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Classes of Ordinary Differential Equations Obtained for the Probability Functions of Lévy Distribution

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***Abstract*— Lévy distribution is one of few stable distributions. The absence of closed form of some of the probability functions of distribution has inspired researchers into finding alternate options such as approximations. In this paper, homogenous ordinary differential equations (ODES) of different orders were obtained for the probability density function, survival function, hazard function and reversed hazard function of Lévy distribution. This is possible since the aforementioned probability functions are differentiable. However, approximation remains the only option for the quantile function and inverse survival function of the distribution. This is because those functions may not be reduced to an ODE as a result of the intractable nature of the cumulative distribution function which is used in obtaining them. . Differentiation and modified product rule were used to obtain the required ordinary differential equations, whose solutions are the respective probability functions. The different conditions necessary for the existence of the ODEs were obtained and it is in consistent with the support that defined the various probability functions considered. The parameters that defined each distribution greatly affect the nature of the ODEs obtained. This method provides new ways of classifying and approximating other probability distributions apart from one considered in this research. Algorithms for implementation can be helpful in improving the results.**

***Index Terms*— Differentiation, product rule, quantile function, survival function, approximation, hazard function, Lévy.**

I. Introduction

DIFFERENT mathematical techniques are viable tools in statistics. In mathematical statistics, different mathematical areas are used heavily in better understanding of probability distributions. Some of these are calculus, differential equations, algebra, measure theory, fixed point and topology and so on. Hitherto most of the use of ordinary differential equation (ODE) is often in mode and parameter estimation and approximation. Approximation of quantile function features prominently in the use of ODE in approximation. However, the use is often restricted to distributions with intractable probability density function (PDF) and/ or cumulative distribution (CDF). This is due to the inability of the inversion method or rejection sampling method to recover the quantile function from the CDF [1-10].

Few available literatures have considered the study of the ODE of different probability functions of Lévy distribution in particular and probability distributions in general. The available ones contain previous works done on the ODE of the following distributions: beta distribution [11], Lomax distribution [12], beta prime distribution [13], Laplace distribution [14] and raised cosine distribution [15].

The aim of this research is to develop homogenous ordinary differential equations for the probability density function (PDF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of Lévy distribution. The ODE for the quantile function and inverse survival function (ISF) of the distribution are complex and not included in the paper. This will also help to provide the answers as to whether there are discrepancies between the support of the distribution and the necessary conditions for the existence of the ODEs. Similar results for other distributions have been proposed for the following probability distributions: Fréchet Distribution [16], exponentiated generalized Exponential distribution [17], Cauchy, standard Cauchy and Log-Cauchy distributions [18], Burr XII and Pareto distributions [19], Gompertz and gamma Gompertz distributions [20], 3-parameter Weibull distribution [21], exponentiated Fréchet distribution [22], half-Cauchy and power Cauchy distributions [23], exponential and truncated exponential distributions [24], exponentiated Pareto distribution [25], Gumbel distribution [26], half-Normal distribution [27], Harris extended exponential distribution [28] and Weibull distribution [29].

Lévy distribution is a continuous probability distribution named after a mathematician called Paul Lévy. The distribution is stable and infinitely divisible. The distribution is a special case of the inverse gamma and type 5 Pearson distributions. The distribution is also related to the normal, scaled inverse-Chi-squared and folded normal distributions. The distribution has been applied in the following areas: spectroscopy, modeling change in a planet's magnetic field, a limiting probability of hitting times in Brownian motion, time series analysis, modeling the change of position of photon in turbid medium and in Cauchy process. Lévy distribution is mostly applied in stock analysis, income distribution [30], extreme values and time series analysis of rainfall [31], molecular physics [32] and

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b-spline curve estimation [33]. The applications are often limited because of absence of closed form of the probability functions of the distribution in particular and stable distributions in general. The nature of the distribution makes it hard for combining, modification, exponentiation, compounding, transmutation, generation and so on. It is widely believed to be a special case or sub model of the inverse gamma distribution.. The behavior of the product and quotient of two Lévy random variables was studied by [34].

The ordinary differential calculus was used to obtain the results presented in different sections.

II. Probability Density Function

The probability density function of the Lévy distribution is given by;

$$f(x)=\sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(x-\mu)}}}{(x-\mu)^{3/2}} \qquad x\in(\mu,\infty) \tag{1}$$

To obtain the first order ordinary differential equation for the probability density function of the Lévy distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\frac{c}{2(x-\mu)^2}-\frac{3}{(x-\mu)}\right\}f(x) \\ =\left[\frac{c-3(x-\mu)}{2(x-\mu)^2}\right]f(x) \tag{2}$$

The condition necessary for the existence of equation is $c>0, x\neq\mu$.

The first order ordinary differential for the probability density function of the Lévy distribution is given as;

$$2(x-\mu)^2 f'(x)-(c-3x+3\mu)f(x)=0 \tag{3}$$

$$f(\mu+1)=\sqrt{\frac{c}{2\pi}}\left(e^{-\frac{c}{2}}\right) \tag{4}$$

Consider some special cases;

Case 1: When $\mu=0$,equation (3) becomes;

$$2x^2 f'(x)-(c-3x)f(x)=0 \tag{5}$$

Case 2: When $\mu=0$ and $c=1/\sigma^2$

$$2\sigma^2x^2 f'(x)-(1-3\sigma^2x)f(x)=0 \tag{6}$$

Case 3: When $\mu=\frac{1}{2}$ and $c=\frac{c}{2}$

$$(2x-1)^2 f'(x)-(c-6x+3)f(x)=0 \tag{7}$$

To obtain the second order ordinary differential equation for the probability density function of the Lévy distribution, differentiate equation (2), to obtain;

$$f''(x)=\frac{(c-3x+3\mu)f'(x)}{2(x-\mu)^2}-\frac{cf(x)}{(x-\mu)^3} \tag{8}$$

The condition necessary for the existence of equation is $c>0, x\neq\mu$.

$$f''(x)=\frac{(x-\mu)(c-3x+3\mu)f'(x)-2cf(x)}{2(x-\mu)^3} \tag{9}$$

$$2(x-\mu)^3 f''(x)-(x-\mu)(c-3x+3\mu)f'(x) \\ +2cf(x)=0 \tag{10}$$

$$f'(\mu+1)=(c-3)\sqrt{\frac{c}{2\pi}}\left(e^{-\frac{c}{2}}\right) \tag{11}$$

Consider some special cases;

Case 1: When $\mu=0$, equation (10) becomes;

$$2x^3 f''(x)-(cx-3x^2)f'(x)+2cf(x)=0 \tag{12}$$

Case 2: When $\mu=0$ and $c=1/\sigma^2$

$$2\sigma^2x^3 f''(x)-(x-3\sigma^2x^2)f'(x)+2f(x)=0 \tag{13}$$

Case 3: When $\mu=\frac{1}{2}$ and $c=\frac{c}{2}$

$$(2x-1)^3 f''(x)-(2x-1)(c-6x+3)f'(x)+4f(x)=0 \tag{14}$$

Alternatively, the second order ordinary differential equation for the probability density function of the Lévy distribution can be derived in this way.

Equation (2) can be written as;

$$\frac{f'(x)}{f(x)}=\left[\frac{c-3x+3\mu)}{2(x-\mu)^2}\right] \tag{15}$$

Substitute equation (15) into equation (8);

$$f''(x)=\frac{(f'(x))^2}{f(x)}-\frac{cf(x)}{(x-\mu)^3} \tag{16}$$

$$f''(x)=\frac{(x-\mu)^3 f'^2(x)-cf^2(x)}{(x-\mu)^3 f(x)} \tag{17}$$

$$(x-\mu)^3 f(x)f''(x)-(x-\mu)^3 f'^2(x)+cf^2(x)=0 \tag{18}$$

To obtain the third order ordinary differential equation for the probability density function of the Lévy distribution, differentiate equation (8), to obtain;

$$f'''(x)=\frac{(c-3x+3\mu)f''(x)}{2(x-\mu)^2}-\frac{2(c-3x+3\mu)f'(x)}{2(x-\mu)^3} \\ -\frac{3f'(x)}{(x-\mu)^2}+\frac{3cf(x)}{(x-\mu)^4}-\frac{cf'(x)}{(x-\mu)^3} \tag{19}$$

The condition necessary for the existence of equation is $c>0, x\neq\mu$.

$$2(x-\mu)^4 f'''(x)-(x-\mu)^2(c-3x+3\mu)f''(x) \\ +2((x-\mu)(c-3x+3\mu)+3(x-\mu)^2 \\ +c(x-\mu))f'(x)-6cf(x)=0 \tag{20}$$

$$f''(\mu+1)=\sqrt{\frac{c}{2\pi}}\left(e^{-\frac{c}{2}}\right)\left(\frac{(c-3)^2-2c}{2}\right) \tag{21}$$

Consider some special cases;

Case 1: When $\mu=0$, equation (20) becomes;

$$2x^4 f'''(x) - x^2(c-3x) f''(x) + 4cxf'(x) - 6cf(x) = 0 \quad (22)$$

Case 2: When $\mu = 0$ and $c = 1/\sigma^2$

$$2\sigma^2 x^4 f'''(x) - (x^2 - 3\sigma^2 x^3) f''(x) + 4xf'(x) - 6f(x) = 0 \quad (23)$$

Case 3: When $\mu = \frac{1}{2}$ and $c = \frac{c}{2}$

$$(2x-1)^4 f'''(x) - (2x-1)^2(c-6x+3) f''(x) + 8c(2x-1) f'(x) - 24cf(x) = 0 \quad (24)$$

Alternatively, the third order ordinary differential equation for the probability density function of the Lévy distribution can be derived in this way.
Substitute equation (15) into equation (19);

$$f'''(x) = \frac{f'(x)f''(x)}{f(x)} - \frac{2f'^2(x)}{f(x)} - \frac{3f'(x)}{(x-\mu)^2} + \frac{3cf(x)}{(x-\mu)^4} - \frac{cf'(x)}{(x-\mu)^3} \quad (25)$$

$$(x-\mu)^4 f(x) f'''(x) - (x-\mu)^4 f'(x) f''(x) + 2(x-\mu)^4 f'^2(x) + (3(x-\mu)^2 + c(x-\mu)) f(x) f'(x) - 3cf^2(x) = 0 \quad (26)$$

When $\mu = 0$, equation (26) becomes;

$$x^4 f(x) f'''(x) - x^4 f'(x) f''(x) + 2x^4 f'^2(x) + x(3x+c) f(x) f'(x) - 3cf^2(x) = 0 \quad (27)$$

The Quantile function of the Lévy distribution may not be reduced to ordinary differential equations.

III. Survival Function

The Survival function of the Lévy distribution is given by;

$$S(t) = 1 - erfc\left(\sqrt{\frac{c}{2(t-\mu)}}\right)$$

(28) To obtain the first order ordinary differential equation for the probability density function of the Lévy distribution, differentiate equation (28), to obtain;

$$S'(t) = -\sqrt{\frac{c}{2\pi}} \frac{e^{-\frac{c}{2(t-\mu)}}}{(t-\mu)^{3/2}} = -f(t) \quad (29)$$

The condition necessary for the existence of equation is $c > 0, t \neq \mu$.

The first order ordinary differential for the Survival function of the Lévy distribution is given as;

$$S'(t) + f(t) = 0 \quad (30)$$

$$S(\mu+1) = 1 - erfc\left(\sqrt{\frac{c}{2}}\right) \quad (31)$$

Consequently higher order ordinary differential equations can be derived;

$$S''(t) = -f'(t) \quad (32)$$

Substitute equations (29) and (32) into equation (3);

$$2(t-\mu)^2 S''(t) - (c-3t+3\mu) S'(t) = 0 \quad (33)$$

$$S'(\mu+1) = -\sqrt{\frac{c}{2\pi}}\left(e^{-\frac{c}{2}}\right) \quad (34)$$

Consider some special cases;
Case 1: When $\mu = 0$,equation (33) becomes;

$$2t^2 S''(t) - (c-3t) S'(t) = 0 \quad (35)$$

Case 2: When $\mu = 0$ and $c = 1/\sigma^2$

$$2\sigma^2 t^2 S''(t) - (1-3\sigma^2 t) S'(t) = 0 \quad (36)$$

Case 3: When $\mu = \frac{1}{2}$ and $c = \frac{c}{2}$

$$(2t-1)^2 S''(t) - (c-6t+3) S'(t) = 0 \quad (37)$$

$$S'''(t) = -f''(t) \quad (38)$$

Substitute equations (29), (32) and (38) into equation (10);

$$2(t-\mu)^3 S'''(t) - (t-\mu)(c-3t+3\mu) S''(t) + 2cS'(t) = 0 \quad (39)$$

$$S''(\mu+1) = -(c-3)\sqrt{\frac{c}{2\pi}}\left(e^{-\frac{c}{2}}\right) \quad (40)$$

Consider some special cases;
Case 1: When $\mu = 0$, equation (39) becomes;

$$2t^3 S'''(t) - t(c-3t) S''(t) + 2cS'(t) = 0 \quad (41)$$

Case 2: When $\mu = 0$ and $c = 1/\sigma^2$

$$2\sigma^2 t^3 S'''(t) - (t - 3\sigma^2 t^2) S''(t) + 2S'(t) = 0 \quad (42)$$

Case 3: When $\mu = \frac{1}{2}$ and $c = \frac{c}{2}$

$$(2t-1)^3 S'''(t) - (2t-1)(c-6t+3) S''(t) + 4S'(t) = 0 \quad (43)$$

Alternatively, an ordinary differential equation of the third order of the Survival function of the Lévy distribution can be obtained by substituting equations (29), (32) and (38) into equation (18);

$$(t-\mu)^3 S'(t) S'''(t) - (t-\mu)^3 S''^2(t) + cS'^2(t) = 0 \quad (44)$$

Case 1: When $\mu = 0$, equation (44) becomes;

$$t^3 S'(t) S'''(t) - t^3 S''^2(t) + cS'^2(t) = 0 \quad (45)$$

Case 2: When $\mu = 0$ and $c = 1/\sigma^2$

$$\sigma^2 t^3 S'(t) S'''(t) - \sigma^2 t^3 S''^2(t) + S'^2(t) = 0 \quad (46)$$

Case 3: When $\mu = \frac{1}{2}$ and $c = \frac{c}{2}$

$$(2t-1)^3 S'(t) S'''(t) - (2t-1)^3 S''^2(t) + 4cS'^2(t) = 0 \quad (47)$$

IV. Hazard Function

The Hazard function of the Lévy distribution is given by;

$$h(t)=\frac{\sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(t-\mu)}}}{(t-\mu)^{3/2}}}{1-erfc\left(\sqrt{\frac{c}{2(t-\mu)}}\right)} \quad (48)$$

$$=\sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(t-\mu)}}}{(t-\mu)^{3/2}}\left(1-erfc\left(\sqrt{\frac{c}{2(t-\mu)}}\right)\right)^{-1}$$

To obtain the first order ordinary differential equation for the Hazard function of the Lévy distribution, differentiate equation (48), to obtain;

$$h'(t)=\left[\frac{c-3(t-\mu)}{2(t-\mu)^2}+h(t)\right]h(t) \quad (49)$$

The condition necessary for the existence of equation is $c>0, t\neq\mu$.

The first order ordinary differential for the Hazard function of the Lévy distribution is given as;

$$2(t-\mu)^2h'(t)-(c-3t+3\mu)h(t)-2(t-\mu)^2h^2(t)=0 \quad (50)$$

$$h(\mu+1)=\frac{\sqrt{\frac{c}{2\pi}}e^{-\frac{c}{2}}}{1-erfc\left(\sqrt{\frac{c}{2}}\right)} \quad (51)$$

Case 1: When $\mu=0$, equation (50) becomes;

$$2t^2h'(t)-(c-3t)h(t)+2t^2h^2(t)=0 \quad (52)$$

Case 2: When $\mu=0$ and $c=1/\sigma^2$

$$2\sigma^2t^2h'(t)-(1-3\sigma^2t)h(t)+2\sigma^2t^2h^2(t)=0 \quad (53)$$

Case 3: When $\mu=\frac{1}{2}$ and $c=\frac{c}{2}$

$$(2t-1)^2h'(t)-(c-6t+3)h(t)+(2t-1)^2h^2(t)=0 \quad (54)$$

The second order ordinary differential for the Hazard function of the Lévy distribution is given as;

$$2(t-\mu)^2h''(t)-(c-7t+7\mu)h'(t)$$
$$+4(t-\mu)^2h(t)h'(t)+4(t-\mu)h^2(t)+3h(t)=0 \quad (55)$$

(56)

Case 1: When $\mu=0$, equation (55) becomes;

$$2t^2h''(t)-(c-7t)h'(t)+4t^2h(t)h'(t)$$
$$+4th^2(t)+3h(t)=0 \quad (57)$$

Case 2: When $\mu=0$ and $c=1/\sigma^2$

$$2\sigma^2t^2h''(t)-(1-7\sigma^2t)h'(t)+4\sigma^2t^2h(t)h'(t)$$
$$+4\sigma^2th^2(t)+3\sigma^2h(t)=0 \quad (58)$$

Case 3: When $\mu=\frac{1}{2}$ and $c=\frac{c}{2}$

$$(2t-1)^2h''(t)-(c-14t+7)h'(t)$$
$$+2(2t-1)^2h(t)h'(t)+4(2t-1)h^2(t)+6h(t)=0 \quad (59)$$

V. Reversed Hazard Function

The reversed Hazard function of the Lévy distribution is given by;

$$j(t)=\frac{\sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(t-\mu)}}}{(t-\mu)^{3/2}}}{erfc\left(\sqrt{\frac{c}{2(t-\mu)}}\right)} \quad (60)$$

$$=\sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(t-\mu)}}}{(t-\mu)^{3/2}}\left(erfc\left(\sqrt{\frac{c}{2(t-\mu)}}\right)\right)^{-1}$$

To obtain the first order ordinary differential equation for the reversed Hazard function of the Lévy distribution, differentiate equation (60), to obtain;

$$j'(t)=\left[\frac{c-3(t-\mu)}{2(t-\mu)^2}-j(t)\right]j(t) \quad (61)$$

The condition necessary for the existence of equation is $c>0, t\neq\mu$.

The first order ordinary differential for the reversed Hazard function of the Lévy distribution is given as;

$$2(t-\mu)^2j'(t)+2(t-\mu)^2j^2(t)-(c-3t+3\mu)j(t)=0 \quad (62)$$

$$j(\mu+1)=\frac{\sqrt{\frac{c}{2\pi}}e^{-\frac{c}{2}}}{erfc\left(\sqrt{\frac{c}{2}}\right)} \quad (63)$$

VI. Concluding Remarks

Differentiation and modified product rule were used to obtain the ordinary differential equations (ODES) of different orders for the probability density function, survival function, hazard function and reversed hazard function of Lévy distribution. This was largely due to differentiability of the probability functions. However the case of the quantile function and inverse survival function of the distribution were not included in this paper because of the complexity of the resulting ODEs obtained from the two functions.. This is because those functions may not be reduced to an ODE as a result of the intractable nature of the cumulative distribution function. Every changes in the parameters result to a unique ODE. Overall, the ODEs are in consistent with the support and parameter domains that characterize the Lévy distribution. In addition, several methods can be used to obtain the solutions of the ODEs [35-41].

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