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A New Operational Matrix Method for Solving Nonlinear Caputo Fractional Derivative Integro-Differential Static Beam Problems via Chebyshev Polynomials

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Abstract—**A new operational matrix method based on Chebyshev polynomials is developed for obtaining approximate numerical solutions of boundary value problems for non-linear fourth order Caputo fractional derivative integro-differential equations. The new Chebyshev operational matrices are applied to reduce the integro-differential equations for the static beam problem to a system of nonlinear algebraic equations. Examples are given to show the simplicity and accuracy of the proposed method.**

Index Terms—**Caputo fractional integro-differential equation (FIDEs), Chebyshev polynomials, Static beam problem.**

I. Introduction

FRACTIONAL differential equations (FDEs) are generalizations of integer order differential equations to arbitrary (non-integer) orders. They have been the focus of many studies because they can give more realistic models of many physical real world problems, for example, static beam problems [1], fluid dynamics [2], chemical kinetics [3] etc.

Modeling and simulation of systems or processes by using fractional derivatives leads to fractional differential equations (FDEs) or fractional integro-differential equations (FIDEs). These FIDEs are usually difficult to solve analytically and therefore numerical methods are usually required. Many numerical methods have been used to solve FIDEs, for example, the Adomian decomposition method [4], the variational iteration method [5], the homotopy perturbation method [6] and predictor-corrector methods [7]. Further, standard methods of approximating solutions of FIDEs using families of basis functions are also being widely used. The most commonly used sets of approximating functions include block pulse functions [8], Bernstein polynomials [9], Chebyshev polynomials [10], Legendre polynomials [11] and Laguerre polynomials [12].

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The main aim of this work is to find approximate solutions of non-local static beam problems [13] which can be modeled as nonlinear fourth order Caputo fractional derivative integro-differential equations. We propose a new operational matrix method for solving these types of problems based on shifted Chebyshev polynomial of the first kind.

Consider the nonlinear fourth order Caputo fractional integro-differential equation of a static beam in the form :

$$u^{(4\alpha)}(x)-\epsilon u^{(2\alpha)}(x)-\frac{2}{L}\int_0^L \left[u^{(\alpha)}(x)\right]^2 dx u^{(2\alpha)}(x)=f(x) \tag{1}$$

with boundary conditions

$$u(0)=u(L)=u^{(2\alpha)}(0)=u^{(2\alpha)}(L)=0, 0<x<L, \tag{2}$$

where $u(x)$ represents the static deflection of the beam at the point x and ϵ is a positive constant. This equation models the bending equilibrium of an extensible beam (of length L) which is simply supported at $x=0$ and attached to a fixed nonlinear torsional spring at $x=L$. The given function, $f(x)$, represents the force exerted by the foundation. The mathematical model for the elastic beam with non-local terms of the type $\int_0^L \left[u^{(\alpha)}\right]^2 dx u^{(2\alpha)}$ can be found in [14], [15]. Due to the presence of the integral over $[0,L]$, the problem is not modeled by a point-wise equation and therefore is called a non-local problem.

II. Preliminaries

In this section, we introduce some necessary definitions of the Caputo fractional derivative and some properties of Chebyshev polynomials.

A. Definition of Caputo fractional derivative

Definition 1. *Let $\alpha\in\mathbb{R}^+, n=\lceil\alpha\rceil$ and $u\in AC^n[a,b]$. Then the Caputo fractional derivative of $u(x)$ is defined by [16]*

$$D_a^\alpha u(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{u^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}d\tau.$$

The Caputo fractional derivative is a linear operator similar to integer order differential operators. Some properties of Caputo fractional derivatives are as follows [16].

$$D_a^\alpha C=0, \qquad \text{where } C \text{ is a constant.} \tag{3}$$

and

$$D_a^\alpha x^\beta=\begin{cases}0, & \beta<\lceil\alpha\rceil,\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, & \beta\geq\lceil\alpha\rceil,\end{cases} \tag{4}$$

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where $\beta \in \mathbb{N} \cup 0$, $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α and $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

B. Some properties of Chebyshev polynomials

In this section, we summarize some elementary formulae for the manipulation of Chebyshev polynomials.

Definition 2. The Chebyshev polynomials $T_n(s)$, n = 1, 2, ..., N of the first kind are orthogonal polynomials of degree n in s defined on the interval $[-1,1]$ by [17], [18]

$$T_n(s) = \cos(n\theta),$$

where $s = \cos(\theta)$, $\theta \in [0,\pi]$ and $s \in [-1,1]$.

For convenience, we first transform to the interval $x \in [0,1]$ by using the transformation $x = \frac{1}{2}(s+1)$ and obtain the shifted Chebyshev polynomials (of the first kind) in the form $T_n^*(x) = T_n(2x-1)$. The shifted polynomials can be generated by using the recurrence relation

$$T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x),$$

with $T_0^*(x) = 1$ and $T_1^*(x) = 2x-1$ and then

$$\begin{aligned} T_2^*(x) &= 8x^2 - 8x + 1, \\ T_3^*(x) &= 32x^3 - 48x^2 + 18x - 1, \\ &\vdots \\ T_N^*(x) &= N\sum_{k=0}^{N}(-1)^{N-k}\frac{2^{2k}(N+k-1)!}{(2k)!(N-k)!}x^k. \end{aligned} \quad (5)$$

The shifted Chebyshev polynomial $T_{N+1}^*(x)$ of degree $N+1$ has exactly $N+1$ real zeroes on the interval $[0,1]$. The n^{th} zero x_n is given by

$$x_n = \frac{1}{2}\left(1 + \cos\left(\frac{[2(N-n)+1]\pi}{2(N+1)}\right)\right). \quad (6)$$

C. Chebyshev operational matrix for Caputo fractional derivatives

We consider the shifted Chebyshev vector given by $\mathbf{T}^*(x) = [T_0^*(x) \;\; T_1^*(x) \;\; T_2^*(x) \;\; \ldots \;\; T_n^*(x)]^T$. Then the shifted Chebyshev polynomial, $\mathbf{T}^*(x)$, in (5) can be expressed in matrix form as

$$\mathbf{T}^*(x) = \mathbf{D}^{-1}\mathbf{Y}(x), \quad (7)$$

where

$$\mathbf{Y}(x) = \begin{bmatrix} 1 & x & x^2 & \ldots & x^n \end{bmatrix}^T, \quad (8)$$

and

$$\mathbf{D} = \begin{bmatrix} 2^0\begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 & \cdots & 0 \\ 2^{-2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} & 2^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k\begin{pmatrix} 2N \\ N \end{pmatrix} & 2k\begin{pmatrix} 2N \\ N-1 \end{pmatrix} & \cdots & 2k\begin{pmatrix} 2N \\ 0 \end{pmatrix} \end{bmatrix} \quad (9)$$

and $k = 2^{-2N}$.

From (4), it is not difficult to show inductively that the Caputo fractional derivative of the vector, $\mathbf{Y}(x)$, of order α is given by the matrix form

$$D^\alpha \mathbf{Y}(x) = \mathbf{B}_\alpha(x)\mathbf{Y}(x), \quad (10)$$

where

$$\mathbf{B}_\alpha(x) = x^{-\alpha}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)} \end{bmatrix}. \quad (11)$$

The fractional derivative of order $j\alpha$ is also given by

$$D^{(j\alpha)}\mathbf{Y}(x) = \mathbf{B}_{j\alpha}(x)\mathbf{Y}(x), \quad (12)$$

where

$$\mathbf{B}_{j\alpha}(x) = x^{-j\alpha}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \frac{\Gamma(j)}{\Gamma(j-j\alpha)} & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N+1)}{\Gamma(N+1-j\alpha)} \end{bmatrix}, \quad (13)$$

and $j = 1, 2, 3, \ldots, N$.

D. Expansions for shifted Chebyshev polynomials of the first kind

In this section, we apply the shifted Chebyshev operational matrix of fractional derivatives to solve nonlinear Caputo fractional integro-differential equations (NFIDEs) in (1). To do this, we first approximate a solution, $u^N(x)$, of the beam problem (1) as

$$\begin{aligned} u^N(x) &= \sum_{n=0}^{N} a_n T_n^*(x) = \mathbf{A}\mathbf{T}^*(x) \\ &= \mathbf{A}\mathbf{D}^{-1}\mathbf{Y}(x), \end{aligned} \quad (14)$$

where $\mathbf{A} = [a_0 \;\; a_1 \;\; \ldots \;\; a_N]$ is an unknown vector and the matrix, $\mathbf{Y}(x)$, in (8).

Theorem 1. *Let $u^N(x)$ be approximated by the shifted Chebyshev polynomials of the first kind as in* (14) *and also suppose $\alpha > 0$. Then the Caputo fractional derivative with order α of the shifted Chebyshev polynomials of the first kind is given by [19]*

$$D^\alpha(u^N(x)) = \sum_{n=\lceil\alpha\rceil}^{N}\sum_{k=\lceil\alpha\rceil}^{n} a_n w_{n,k}^{(\alpha)} x^{k-\alpha}, \quad (15)$$

where

$$w_{n,k}^{(\alpha)} = (-1)^{n-k}\frac{2^{2k}n(n+k-1)!\Gamma(k+1)}{(2k)!(n-k)!\Gamma(k+1-\alpha)}. \quad (16)$$

From theorem 1 and (10), the Caputo fractional derivative matrix that represents $D^\alpha(u^N(x))$ is given in matrix form as

$$\begin{aligned} D^\alpha\left(u^N(x)\right) &= D^\alpha\left(\mathbf{A}\mathbf{D}^{-1}\mathbf{Y}(x),\right) \\ &= \mathbf{A}\mathbf{D}^{-1}D^\alpha\mathbf{Y}(x) \\ &= \mathbf{A}\mathbf{D}^{-1}\mathbf{B}_\alpha(x)\mathbf{Y}(x). \end{aligned} \quad (17)$$

We also have that the j^{th} Caputo fractional derivative matrix with order α is given by

$$D^{(j\alpha)}\left(u^N(x)\right) = \mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{j\alpha}(x)\mathbf{Y}(x), \qquad (18)$$

where $\mathbf{D}$, $\mathbf{B}_\alpha(x)$, and $\mathbf{B}_{j\alpha}(x)$ are defined in (9), (11), and (13) respectively.

III. Chebyshev solutions of the static beam

Consider the Caputo fractional integro-differential equation (1)

$$u^{(4\alpha)}(x) - \epsilon u^{(2\alpha)}(x) - \frac{2}{L}\int_0^L \left[u^{(\alpha)}(x)\right]^2 dx u^{(2\alpha)}(x) = f(x), \qquad (19)$$

with the boundary conditions

$$u(0) = u(L) = u^{(2\alpha)}(0) = u^{(2\alpha)}(L) = 0, 0 < x < L. \quad (20)$$

First, we assume that the approximate solution, $u^N(x)$, in (19) can be written as an expansion in Chebyshev polynomials of the first kind in the form:

$$u^N(x) = \sum_{n=0}^{N} a_n T_n^*(x) = \mathbf{A}\mathbf{D}^{-1}\mathbf{Y}(x), \qquad (21)$$

where the coefficient $\mathbf{A} = [a_0 \quad a_1 \quad ... \quad a_N]$ is a known vector. Now using (18), we can approximate the Caputo fractional derivative of order α in (17). The order 2α and 4α of shifted Chebyshev polynomials of the first kind are given by

$$D^{(2\alpha)}\left(u^N(x)\right) = \mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{2\alpha}(x)\mathbf{Y}(x), \qquad (22)$$

$$D^{(4\alpha)}\left(u^N(x)\right) = \mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{4\alpha}(x)\mathbf{Y}(x), \qquad (23)$$

where the matrices $\mathbf{D}^{-1}, \mathbf{B}_{2\alpha}$ and $\mathbf{B}_{4\alpha}$ are given in (9) and (13). Substituting (17), (21), (22) and (23) into (19), we obtain the following matrix equation

$$\begin{aligned}&\mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{4\alpha}(x)\mathbf{Y}(x) - \epsilon\mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{2\alpha}(x)\mathbf{Y}(x)\\ &-\frac{2}{L}\int_0^L \left[\mathbf{A}\mathbf{D}^{-1}\mathbf{B}_\alpha(x)\mathbf{Y}(x)\right]^2 dx\mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{2\alpha}(x)\mathbf{Y}(x)\\ &= f(x).\end{aligned} \qquad (24)$$

The matrices of the boundary conditions in (20) are then

$$\begin{aligned} u(0) &= \mathbf{A}\mathbf{D}^{-1}\mathbf{Y}(0), \quad u(L) = \mathbf{A}\mathbf{D}^{-1}\mathbf{Y}(L),\\ u^{(2\alpha)}(0) &= \mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{2\alpha}(0)\mathbf{Y}(0), \\ u^{(2\alpha)}(L) &= \mathbf{A}\mathbf{D}^{-1}\mathbf{B}_{2\alpha}(L)\mathbf{Y}(L).\end{aligned} \qquad (25)$$

Next, we substitute the real zeroes on the interval $[0, 1]$ in (6), x_n, $n = 0, 1, 2, \ldots, N$ into (24) and obtain a system of $(N + 1)$ algebraic equations. The first two equations of the system from (24) can be replaced by the boundary condition $u(0), u^{(2\alpha)}(0)$ in (25) and the last two equations in (24) can be replaced by the boundary condition $u(L), u^{(2\alpha)}(L)$ from (25). Next, we use Newton's iterative method to solve the system of $(N + 1)$ nonlinear algebraic equations to obtain the unknown vector $\mathbf{A}$. Then the approximate solution $u^N(x)$ given in (14) can be calculated to obtain a numerical approximate solution of the static beam problem.

IV. Error bound

In this section, we derive an error bound for the approximate solution $u(x)$ in (19). Diethelm(2010) [20] has proved the smoothness of solutions of Caputo fractional differential equations and shown that very good results can be obtained by differentiation of the solution in the interval $[0, L]$ [21].

Lemma 1. If $u(x)$ is the exact solution and $u^N(x) = \mathbf{C}\mathbf{D}^{-1}\mathbf{Y}(x)$ is the best shifted Chebyshev approximate solution of (19), then the error bound is given by

$$\begin{aligned}\|u(x) - u^N(x)\| &\leq \frac{h^{\frac{n2+3}{2}} R}{(n + 1)!\sqrt{2n + 3}},\\ x &\in [x_i, x_{i+1}] \subseteq [0, 1],\end{aligned} \qquad (26)$$

where

$$R = \max_{x\in[x_i,x_{i+1}]} |u^{(n+1)}(x)|, \qquad h = x_{i+1} - t_i.$$

Proof : Applying Taylor's expansion, we set

$$\begin{aligned} u_1(x) &= u(x_i) - u'(x_i)(x - x_i) + u''(x_i)\frac{(x - x_i)^2}{2!}\\ &\quad + \ldots + u^{(n)}(x_i)\frac{(x - x_i)^n}{n!}.\end{aligned}$$

It is clear that

$$|u(x) - u_1(x)| \leq |u^{(n+1)}(\xi)|\frac{(x - x_i)^{n+1}}{(n + 1)!},$$

where $\xi \in [x_i, x_{i+1}]$. Since $u^N(x) = \mathbf{C}\mathbf{D}^{-1}\mathbf{Y}(x)$ is the best approximation of $u(x)$, we have

$$\begin{aligned}\|u(x) - u^N(x)\|^2 &= \|u(x) - \mathbf{C}\mathbf{D}^{-1}\mathbf{Y}(x)\|^2\\ &\leq \|u(x) - u_1(x)\|^2\\ &= \int_{x_i}^{x_{i+1}} |u(\tau) - u_1(\tau)|^2 d\tau\\ &\leq \int_{x_i}^{x_{i+1}} |u^{(n+1)}(\xi)|^2\frac{(\tau - x_i)^{2(n+1)}}{(n + 1)!^2} d\tau\\ &\leq \int_{x_i}^{x_{i+1}} R^2\frac{(\tau - x_i)^{2(n+1)}}{(n + 1)!^2} d\tau\end{aligned}$$

where $R = max_{x\in[x_i,x_{i+1}]}|u^{(n+1)}(x)|$,

$$\begin{aligned}&= \frac{R^2}{(n + 1)!^2}\frac{(\tau - x_i)^{2(n+2)}}{2(n + 2)}\Big|_{x_i}^{x_{i+1}}\\ &= \frac{R^2}{(n + 1)!^2}\frac{h^{2n+3}}{2n + 3}, \quad \text{where } h = x_{i+1} - x_i.\end{aligned}$$

Take the square root of both sides, we have

$$\|u(x) - u^N(x)\| \leq \frac{h^{\frac{2n+3}{2}} R}{(n + 1)!\sqrt{2n + 3}},$$

which is the desired result for each sub interval $[x_i, x_{i+1}], i = 1, 2, \ldots, n$. Then, the solution $u(x)$ has a local error bound of $\mathcal{O}(h^{\frac{2n+3}{2}})$, and a global error bound of $\mathcal{O}(h^{\frac{2n+1}{2}})$ on the interval $[0, 1]$.

V. Numerical Results

In this section, we give some examples to illustrate the applicability and accuracy of the proposed method. All numerical computations were carried out using the Maple program.

Example 1 : Consider the nonlinear fourth order Caputo fractional integro-differential equation of the static beam

$$u^{(iv)}(x) - u^{''}(x) - 2\int_0^1 \left[u^{'}(x)\right]^2 dt u^{''}(x)$$
$$= 2sin(\pi x)\pi^4 + sin(\pi x)\pi^2, \quad 0 < x < 1,$$

with the boundary conditions

$$u(0) = u(1) = u^{''}(0) = u^{''}(1) = 0,$$

which has the exact solution $u(x) = \sin(\pi x)$.

By applying the Chebyshev method, we obtain the following approximate solution from the shifted Chebyshev expansion with seven terms (N = 7):

$$u^{(7)}(x) \quad = \quad \sum_{n=0}^{7} a_n T_n^*(x) = \mathbf{A}\mathbf{D}^{-1}\mathbf{Y}(x),$$

where the matrices $\mathbf{A} = [a_0 \quad a_1 \quad ... \quad a_7]^T$, $\mathbf{Y}(x) = \begin{bmatrix}1 & x & x^2 & \dots & x^7\end{bmatrix}^T$ and

$$\mathbf{D}^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 2 & 0 & \dots & 0 \\ 1 & -8 & 8 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 98 & -1568 & \dots & 8192 \end{bmatrix}.$$

Using the real zeroes of shifted Chebyshev polynomial $T_8^*(x)$ in (6), we obtain

$$x_0 = 0.0096, \quad x_1 = 0.0843, \quad x_2 = 0.2222, \quad x_3 = 0.4025,$$
$$x_4 = 0.5975, \quad x_5 = 0.7778, \quad x_6 = 0.9157, \quad x_7 = 0.9904.$$

For each x, we calculate all matrices $\mathbf{Y}(x)$ in (8), $\mathbf{B}_\alpha(x)$ in (11), $\mathbf{B}_2\alpha(x))$ and $\mathbf{B}_4\alpha(x)$ in (13) as follows:

$$\mathbf{B}_\alpha = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{B}_{6\times 6}\end{bmatrix}, \mathbf{B}_{2\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{B}_{5\times 5}\end{bmatrix}, \mathbf{B}_{4\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{B}_{3\times 3}\end{bmatrix},$$

$$\mathbf{B}_{6\times 6} \quad = \quad \text{diag}\left[0 \quad \frac{1}{x} \quad \frac{2}{x} \quad \frac{6}{x} \quad \frac{12}{x} \quad \frac{20}{x} \quad \frac{30}{x} \quad \frac{42}{x}\right],$$
$$\mathbf{B}_{5\times 5} \quad = \quad \text{diag}\left[0 \quad 0 \quad \frac{2}{x^2} \quad \frac{6}{x^2} \quad \frac{24}{x^2} \quad \frac{60}{x^2} \quad \frac{120}{x^2} \quad \frac{210}{x^2}\right],$$
$$\mathbf{B}_{3\times 3} \quad = \quad \text{diag}\left[0 \quad 0 \quad 0 \quad 0 \quad \frac{24}{x^4} \quad \frac{120}{x^4} \quad \frac{360}{x^4} \quad \frac{840}{x^4}\right].$$

We then construct the system of 8 algebraic equations including all boundary conditions. After that, we use the Maple program to solve for the matrix $\mathbf{A} = [a_0 \quad a_1 \quad ... \quad a_7]^T$ which gives

$$\mathbf{A} = \begin{bmatrix} 4.7109 \times 10^{-1} \\ 5.4476 \times 10^{-12} \\ -4.9860 \times 10^{-1} \\ -6.0979 \times 10^{-12} \\ 2.8110 \times 10^{-2} \\ 6.2246 \times 10^{-13} \\ -6.0570 \times 10^{-4} \\ 2.7878 \times 10^{-14} \end{bmatrix},$$

Therefore, the approximate shifted Chebyshev solution for $N = 7$ is given by

$$\begin{aligned} u^{(N)}(x) \quad = \quad & -1.8928 \times 10^{-11} + 3.1329\,x + 5.5629 \times 10^{-10}\,x^2 \\ & -5.0253\,x^3 - 0.5885\,x^4 + 3.7214\,x^5 - 1.2405\,x^6 \\ & +2.2838 \times 10^{-10}\,x^7 \end{aligned}$$

and the graph of the solution is as shown in Fig. 1.

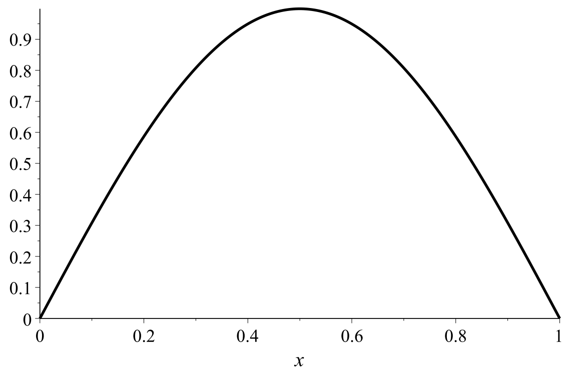

Fig. 1. Graph of solution of the static deflection of the beam for Ex.1

The accuracy of this method is shown by comparing the approximate solution with the exact solution $N = 14, 15, 16$ in Table I and Table II.

TABLE I
Solution of Example 1

x	Exact	$N = 14$	$N = 15$	$N = 16$
0.1	0.3082	0.3090	0.3090	0.3090
0.2	0.5865	0.5878	0.5878	0.5878
0.3	0.8076	0.8090	0.8090	0.8090
0.4	0.9495	0.9511	0.9511	0.9511
0.5	0.9984	0.9999	1.0000	0.9999
0.6	0.9495	0.9511	0.9511	0.9511
0.7	0.8076	0.8090	0.8090	0.8090
0.8	0.5865	0.5878	0.5878	0.5878
0.9	0.3082	0.3090	0.3090	0.3090

We also show the numerical absolute errors, $|u_{exact}(x) - u^N(x)|$ in Table II and graphs of absolute errors in Fig. 2

TABLE II
Absolute error of Example 1

x	$N = 14$	$N = 15$	$N = 16$
0.1	0.0000	0.0000	2.0×10^{-10}
0.2	1.0×10^{-10}	0.0000	4.0×10^{-10}
0.3	1.0×10^{-10}	1.0×10^{-10}	2.0×10^{-10}
0.4	3.0×10^{-10}	1.0×10^{-10}	4.0×10^{-10}
0.5	2.0×10^{-10}	0.0000	1.7×10^{-10}
0.6	0.0000	5.0×10^{-10}	5.0×10^{-10}
0.7	1.0×10^{-9}	1.0×10^{-10}	6.0×10^{-10}
0.8	1.1×10^{-9}	7.0×10^{-10}	1.1×10^{-9}
0.9	2.5×10^{-9}	1.0×10^{-10}	6.0×10^{-10}

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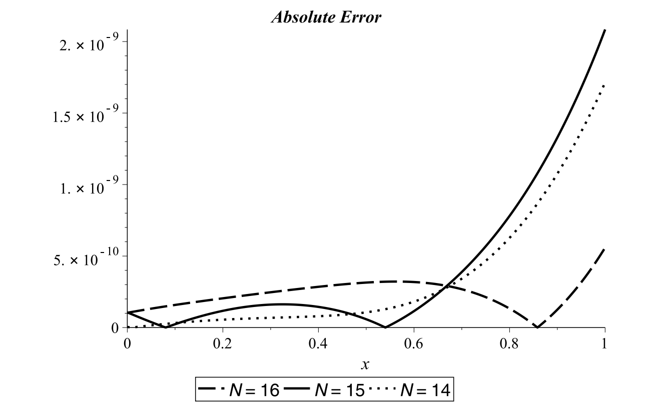

Fig. 2. Graph of absolute error of the static deflection of the beam for Ex.1

Example 2 : Consider the following nonlinear fourth order Caputo fractional integro-differential static beam problem:

$$u^{(4\alpha)}(x) - u^{(2\alpha)}(x) - 2\int_0^1 \left[u^{(\alpha)}(x)\right]^2 dx u^{(2\alpha)}(x) = -x, \quad 0 < x < 1,$$

and the boundary conditions

$$u(0) = u(1) = u^{(2\alpha)}(0) = u^{(2\alpha)}(1) = 0.$$

Applying Chebyshev method, we use the shifted Chebyshev operational matrices to obtain the approximate solutions for $\alpha = 1$ and values $N = 7, 10, 16$ respectively as follows.

$$\begin{aligned} u^{(7)}(x) &= -3.21 \times 10^{-13} - 0.02\,x - 7.61 \times 10^{-12}\,x^2 \\ &\quad +0.25 \times 10^{-1}\,x^3 + \ldots - 0.19 \times 10^{-3}\,x^7 \\ u^{(10)}(x) &= -7.59 \times 10^{-13} - 0.02\,x + 3.96 \times 10^{-12}\,x^2 \\ &\quad +0.02\,x^3 + \ldots - 1.23 \times 10^{-7}\,x^{10} \\ u^{(16)}(x) &= 1.09 \times 10^{-12} - 0.02\,x + 1.01 \times 10^{-11}\,x^2 \\ &\quad +0.02\,x^3 + \ldots - 1.83 \times 10^{-8}\,x^{16}. \end{aligned}$$

We define an absolute residual error given by

$$E_r = |Lu - f|,$$

where $Lu = u^{(4\alpha)} - \epsilon u^{(2\alpha)} - \frac{2}{L}\int_0^L \left[u^{(\alpha)}\right]^2 dx u^{(2\alpha)}$ and $f(x)$ is known function. The results of the errors for different values of $N = 6, 7, 10$ are shown in Table III.

TABLE III
ABSOLUTE RESIDUAL ERROR OF EXAMPLE 2

x	$N = 7$	$N = 10$	$N = 16$
0.1	1.92×10^{-4}	1.55×10^{-8}	3.00×10^{-11}
0.2	1.67×10^{-5}	3.60×10^{-9}	2.00×10^{-10}
0.3	1.92×10^{-5}	2.50×10^{-9}	2.00×10^{-10}
0.4	5.81×10^{-7}	1.70×10^{-9}	1.00×10^{-10}
0.5	1.37×10^{-5}	2.00×10^{-10}	2.00×10^{-10}
0.6	6.33×10^{-7}	1.40×10^{-9}	7.00×10^{-10}
0.7	2.29×10^{-5}	2.28×10^{-9}	1.00×10^{-10}
0.8	2.17×10^{-5}	3.60×10^{-9}	1.00×10^{-10}
0.9	2.72×10^{-4}	1.67×10^{-8}	2.00×10^{-10}

Graphs of the numerical approximate solutions with $N = 7$ and different values of order $\alpha = 0.6, 0.7, 0.8, 0.9, 1$ for Caputo fractional derivative are shown in Fig. 3.

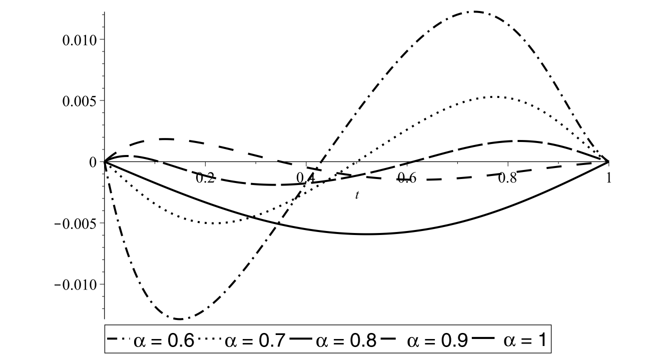

Fig. 3. Graphical solutions for deflection of the static beam for fractional order $\alpha = 0.6, 0.7, 0.8, 0.9.1$

Example 3 : Consider Caputo non-local fractional integro-differential equation of the static beam on elastic bearings:

$$u^{(4\alpha)}(x) - u^{(2\alpha)}(x) - 2\int_0^1 \left[u^{(\alpha)}(x)\right]^2 dx u^{(2\alpha)}(x) = \frac{1}{x+1}, \quad 0 < x < 1,$$

and the boundary conditions

$$u(0) = u(1) = u^{(2\alpha)}(0) = u^{(2\alpha)}(1) = 0.$$

This example is solved by our method with $N = 7$ when fractional orders $\alpha = 0.6, 0.7, 0.8, 0.9, 1$ respectively. The approximate shifted Chebyshev solutions which depend on order α given by

$$\begin{aligned} u^{(7)}_{\alpha=0.6}(x) &= 1.49 \times 10^{-12} + 0.33\,x - 2.13\,x^2 + 5.45\,x^3 \\ &\quad -7.40\,x^4 + \ldots - 0.25\,x^7 \\ u^{(7)}_{\alpha=0.7}(x) &= -6.00 \times 10^{-13} + 0.08\,x - 0.18\,x^2 - 0.60\,x^3 \\ &\quad +2.80\,x^4 + \ldots - 1.11\,x^7 \\ u^{(7)}_{\alpha=0.8}(x) &= -3.00 \times 10^{-13} - 8.54 \times 10^{-3}\,x + 0.30\,x^2 - 1.56\,x^3 \\ &\quad +3.61\,x^4 + \ldots - 0.83\,x^7 \\ u^{(7)}_{\alpha=0.9}(x) &= -3.30 \times 10^{-13} - 0.04\,x + 0.30\,x^2 - 0.92\,x^3 \\ &\quad +1.63\,x^4 + \ldots - 0.26\,x^7 \\ u^{(7)}_{\alpha=1}(x) &= 5.86 \times 10^{-13} + 0.03\,x + 7.42 \times 10^{-12}\,x^2 - 0.06\,x^3 \\ &\quad +0.04\,x^4 + \ldots - 3.31 \times 10^{-4}\,x^7. \end{aligned}$$

Graphs of numerical solutions for order $\alpha = 0.6$, 0.7, 0.8, 0.9 and 1 are shown in Fig. 4

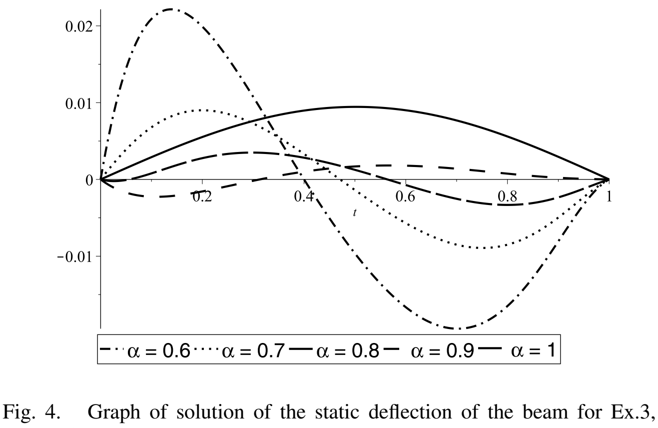

Fig. 4. Graph of solution of the static deflection of the beam for Ex.3, $\alpha = 0.6$, 0.7, 0.8, 0.9 and 1

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VI. Conclusion

In this paper, a new operational method based on Chebyshev polynomials for Caputo fractional derivative is applied to solve boundary value problems of the non-local Caputo fractional integro-differential static beam equation. This method is simple and is a good mathematical method for finding numerical solutions of NFDEs. The validity, accuracy and applicability of our Chebyshev method have been illustrated through numerical results. The approximate solutions have been compared with known exact solutions for some problems. The comparisons show that the Chebyshev method gives good accuracy and is efficient for a class of nonlinear fractional integro-differential equations (FIDEs) which arise in the study of transverse vibrations of a hinged beam.

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