Discussion for Specially Quasi Alpha-diagonally Dominant Matrices and Its Application

Guichun Han, Huishuang Gao, and Chunsheng Zhang

***Abstract*—In this paper, several new subclasses of nonsingular *H*-matrices are defined, the concepts of quasi α -diagonally dominant matrices are introduced, and two equivalent conditions of strictly specially quasi α_2 -diagonally dominant matrices are given. By those theorems, some practical criteria for nonsingular *H*-matrices are obtained and the obtained result is introduced into identifying the stability of neural networks. In the end, effectiveness of the results is illustrated by numerical example.**

***Index Terms*—Dirichlet problem, Approximate solution, H-matrices, quasi α -diagonally dominant matrices, stability of neural networks**

I. Introduction

Nonsingular *H*-matrices are special class matrices with a vital role in many fields such as computational mathematics, mathematic physics, stability of control systems, and so on. The study on nonsingular *H*-matrices has been a hot issue and in recent years, there have been some new results (see [6-10]).

As we all known, many practical problems can be summarized to the solutions of large linear equations with special coefficient matrices. And the solution of linear equations that we use mostly is classical iterative methods including Jacobi-type iterative method, Gauss-Seidel-type iterative method, SOR-type iterative method, and so on. For large linear equations $AX = b$, when its coefficient matrix *A* is a nonsingular *H*-matrix, many classical iterative methods are convergent. For example (see [11]), in the solving Dirichlet problem on the unite square, we need to find the approximate solution of the function $u(x, y)$ defined on the unite square, satisfying Laplase equation.

Manuscript received January 9, 2018; revised February 3, 2018. This work was supported in part the National Natural Science Foundation under Grant No. 81460656 of China, by Natural Science Foundation of Inner Mongolia Autonomous Region under Grant No. 2016MS0118 of China, and by the Science Research Funding of the Inner Mongolia University for the Nationalities under Grant No. NMDYB1778 of China.

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$$\frac{\partial^2 u(x,y)}{\partial x} + \frac{\partial^2 u(x,y)}{\partial y} = u_{xx}(x,y) + u_{yy}(x,y) = 0$$

$$(0 < x, y < 1),$$

with boundary value condition $u(x, y) = g(x, y)$, $(x, y) \in \Gamma$, where Γ denoted square boundary, and $g(x, y)$ defined special function in Γ.

By uniform grid mesh and Taylor expansion, we can obtain the following equations

$$\begin{cases} \omega_1 = \frac{1}{4}(\omega_3 + \omega_4 + g_1 + g_{11}) \\ \omega_2 = \frac{1}{4}(\omega_3 + \omega_4 + g_5 + g_7) \\ \omega_3 = \frac{1}{4}(\omega_1 + \omega_2 + g_2 + g_4) \\ \omega_4 = \frac{1}{4}(\omega_1 + \omega_2 + g_8 + g_{10}) \end{cases}$$

where ω_i are approximate values for u_i ($i = 1, 2, 3, 4$), respectively.

Therefore, the definite solution of differential equations is transformed into the solving linear equations, and written in matrix form:

$$AW = K,$$

where

$$A = \begin{bmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{bmatrix},$$

$$W = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix}, \quad k = \begin{bmatrix} g_1 + g_{11} \\ g_5 + g_7 \\ g_2 + g_4 \\ g_8 + g_{10} \end{bmatrix}$$

From the above example, it is obvious to have the coefficient matrix *A* is a nonsingular *H*-matrix.

Let $N = \{1, 2, \cdots, n\}$, $C^{n,n}$ denote the set of all *n* by *n* complex matrices and $A = (a_{ij}) \in C^{n,n}$, $\forall i \in N$,

$$R_i(A)=\sum_{j\neq i}|a_{ij}|,\ C_i(A)=\sum_{j\neq i}|a_{ji}|,$$

$$P_i(A)=\sum_{k\in N\setminus\{i\}}^{n}|a_{ik}|\frac{R_k(A)}{|a_{kk}|},\Lambda_i(A)=\sum_{k\in N\setminus\{i\}}^{n}|a_{ki}|\frac{R_i(A)}{|a_{ii}|}.$$

$A=(a_{ij})\in C^{n,n}$ is a diagonally dominant matrix (by rows) iff $|a_{ii}|>R_i(A)$ ($\forall i\in N$) and is denoted by $A\in D$. $A=(a_{ij})\in C^{n,n}$ is a generalized strictly diagonally dominant matrix (by rows) if there exists positive numbers $x_1,\cdots,x_n$ such that

$$x_i|a_{ii}|>\sum_{j\neq i}x_j|a_{ij}|\ (\forall i\in N),$$

i.e., there exists an positive diagonal matrix $X=diag(x_1,\cdots,x_n)$ such that $AX\in D$, and denoted by $A\in\tilde{D}$. Well known characterization of nonsingular *H*-matrices is given by the matrix is a nonsingular *H*-matrix if and only if $A\in\tilde{D}$ (see [1] and [2]). So, we always assume that all diagonal entries of *A* are nonzero and $R_i(A)>0\ (\forall i\in N)$.

II. STRICTLY QUASI α (ALPHA)-MATRICES

α_1-matrices and α_2-matrices introduced by Ostrowski, are both generalizations of strictly diagonally dominant matrix property, and are both subclasses of nonsingular *H*-matrices.

Definition 2.1.[3] A matrix $A=(a_{ij})\in C^{n,n}$ is called an α_1-matrix if there exists $\alpha\in[0,1]$, such that

$$|a_{ii}|>\alpha R_i(A)+(1-\alpha)C_i(A)\ (\forall i\in N).$$

Definition 2.2.[3] A matrix $A=(a_{ij})\in C^{n,n}$ is called an α_2-matrix if there exists some $\alpha\in[0,1]$, such that

$$|a_{ii}|>[R_i(A)]^{\alpha}[C_i(A)]^{1-\alpha}\ \ (\forall i\in N).$$

As we have mentioned above, the following nonsingular result is valid.

Lemma 2.1.[3] If a matrix $A=(a_{ij})\in C^{n,n}$ is an α_1 or α_2-matrix, then *A* is nonsingular, moreover it is an *H*-matrix.

Definition 2.3. A matrix $A=(a_{ij})\in C^{n,n}$ is called a specially quasi α_1-matrix, if there exists some $\alpha\in[0,1]$, such that

$$R_i(A)>\alpha P_i(A)+(1-\alpha)\Lambda_i(A)\ \ (\forall i\in N).$$

Definition 2.4. A matrix $A=(a_{ij})\in C^{n,n}$ is called a specially quasi α_2-matrix if there exists some $\alpha\in[0,1]$, such that

$$R_i(A)>[P_i(A)]^{\alpha}[\Lambda_i(A)]^{1-\alpha}\ \ (\forall i\in N).$$

Lemma 2.2.[4] Let $A=(a_{ij})\in C^{n,n}$, if there exists a positive diagonal matrix $X=diag(x_1,\cdots,x_n)$, such that AX is a nonsingular *H*-matrix, then *A* is a nonsingular *H*-matrix.

Lemma 2.3.[5] Let σ and τ are two nonnegative real numbers, then for $\forall\alpha\in[0,1]$, we have

$$\alpha\tau+(1-\alpha)\sigma\geq\tau^{\alpha}\sigma^{1-\alpha},$$

and $\alpha\tau+(1-\alpha)\sigma=\tau^{\alpha}\sigma^{1-\alpha}$ if and only if $\tau=\sigma$.

III. EQUAIVALENT REPRESN OF STRICTLY QUASI α_1-DIAGONALL DOMINANT MATRICES

From now on, we will use the following notations.

$$N_1=\{i\in N\,|\,P_i(A)<R_i(A)<\Lambda_i(A)\};$$
$$N_2=\{i\in N\,|\,\Lambda_i(A)<R_i(A)<P_i(A)\};$$
$$N_3=\{i\in N\,|\,R_i(A)\geq\Lambda_i(A)>P_i(A)\};$$
$$N_4=\{i\in N\,|\,R_i(A)\geq P_i(A)>\Lambda_i(A)\};$$
$$N_5=\{i\in N\,|\,R_i(A)>\Lambda_i(A)=P_i(A)\};$$
$$N_0=\{i\in N\,|\,P_i(A)\geq R_i(A),\Lambda_i(A)\geq R_i(A)\}.$$

Theorem 3.1. Let $A=(a_{ij})\in C^{n,n}$, then A is a specially quasi α_1-matrix if and only if $N_0=\varnothing$, and for $\forall i\in N_1,\forall j\in N_2$, such

$$\max_{i\in N_1}\frac{\Lambda_i(A)-R_i(A)}{\Lambda_i(A)-P_i(A)}<\min_{j\in N_2}\frac{R_j(A)-\Lambda_j(A)}{P_j(A)-\Lambda_j(A)}\qquad(1)$$

Proof. (**Sufficiency**) For $\forall i\in N_1,\forall j\in N_2$, we get

$$0<\frac{\Lambda_i(A)-R_i(A)}{\Lambda_i(A)-P_i(A)}<1;$$

and

$$0<\frac{R_j(A)-\Lambda_j(A)}{P_j(A)-\Lambda_j(A)}<1.$$

From condition (1) and the above two inequalities, there exists some $\alpha\in[0,1]$, such that

$$0<\max_{i\in N_1}\frac{\Lambda_i(A)-R_i(A)}{\Lambda_i(A)-P_i(A)}<\alpha<\min_{j\in N_2}\frac{R_j(A)-\Lambda_j(A)}{P_j(A)-\Lambda_j(A)}<1.\qquad(2)$$

For $\forall i\in N_1$, from $\max_{i\in N_1}\frac{\Lambda_i(A)-R_i(A)}{\Lambda_i(A)-P_i(A)}<\alpha$ of (2), we obtain

$$\Lambda_i(A)-R_i(A)<\alpha\Lambda_i(A)-\alpha P_i(A),$$

i.e.

$$R_j(A)>\alpha P_j(A)+(1-\alpha)\Lambda_j(A).$$

For $\forall j\in N_2$, from $\alpha<\min_{j\in N_2}\frac{R_j(A)-\Lambda_j(A)}{P_j(A)-\Lambda_j(A)}$ of (2), we obtain

$$\alpha P_j(A)-\alpha\Lambda_j(A)<R_j(A)-\Lambda_j(A),$$

i.e.

$$R_j(A)>\alpha P_j(A)+(1-\alpha)\Lambda_j(A).$$

Moreover, for $\forall k\in N_3\cup N_4\cup N_5$, and for $\forall\alpha\in[0,1]$, it is obvious that

$$R_k(A)>\alpha P_k(A)+(1-\alpha)\Lambda_k(A).$$

ISBN: 978-988-14047-8-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Combining the discussion above with condition that $N_0=\varnothing$, for $\forall i\in N_1\bigcup N_2\bigcup N_3\bigcup N_4\bigcup N_5=N$, we have that there exists some $\alpha\in[0,1]$, such that

$$R_i(A)>\alpha P_i(A)+(1-\alpha)\Lambda_i(A).$$

Therefore by definition 1, we have A is a specially quasi α_1 -matrix.

(**Necessity**) Suppose A is a specially quasi α_1 -matrix, then $N_0=\varnothing$ and there exists some $\alpha\in[0,1]$, such that

$$R_i(A)\ge\alpha P_i(A)+(1-\alpha)\Lambda_i(A)\quad(\forall i\in N).$$

i.e.

$$\frac{\Lambda_i(A)-R_i(A)}{\Lambda_i(A)-P_i(A)}<\alpha.$$

These show inequality (1) holds. □

The row and column of the matrix are of the same property, then we have the similar result with Theorem 3.1.

Theorem 3.2. Let $A=(a_{ij})\in C^{n,n}$, then A is a specially quasi α_1 -matrix if and only if $N_0=\varnothing$, and for $\forall i\in N_1,\forall j\in N_2$, such

$$\max_{j\in N_2}\frac{P_j(A)-R_j(A)}{P_j(A)-\Lambda_j(A)}<\min_{i\in N_1}\frac{R_i(A)-P_i(A)}{\Lambda_i(A)-P_i(A)}\quad(3)$$

As its application, some new practical criteria for nonsingular *H*-matrices are obtained.

Theorem 3.3. Let $A=(a_{ij})\in C^{n,n}$, $N_0=\varnothing$, and for $\forall i\in N_1$, $\forall j\in N_2$, such that

$$\frac{\Lambda_i(A)-R_i(A)}{\Lambda_i(A)-P_i(A)}\le\frac{R_j(A)-\Lambda_j(A)}{P_j(A)-\Lambda_j(A)},$$

then A is a nonsingular *H*-matrix.

Proof. Let $X=diag(x_1,\cdots,x_n)$, where $x_i=\frac{R_i(A)}{|a_{ii}|}>0\ (\forall i\in N)$. For $\forall i\in N$, we have

$$|(AX)_{ii}|=|a_{ii}|\frac{R_i(A)}{|a_{ii}|}=R_i(A);$$

$$R_i(AX)=\sum_{k\in N\backslash\{i\}}^{n}|a_{ik}|\frac{R_k(A)}{|a_{kk}|}=P_i(A);$$

$$C_i(AX)=\sum_{k\in N\backslash\{i\}}^{n}|a_{ki}|\frac{R_i(A)}{|a_{ii}|}=\Lambda_i(A).$$

For $\forall i\in N_1$, $\forall j\in N_2$, similar discussion as in the sufficiency proof of Theorem 3.1, there exists some $\alpha\in[0,1]$, such that

$$R_i(A)\ge\alpha P_i(A)+(1-\alpha)\Lambda_i(A),$$

and

$$R_j(A)\ge\alpha P_j(A)+(1-\alpha)\Lambda_j(A).$$

On the base of Lemma 2.3, the above inequalities can be expressed as

$$|(AX)_{ii}|\ge\alpha R_i(AX)+(1-\alpha)C_i(AX)$$
$$>[R_i(AX)]^{\alpha}[C_i(AX)]^{(1-\alpha)},$$

and

$$|(AX)_{jj}|\ge\alpha R_j(AX)+(1-\alpha)C_j(AX)$$
$$>[R_j(AX)]^{\alpha}[C_j(AX)]^{(1-\alpha)}.$$

For $\forall k\in N_3\bigcup N_4\bigcup N_5$, and $\forall\alpha\in[0,1]$, it is obvious that

$$R_k(A)>\alpha P_k(A)+(1-\alpha)\Lambda_k(A),$$

i.e.,

$$|(AX)_{kk}|>\alpha R_k(AX)+(1-\alpha)C_k(AX)$$
$$\ge[R_k(AX)]^{\alpha}[C_k(AX)]^{(1-\alpha)}.$$

In a word, we can conclude that A is a specially quasi α_1 - matrix. By Lemma 2.1, AX is a nonsingular *H*-matrix, and then by Lemma 2.2, A is a nonsingular *H*-matrix. □

Theorem 3.4. Let $A=(a_{ij})\in C^{n,n}$, $N_0=\varnothing$, and for $\forall i\in N_1$, $\forall j\in N_2$, such that

$$\frac{R_i(A)-P_i(A)}{\Lambda_i(A)-P_i(A)}\ge\frac{P_j(A)-R_j(A)}{P_j(A)-\Lambda_j(A)},$$

then A is a nonsingular *H*-matrix.

Proof. With the similar discussion as in the proof of Theorem 3.3, the result is obtained. □

IV. Numerical Example

Example 4.1. Let

$$A=\begin{bmatrix}1&-0.3&0.8\\-0.4&1&0\\0.7&-0.3&1\end{bmatrix}.$$

Then we have

$$R_1(A)=1.1,\ R_2(A)=0.4,\ R_3(A)=1;$$
$$C_1(A)=1.1,\ C_2(A)=0.6,\ C_3(A)=0.8;$$
$$|a_{11}|=1;\ |a_{22}|=1;\ |a_{33}|=1.$$

But, we notice $|a_{11}|=1<1.1=R_1(A)=C_1(A)$. The condition doesn't satisfy Theorem 2 in [6], so we can't obtain the conclusion that A is a nonsingular *H*-matrix.

Nevertheless, by this paper, let $X=\mathrm{diag}(1.1,0.4,1)$, then according to notations of this paper, we have

$$P_1(A)=0.92,P_2(A)=0.44,\ P_3(A)=0.89;$$
$$\Lambda_1(A)=1.21,\Lambda_2(A)=0.24,\ \Lambda_3(A)=0.8.$$

By calculation, we obtain

$$\frac{\Lambda_1(A)-R_1(A)}{\Lambda_1(A)-P_1(A)}\approx0.3448<\frac{R_2(A)-\Lambda_2(A)}{P_2(A)-\Lambda_2(A)}=0.8,$$

and

$$\frac{R_1(A)-P_1(A)}{\Lambda_1(A)-P_1(A)}\approx0.6206>\frac{P_2(A)-R_2(A)}{P_2(A)-\Lambda_2(A)}=0.2.$$

It satisfies condition of Theorem 3.2, and then A is a nonsingular *H*-matrix.

We consider the following Hopfield type continuous neural networks:

$$C_i\frac{du_i}{dt}=-\frac{u_i}{R_i}+\sum_{j=1}^{3}T_{ij}g_j(u_j(t-\tau))+I_i\ (i=1,2,3),$$

ISBN: 978-988-14047-8-7
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

where,

$$g_i(u_i) > 0,\ u_i \neq 0,\ 0 < g_i \leq 1,$$

$$g_i(\pm\infty) = \pm 1,\ C_i = 1\ (i = 1,2,3);$$

$$R_1 = \frac{1}{2},\ R_2 = \frac{1}{2},\ R_3 = \frac{1}{2};$$

$$(T_{ij})_{3\times 3} = \begin{bmatrix} 1 & 0.3 & -0.8 \\ 0.4 & 1 & 0 \\ -0.7 & 0.3 & 1 \end{bmatrix}.$$

Notice that $diag(\frac{1}{R_1}, \frac{1}{R_2}, \frac{1}{R_3}) - (|T_{ij}|)_{3\times 3} = A$ is a nonsingular *H*-matrix, and then *A* is a nonsingular *M*-matrix, which ensures existence, uniqueness, and global exponential stability of the equilibrium point of the above neural networks by [12].

V. Conclusion

In conclusion, we define several new subclasses of nonsingular *H*-matrices by the concepts of quasi α -diagonally dominant matrices introduced, and give two equivalent conditions of strictly specially quasi α_2 -diagonally dominant matrices in this paper. In the end, numerical example illustrates effectiveness of the results.

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