Solving Linear Homogeneous Recurrence Relation via the Inverse of Vandermonde Matrix

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***Abstract*—We present a simple and efficient method for solving linear homogeneous recurrence relation via the inverse of Vandermonde matrix, which mainly involves synthesis divisions. This method does not need to use generating function techniques or solve a system of linear equations to determine the unknown coefficients of the solution. Some illustrative examples are provided.**

***Index Terms*—synthetic division, Vandermonde matrix, linear homogeneous recurrence relation, matrix inverse.**

I. INTRODUCTION

THE applications of Vandermonde matrix (VDM) in various areas are well-known, such as polynomial interpolations, signal processing, curve fitting, coding theory and control theory [1, 3, 4, 6, 9], etc. However, the study of more efficient algorithms for computing the inverse of VDM or the generalized VDM (also called the confluent Vandermonde matrix) and their applications is still an important research topic in the areas of applied mathematics, engineering or computer sciences. In this paper, we present a simple and efficient method for solving linear homogeneous recurrence relation via the inverse of VDM, which mainly involves synthesis divisions. Unlike the other common approaches for solving linear homogeneous recurrence relations, this method does not involve generating function techniques or solving a system of linear equations to determine the unknown coefficients of the solution [2]. This method is a further development of the author's previous works published in [5, 7, 8, 10, 11]. We expect it will be useful for reference by engineers or researchers working in the similar or related research areas.

The whole paper is organized like this. The mathematical background on using synthetic divisions to find the inverse of VDM is described in section 2. The application of the technique to solving linear homogeneous recurrence relation is described in section 3. Then, some examples and concluding remarks are provided in section 4 and section 5 respectively.

II. FINDING THE VDM INVERSE BY SYNTHETIC DIVISIONS

Consider the polynomial

$$\begin{aligned} f(x) &= (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \\ &= x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n \end{aligned}$$

where a_i and λ_i are constants. We are interested to find the inverse of the Vandermonde matrix below:

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$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1{}^2 & \lambda_2{}^2 & \cdots & \lambda_n{}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1{}^{n-1} & \lambda_2{}^{n-1} & \cdots & \lambda_n{}^{n-1} \end{pmatrix}$$

According to [10], we can apply the formula $V^{-1} = W \times A$ to compute the inverse of *V*, where the matrices *W* and *A* are defined as follows:

$$W = \begin{pmatrix} \dfrac{\lambda_1{}^{n-1}}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} & \dfrac{\lambda_1{}^{n-2}}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} & \cdots & \dfrac{1}{\prod_{j\neq 1}(\lambda_1-\lambda_j)} \\ \dfrac{\lambda_2{}^{n-1}}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} & \dfrac{\lambda_2{}^{n-2}}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} & \cdots & \dfrac{1}{\prod_{j\neq 2}(\lambda_2-\lambda_j)} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\lambda_n{}^{n-1}}{\prod_{j\neq n}(\lambda_n-\lambda_j)} & \dfrac{\lambda_n{}^{n-2}}{\prod_{j\neq n}(\lambda_n-\lambda_j)} & \cdots & \dfrac{1}{\prod_{j\neq n}(\lambda_n-\lambda_j)} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix},$$

where $a_1 = -\sum \lambda_j$, $a_2 = \sum_{j\neq m} \lambda_j\lambda_m$, $a_3 = -\sum_{j\neq m\neq s} \lambda_j\lambda_m\lambda_s$, ..., and $a_n = (-1)^n \prod \lambda_j$. Instead of using the formula $V^{-1} = W \times A$ directly, we can apply synthetic divisions to compute V^{-1} row by row, as illustrated below [11].

(i) Consider a Vandermonde matrix with order 2 below,

$$V = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

Define $f(x) = (x-\lambda_1)(x-\lambda_2) = x^2 + a_1x + a_2$, where $a_1 = -\lambda_1 - \lambda_2$ and $a_2 = \lambda_1\lambda_2$. Applying synthetic divisions to $[f(x)-a_2] \div (x-\lambda_1)$, we have

$$\begin{array}{r|cc} \lambda_1 & 1 & a_1 \\ & & \lambda_1 \\ \hline & 1 & \lambda_1 + a_1 \\ & & \lambda_1 \\ \hline & 1 & \boxed{\lambda_1 - \lambda_2} \end{array}$$

By dividing each element obtained by the first synthetic division by the element in bracket, namely $\lambda_1 - \lambda_2$, we can determine the first row of V^{-1}, except in the reverse order. Similarly, we can apply synthetic divisions to $[f(x) - a_2] \div (x - \lambda_2)$ to determine the second row of V^{-1} as follows:

$$\begin{array}{c|cc} \lambda_2 & 1 & a_1 \\ & & \lambda_2 \\ \hline & 1 & \lambda_2 + a_1 \\ & & \lambda_2 \\ \hline & 1 & \boxed{\lambda_2 - \lambda_1} \end{array}$$

Hence,

$$V^{-1} = \begin{pmatrix} \dfrac{\lambda_1 + a_1}{\lambda_1 - \lambda_2} & \dfrac{1}{\lambda_1 - \lambda_2} \\ \dfrac{\lambda_2 + a_1}{\lambda_2 - \lambda_1} & \dfrac{1}{\lambda_2 - \lambda_1} \end{pmatrix}$$

(ii) Consider a Vandermonde matrix with order 3 below,

$$V = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}.$$

Define $f(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = x^3 + a_1 x^2 + a_2 x + a_3$, where $a_1 = -\lambda_1 - \lambda_2 - \lambda_3$, $a_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, and $a_3 = -\lambda_1\lambda_2\lambda_3$. Applying synthetic divisions to the $[f(x) - a_3] \div (x - \lambda_1)$, we have

$$\begin{array}{c|ccc} \lambda_1 & 1 & a_1 & a_2 \\ & & \lambda_1 & \lambda_1^2 + a_1\lambda_1 \\ \hline & 1 & \lambda_1 + a_1 & \lambda_1^2 + a_1\lambda_1 + a_2 \\ & & \lambda_1 & \lambda_1^2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 \\ \hline & 1 & \lambda_1 - \lambda_2 - \lambda_3 & \boxed{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \end{array}$$

since $2\lambda_1^2 + a_1\lambda_1 + a_2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$.

Next,

$$\begin{array}{c|ccc} \lambda_2 & 1 & a_1 & a_2 \\ & & \lambda_2 & \lambda_2^2 + a_1\lambda_2 \\ \hline & 1 & \lambda_2 + a_1 & \lambda_2^2 + a_1\lambda_2 + a_2 \\ & & \lambda_2 & \lambda_2^2 - \lambda_1\lambda_2 - \lambda_2\lambda_3 \\ \hline & 1 & \lambda_2 - \lambda_1 - \lambda_3 & \boxed{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \end{array}$$

since $2\lambda_2^2 + a_1\lambda_2 + a_2 - \lambda_1\lambda_2 - \lambda_1\lambda_3 = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)$.

Also,

$$\begin{array}{c|ccc} \lambda_3 & 1 & a_1 & a_2 \\ & & \lambda_3 & \lambda_3^2 + a_1\lambda_3 \\ \hline & 1 & \lambda_3 + a_1 & \lambda_3^2 + a_1\lambda_3 + a_2 \\ & & \lambda_3 & \lambda_3^2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 \\ \hline & 1 & \lambda_3 - \lambda_1 - \lambda_2 & \boxed{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \end{array}$$

since $2\lambda_3^2 + a_1\lambda_3 + a_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$.

Hence,

$$V^{-1} = \begin{pmatrix} \dfrac{\lambda_1^2 + a_1\lambda_1 + a_2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} & \dfrac{\lambda_1 + a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} & \dfrac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \\ \dfrac{\lambda_2^2 + a_1\lambda_2 + a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} & \dfrac{\lambda_2 + a_1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} & \dfrac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \\ \dfrac{\lambda_3^2 + a_1\lambda_3 + a_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} & \dfrac{\lambda_3 + a_1}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} & \dfrac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \end{pmatrix}.$$

It can be proved that the method of synthetic divisions can be applied to determine the inverse of Vandermonde matrix of order n (≥ 2), but the details will not be included here.

III. LINEAR HOMOGENEOUS RECURRENCE RELATIONS

Consider a linear homogeneous recurrence relation with constant coefficients as follows:

$$u_n + a_1 u_{n-1} + \cdots + a_k u_{n-k} = 0,$$

such that $u_0 = b_0, u_1 = b_1, \ldots, u_{k-1} = b_{k-1}$. The associated auxiliary equation (or characteristic equation) is given by $x^k + a_1 x^{k-1} + \cdots + a_k = 0$. If the roots x_i of the auxiliary equation are distinct, the solution of the homogeneous recurrence relation is represented by

$$c_1 x_1^n + c_2 x_2^n + \cdots + c_k x_k^n,$$

where c_i are unknown coefficients to be determined. By using the given initial conditions, we can solve the following system to determine the coefficients c_i

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ \vdots & & & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \end{pmatrix}.$$

Then, we can solve the system by computing the inverse of the Vandermonde matrix on the left hand side of the equation. Hence, the method of synthetic divisions can be applied to solve the linear homogeneous recurrence relations concerned.

IV. EXAMPLES

Example 1. Solve the linear homogeneous recurrence relation

$$u_{n+2} - 3u_{n+1} - 4u_n = 0, \quad (n \geq 0)$$

such that $u_0 = 1$ and $u_1 = 3$.

Solution. Consider the auxiliary equation

$$x^2 - 3x - 4 = (x - 4)(x + 1).$$

Applying synthetic divisions, we have:

$$\begin{array}{c|cc} -1 & 1 & -3 \\ & & -1 \\ \hline & 1 & -4 \\ & & -1 \\ \hline & 1 & \boxed{-5} \end{array}$$

So, the first row of V^{-1} is (4/5 -1/5).

Next,

$$\begin{array}{c|cc} 4 & 1 & -3 \\ & & 4 \\ \hline & 1 & 1 \\ & & 4 \\ \hline & 1 & \boxed{5} \end{array}$$

So, the second row of V^{-1} is (1/5 1/5).

Hence,

$$V^{-1} = \begin{pmatrix} 4/5 & -1/5 \\ 1/5 & 1/5 \end{pmatrix}.$$

Therefore,

$$V^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/5 \\ 4/5 \end{pmatrix}.$$

The solution of the linear homogeneous recurrence relation is given by $\frac{1}{5}(-1)^n + \frac{4}{5}(4)^n = \frac{1}{5}[(-1)^n + 4^{n+1}]$.

<u>Example 2</u>. Solve the linear homogeneous recurrence relation

$$u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 0, \quad (n \geq 0).$$

such that $u_0 = 2, u_1 = 0$ and $u_2 = -2$.

Solution. Consider the auxiliary equation:

$$x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3) = 0.$$

Applying synthetic divisions, we have:

$$\begin{array}{r|rrr}
1 & 1 & -6 & 11 \\
 & & 1 & -5 \\ \hline
 & 1 & -5 & 6 \\
 & & 1 & -4 \\ \hline
 & 1 & -4 & \boxed{2}
\end{array}$$

So, the first row of V^{-1} is (3 -5/2 1/2).
Next,

$$\begin{array}{r|rrr}
2 & 1 & -6 & 11 \\
 & & 2 & -8 \\ \hline
 & 1 & -4 & 3 \\
 & & 2 & -4 \\ \hline
 & 1 & -2 & \boxed{-1}
\end{array}$$

So, the second row of V^{-1} is (-3 4 -1).
Also,

$$\begin{array}{r|rrr}
3 & 1 & -6 & 11 \\
 & & 3 & -9 \\ \hline
 & 1 & -3 & 2 \\
 & & 3 & 0 \\ \hline
 & 1 & 0 & \boxed{2}
\end{array}$$

So, the third row of V^{-1} is (1 -3/2 1/2).
Hence,

$$V^{-1} = \begin{pmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{pmatrix}$$

Therefore,

$$V^{-1}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$$

The solution of the linear homogeneous recurrence relation is given by $5(1)^n - 4(2)^n + 3^n = 5 - 2^{n+2} + 3^n$.

V. Concluding Remarks

In this paper, a simple and efficient method for solving linear homogeneous recursive relations via Vandermonde matrix has been introduced, which mainly involves synthetic divisions. This method has the advantage that it does not need to use generating function techniques or solve a system of linear equations to determine the unknown coefficients of the solution [2, 7]. Also, the total number of arithmetic operations involved in synthetic divisions is comparatively less than that by applying direct matrix multiplication to $W \times A$ to compute the inverse of VDM, which means this method requires less computational cost. More in-depth analysis of the complexity of this method will be reported elsewhere in the near future.

Acknowledgement

This work is supported by a research grant of EdUHK in 2017/18.

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