{a})}-\frac{abt^{a-1}(1-t^{a})^{b-1}}{(1-(1-t^{a})^{b})}\right\}h(t) \quad (61)$$

The necessary condition for the existence of equation is that $a,b>0,0<t<1$.

$$h'(t)=\left\{\frac{(a-1)}{t}-\frac{a(b-1)t^{a-1}}{(1-t^{a})}-h(t)\right\}h(t) \quad (62)$$

The necessary condition for the existence of equation (62) is that: $b,t>0$.

Differentiate equation (62), to obtain;

$$h''(t)=-\left\{\begin{array}{l}\frac{(a-1)}{t^{2}}+\frac{a^{2}(b-1)(t^{a-1})^{2}}{(1-t^{a})^{2}}\\+\frac{a(a-1)(b-1)t^{a-1}}{(1-t^{a})}+h'(t)\end{array}\right\}h(t)+\left\{\frac{(a-1)}{t}-\frac{a(b-1)t^{a-1}}{(1-t^{a})}-h(t)\right\}h'(t) \quad (63)$$

The necessary condition for the existence of equation is that $a,b>0,0<t<1$.

The following equations obtained from equation (62) are needed in the simplification of equation (63);

$$\frac{(a-1)}{t}-\frac{a(b-1)t^{a-1}}{(1-t^{a})}-h(t)=\frac{h'(t)}{h(t)} \quad (64)$$

$$\frac{a(b-1)t^{a-1}}{(1-t^{a})}=\frac{(a-1)}{t}-\frac{h'(t)}{h(t)}-h(t) \quad (65)$$

$$\left(\frac{a(b-1)t^{a-1}}{1-t^{a}}\right)^{2}=\left(\frac{a-1}{t}-\frac{h'(t)}{h(t)}-h(t)\right)^{2} \quad (66)$$

$$\frac{(b-1)(at^{a-1})^{2}}{(1-t^{a})^{2}}=\frac{1}{b-1}\left(\frac{a-1}{t}-\frac{h'(t)}{h(t)}-h(t)\right)^{2} \quad (67)$$

$$\frac{a(a-1)(b-1)t^{a-1}}{(1-t^{a})}=a-1\left(\frac{a-1}{t}-\frac{h'(t)}{h(t)}-h(t)\right) \quad (68)$$

$$\frac{a(a-1)(b-1)t^{a-2}}{(1-t^{a})}=\frac{a-1}{t}\left(\frac{a-1}{t}-\frac{h'(t)}{h(t)}-h(t)\right) \quad (69)$$

Substitute equations (64), (67) and (69) into equation (63);

$$h''(t)=-\left\{\begin{array}{l}\frac{(a-1)}{t^{2}}+\frac{1}{b-1}\left(\frac{a-1}{t}-\frac{h'(t)}{h(t)}-h(t)\right)^{2}\\+\frac{a-1}{t}\left(\frac{a-1}{t}-\frac{h'(t)}{h(t)}-h(t)\right)+h'(t)\end{array}\right\}h(t)+\frac{h'^{2}(t)}{h(t)} \quad (70)$$

The necessary condition for the existence of equation is that $a>0,0<t<1,b>1$.

VII. Reversed Hazard Function

The reversed hazard function of the Kumaraswamy distribution is given as;

$$j(t)=\frac{abt^{a-1}}{1-t^{a}} \quad (71)$$

To obtain the first order ordinary differential equation for the reversed hazard function of the Kumaraswamy distribution, differentiate equation (71), to obtain;

$$j'(t)=\left\{\frac{(a-1)t^{a-1}}{t^{a-2}}+\frac{at^{a-1}(1-t^{a})^{-2}}{(1-t^{a})^{-1}}\right\}j(t) \quad (72)$$

The necessary condition for the existence of equation is that $a,b>0, 0<t<1$.

$$j'(t)=\left(\frac{a-1}{t}+\frac{at^{a-1}}{1-t^{a}}\right)j(t) \tag{73}$$

$$j'(t)=\left(\frac{a-1}{t}+\frac{h(t)}{b}\right)j(t) \tag{74}$$

The first order ordinary differential equation for the reversed Hazard function of the Kumaraswamy distribution is given by;

$$btj'(t)-tj^{2}(t)-(a-1)bj(t)=0 \tag{75}$$

$$j(0.1)=\frac{10ab(0.1)^{a}}{(1-(0.1)^{a})} \tag{76}$$

VIII. Concluding Remarks

Ordinary differential equations (ODEs) has been obtained for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Kumaraswamy distribution. This differential calculus and efficient algebraic simplifications were used to derive the various classes of the ODEs. The parameter and the supports that characterize the Kumaraswamy distribution determine the nature, existence, orientation and uniqueness of the ODEs. The results are in agreement with those available in scientific literature. Furthermore several methods can be used to obtain desirable solutions to the ODEs [77-86]. This method of characterizing distributions cannot be applied to distributions whose PDF or CDF are either not differentiable or the domain of the support of the distribution contains singular points.

Acknowledgment

The comments of the reviewers were very helpful and led to an improvement of the paper. This research benefited from sponsorship from the Statistics sub-cluster of the *Industrial Mathematics Research Group* (TIMREG) of Covenant University and *Centre for Research, Innovation and Discovery* (CUCRID), Covenant University, Ota, Nigeria.

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