Comparison of Analytical and Numerical Solutions of Fractional-Order Bloch Equations using Reliable Methods

Sekson Sirisubtawee

***Abstract*—In this paper, we solve Caputo fractional-order Bloch equations. The Bloch equations are a model for nuclear magnetic resonance (NMR), which is a physical phenomenon arising in engineering, medicine, and the physical sciences. A revised variational iteration method and an Adams-Bashforth-Moulton type predictor-corrector scheme are used to obtain analytical solutions and numerical solutions, respectively, for the fractional-order equations. We compare the analytical and numerical solutions for selected fractional orders.**

***Index Terms*—Revised variational iteration method, Adams-Bashforth-Moulton type predictor-corrector method, Caputo fractional-order Bloch equations.**

I. Introduction

FRACTIONAL-order differential equations (FDEs) are based on various definitions of fractional derivatives. They are generalizations of classical differential equations, and they have been of interest to mathematicians, scientists and engineers for more than thirty years because they can be used to describe the memory and hereditary properties of various physical processes [1]. FDEs have been efficiently used to model many real phenomena in fields such as engineering [2], physics [3], applied mathematics [4], and disease models [5]. In general, most nonlinear FDEs do not have exact solutions, and hence approximate analytic solutions and numerical solutions are usually required. Some techniques for obtaining approximate analytic solutions of FDEs include the Laplace-Adomian decomposition methods (LADM) [6], the Duan-Rach modified decomposition method [7], the homotopy analysis method (HAM) [8], and the Laplace-variational iteration method (LVIM) [9]. The LADM and HAM are based on the assumption that the solutions of FDEs are in the form of infinite series, whereas the LVIM constructs a correction functional by a generalized Lagrange multiplier method. Numerical methods for obtaining approximate solutions of FDEs are based on discretization of the independent variable and include the Adams-Bashforth-Moulton type predictor-corrector or PECE (Predict, Evaluate, Correct and Evaluate) method [10], the simulink model [11], and the finite element method [12].

The aim of this paper is to investigate and compare analytical and numerical solutions of the following Caputo fractional-order Bloch equation

$$\begin{aligned} {}_{C}D_a^{\alpha} M_x(t) &= \omega_0 M_y(t) - \frac{M_x(t)}{T_2}, \\ {}_{C}D_a^{\alpha} M_y(t) &= -\omega_0 M_x(t) - \frac{M_y(t)}{T_2}, \\ {}_{C}D_a^{\alpha} M_z(t) &= \frac{M_0 - M_z(t)}{T_1}, \end{aligned} \tag{1}$$

with initial conditions

$$M_x(a) = M_x^0,\ M_y(a) = M_y^0,\ M_z(a) = M_z^0. \tag{2}$$

System (1) is a generalization of the classical system of first-order Bloch equations [13] with initial conditions given at $t = a$. The fractional-order system (1) is developed from the first-order system by replacing the first-order time derivatives with the Caputo fractional derivatives of order $\alpha \in (0, 1]$ denoted by ${}_{C}D_a^{\alpha}$. It is important to maintain a consistent set of units for both sides of each equation in the system via fractional time constants. In Eq. (1), the states $M_x(t)$, $M_y(t)$, and $M_z(t)$ represent the system magnetization in x, y and z components, respectively. The meanings of the parameters in the system are as follows: ω_0 is the resonant frequency, T_1 is the spin-lattice relaxation time, T_2 is the spin-spin relaxation time, and M_0 is the equilibrium magnetization. The applications of system (1) in some specific fields can be found in [13]–[15].

In this paper, we obtain an approximate analytical solution of the initial value problem in Eqns. (1)–(2) using the revised variational iteration method (RVIM) [16] and a numerical solution using the Adams-Bashforth-Moulton type predictor-corrector scheme (PECE) [17]. These two methods are currently regarded as the most reliable and efficient analytical and numerical methods for solving FDEs.

The paper is organized as follows. In section 2, preliminary definitions and necessary properties are given. In section 3, a description of the methods used in the paper are briefly given. In section 4, the solutions of the IVP (1)–(2) are obtained using the analytical and numerical methods. Finally, section 5 includes a discussion and conclusions.

II. Preliminary definitions and properties

In this section, we provide necessary definitions of fractional-order operators including the Riemann-Liouville fractional integral and the Caputo fractional derivative. The important properties of the operators are briefly given.

A function $f(t)$ $(t > 0)$ is said to be in the space C_α $(\alpha \in \mathbb{R})$ if it can be written as $f(t) = t^p g(t)$ for some $p > \alpha$, where $g(t)$ is continuous in $[0, \infty)$. The function is also said

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to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in \mathbb{N}$ (for more details see [1]).

Definition 2.1: [1]. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\alpha$ with $a \geq 0$ is defined as

$$ {}_{RL}J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. If $\alpha = 0$, then ${}_{RL}J_a^\alpha f(t) = f(t)$.

Definition 2.2: [1]. Given a positive real number α, the Caputo fractional derivative of order α with $a \geq 0$ is defined in terms of the Riemann-Liouville fractional integral, i.e., ${}_C D_a^\alpha f(t) = {}_{RL}J_a^{m-\alpha} f^{(m)}(t)$, where $m-1 < \alpha < m$, $m \in \mathbb{N}$, or it can be expressed as

$$ {}_C D_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \tag{4}$$

where $t \geq a$ and $f \in C_{-1}^m$. If $\alpha = m$, then ${}_C D_a^\alpha f(t) = f^{(m)}(t)$.

Remark 2.1: [1] For $f(t) \in C_1^n$, $\alpha, \beta \geq 0$, $m-1 < \alpha \leq m$, $\alpha + \beta \leq n$, where $m, n \in \mathbb{N}$, $a \geq 0$ and $\gamma \geq -1$, we have the following important properties

1. ${}_{RL}J_a^\alpha \; {}_{RL}J_a^\beta f(t) = {}_{RL}J_a^\beta \; {}_{RL}J_a^\alpha f(t) = {}_{RL}J_a^{\alpha+\beta} f(t)$,
2. ${}_{RL}J_a^\alpha (t-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}$,
3. ${}_{RL}J_a^\alpha \; {}_C D_a^\alpha f(t) = f(t) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(a) \frac{(t-a)^k}{k!}$.

III. Descriptions of the methods

In this section, we give descriptions of the RVIM and the PECE methods that we use to solve the IVP (1)-(2).

A. The revised variational iteration method

We first provide the general principle of the variational iteration method (VIM) [16], [18] for solving a fractional-order differential equation. Then we describe the RVIM for solving a system of FDEs.

Consider the following fractional-order differential equation:

$$ {}_C D_a^\alpha u(t) + N(u(t)) = f(t), \; 0 < \alpha \leq 1, \tag{5}$$

where N is a nonlinear operator with respect to $u(t)$ and $f(t)$ is a source function. According to the VIM, the correction functional for Eq. (5) is constructed as follows:

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + {}_{RL}J_a^\alpha \left[\lambda \left({}_C D_a^\alpha u_n(t) + N(u_n(t)) - f(t)\right)\right] \\ &= u_n(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \lambda(\tau) \\ &\quad \times \left({}_C D_a^\alpha u_n(\tau) - N(u_n(\tau)) - f(\tau)\right) d\tau, \end{aligned} \tag{6}$$

where λ is the Lagrange multiplier, which can be optimally determined via variational theory [19].

Some approximations are required to identify the Lagrange multiplier. The correction functional equation (6) can be approximated by the following equation:

$$\begin{aligned} u_{n+1}(t) &= u_n(t) \\ &\int_a^t \lambda(\tau) \left(u'_n(\tau) + N(\tilde{u}_n(\tau)) - f(\tau)\right) d\tau. \end{aligned} \tag{7}$$

If we now apply restricted variations $\tilde{u}_n$ to the term $N(u)$, then we can easily determine the multiplier. Assuming the aforementioned functional to be stationary, i.e., $\delta \tilde{u}_n = 0$, we obtain

$$ \delta u_{n+1}(t) = \delta u_n(t) + \delta \int_a^t \lambda(\tau) \left(u'_n(\tau) - f(\tau)\right) d\tau. \tag{8}$$

This yields the Lagrange multiplier $\lambda = -1$. Substituting $\lambda = -1$ into the functional equation (6), we obtain the following iteration formula:

$$ u_{n+1}(t) = u_n(t) - {}_{RL}J_a^\alpha \left({}_C D_a^\alpha u_n(t) + N(u_n(t)) - f(t)\right). \tag{9}$$

The initial approximation $u_0(t)$ can be selected to satisfy the initial conditions of the problem. Finally, we can approximate the solution $u(t) = \lim_{n\to\infty} u_n(t)$ by the Nth term approximation $u_N(t)$.

Now we describe the RVIM for solving a system of fractional-order differential equations. Consider the following incommensurate system of fractional-order differential equations:

$$\begin{aligned} {}_C D_a^{\alpha_i} u_i(t) + N_i(u_1(t), ..., u_m(t)) = f_i(t), \\ 0 < \alpha_i \leq 1, i = 1, 2, ..., m, \end{aligned} \tag{10}$$

where N_i are operators of $u_j(t)$, $j = 1, 2, 3, ..., m$ and $f_i(t)$ are known functions. The correction functionals for this case are as follows:

$$\begin{aligned} u_{i,n+1}(t) = u_{i,n}(t) - {}_{RL}J_a^{\alpha_i} \big({}_C D_a^{\alpha_i} u_{i,n}(t) \\ + N_i(u_{1,n}(t), ..., u_{m,n}(t)) - f_i(t) \big), \\ i = 1, 2, ..., m. \end{aligned} \tag{11}$$

Similarly, the initial approximations $u_{i,0}(t)$, $i = 1, 2, ..., m$ can be independently selected as long as they satisfy the initial conditions of the system. The Nth order terms $u_{i,N}(t)$, $i = 1, 2, ..., m$ can then be used to represent approximations of the solutions $u_i(t) = \lim_{n\to\infty} u_{i,n}(t)$, $i = 1, 2, ..., m$ of the system.

The iterative formula for the RVIM can be constructed by modifying the iteration formula (11) obtained by the standard VIM as above. The modification can be done by replacing $u_{1,n}(t), ..., u_{i-1,n}(t)$ in the formula of $u_{i,n+1}(t)$ with the updated values $u_{1,n+1}(t), ..., u_{i-1,n+1}(t)$, respectively. In consequence, the recursive formula for the RVIM employed to solve system (10) is expressed as

$$\begin{aligned} u_{i,n+1}(t) = u_{i,n}(t) - {}_{RL}J_a^{\alpha_i} \big({}_C D_a^{\alpha_i} u_{i,n}(t) \\ + N_i(u_{1,n+1}(t), ..., u_{i-1,n+1}(t), u_{i,n}(t), ..., u_{m,n}(t)) \\ - f_i(t) \big), \; i = 1, 2, ..., m. \end{aligned} \tag{12}$$

The modified technique in the RVIM formula (12) can accelerate the convergence of iterative approximate solutions comparing to the approximate solutions obtained using the standard VIM. Hence, the corrected solution $u_{i,n+1}(t)$ of the RVIM is more accurate than the solution $u_{i,n+1}(t)$ from the standard VIM because updated values are used to compute the RVIM solution.

B. The predictor-corrector scheme

The Adams-Bashforth-Moulton type predictor-corrector scheme or the PECE method [10] is now widely used to solve FDEs. Hence, we will use this technique to numerically solve the IVP (1)–(2). The formulas of the method are briefly given as follows.

Consider the fractional-order initial value problem

$$\begin{aligned} {}_C D_0^\alpha u(t) = f(t, u(t)), \quad 0 \le t \le T, \\ u^{(k)}(0) = u_0^{(k)}, k = 0, 1, ...m-1, \quad \alpha \in (m-1, m), \end{aligned} \tag{13}$$

where f is a nonlinear function and m is a positive integer. The IVP (13) can be converted to the following Volterra integral equation

$$u(t) = \sum_{k=0}^{m-1} u_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau. \tag{14}$$

To estimate the integral in (14), we uniformly discretize the whole time T by a uniform grid $\{t_n = nh : n = 0, 1, ...N\}$ for some integer N with the step size $h := T/N$. Let $u_h(t_n)$ denote the approximation to $u(t_n)$. Suppose that we have already calculated approximations $u_h(t_j), j = 1, 2, ..., n$, then the approximation $u_h(t_{n+1})$ of the IVP (13) can be computed using the PECE method as follows:

$$\begin{aligned} u_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} u_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, u_h^P(t_{n+1})) + \\ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, u_h(t_j)), \end{aligned} \tag{15}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ -2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1. \end{cases} \tag{16}$$

The preliminary approximation $u_h^P(t_{n+1})$ in Eq. (15) is called a predictor and is given by

$$u_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} u_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, u_h(t_j)), \tag{17}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha). \tag{18}$$

IV. Main results

In this section, we exhibit the use of the RVIM and the PECE as described above to solve the FIVP in Eqs. (1)-(2) with the starting point $t = a = 0$. However, we first demonstrate the exact solution, which can be discovered in [13], of the problem for $\alpha = 1$ as follows.

$$\begin{aligned} M_x(t) &= e^{-t/T_2} \left(M_x^0 \cos(\omega_0 t) + M_y^0 \sin(\omega_0 t) \right), \\ M_y(t) &= e^{-t/T_2} \left(M_y^0 \cos(\omega_0 t) + M_x^0 \sin(\omega_0 t) \right), \\ M_z(t) &= M_z^0 e^{-t/T_1} + M_0(1 - e^{-t/T_1}). \end{aligned} \tag{19}$$

The initial conditions and parameter values, which are employed in our simulations, are as follows.

$$\begin{aligned} M_x^0 = 0, \ M_y^0 = 100, \ M_z^0 = 0, \ \omega_0 = 60 \times \pi, \\ T_1 = 1, \ T_2 = 20 \times 10^{-3}, \ M_0 = 100. \end{aligned} \tag{20}$$

A. The application of the RVIM

The current section is devoted to the use of the RVIM to obtain an analytical solution of the IVP (1)-(2). Applying the iteration formula (12) of the RVIM with $\lambda(\tau) = -1$ to the problem, we obtain the following iteration formulas:

$$\begin{aligned} M_x^{n+1}(t) &= M_x^n(t) + {}_{RL}J_a^\alpha \Big[\lambda(\tau) \Big({}_C D_a^\alpha M_x^n(\tau) \\ &\quad -\omega_0 M_y^n(\tau) + \frac{M_x^n(\tau)}{T_2} \Big) \Big], \\ &= M_x^n(a) - {}_{RL}J_a^\alpha \left(-\omega_0 M_y^n(\tau) + \frac{M_x^n(\tau)}{T_2} \right), \end{aligned} \tag{21}$$

$$\begin{aligned} M_y^{n+1}(t) &= M_y^n(t) + {}_{RL}J_a^\alpha \Big[\lambda(\tau) \Big({}_C D_a^\alpha M_y^n(\tau) + \omega_0 M_x^{n+1}(\tau) \\ &\quad + \frac{M_y^n(\tau)}{T_2} \Big) \Big], \\ &= M_y^n(a) - {}_{RL}J_a^\alpha \left(\omega_0 M_x^{n+1}(\tau) + \frac{M_y^n(\tau)}{T_2} \right), \end{aligned} \tag{22}$$

$$\begin{aligned} M_z^{n+1}(t) &= M_z^n(t) + {}_{RL}J_a^\alpha \Big[\lambda(\tau) \Big({}_C D_a^\alpha M_z^n(\tau) \\ &\quad - \frac{M_0 - M_z^n(\tau)}{T_1} \Big) \Big], \\ &= M_z^n(a) + {}_{RL}J_a^\alpha \left(\frac{M_0 - M_z^n(\tau)}{T_1} \right), \ n = 0, 1, 2, ..., \end{aligned} \tag{23}$$

in which the initial approximations are chosen to be $M_x^0(t) = M_x^0$, $M_y^0(t) = M_y^0$ and $M_z^0(t) = M_z^0$.

B. The application of the PECE method

Applying the PECE method in Eqs. (15)-(18) to the IVP (1)-(2), we discretize the time interval with points $\{t_n\}$ and obtain the formulas for approximations $M_x^{h,n} = M_x^h(t_n)$, $M_y^{h,n} = M_y^h(t_n)$, $M_z^{h,n} = M_z^h(t_n)$ as follows:

$$\begin{aligned} M_x^{h,n+1} &= M_x^0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\omega_0 M_y^{P,h,n+1} - \frac{M_x^{P,h,n+1}}{T_2} \right) \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{1,j,n+1} \left(\omega_0 M_y^{h,j} - \frac{M_x^{h,j}}{T_2} \right), \end{aligned} \tag{24}$$

$$\begin{aligned} M_y^{h,n+1} &= M_y^0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(-\omega_0 M_x^{P,h,n+1} - \frac{M_y^{P,h,n+1}}{T_2} \right) \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{2,j,n+1} \left(-\omega_0 M_x^{h,j} - \frac{M_y^{h,j}}{T_2} \right), \end{aligned} \tag{25}$$

$$\begin{aligned} M_z^{h,n+1} &= M_z^0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(\frac{M_0 - M_z^{P,h,n+1}}{T_1} \right) \\ &\quad + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{3,j,n+1} \left(\frac{M_0 - M_z^{h,j}}{T_1} \right), \end{aligned} \tag{26}$$

in which

$$M_x^{P,h,n+1} = M_x^0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{1,j,n+1} \left(\omega_0 M_y^{h,j} - \frac{M_x^{h,j}}{T_2} \right), \tag{27}$$

$$M_y^{P,h,n+1} = M_y^0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{2,j,n+1} \left(-\omega_0 M_x^{h,j} - \frac{M_y^{h,j}}{T_2} \right), \tag{28}$$

$$M_z^{P,h,n+1} = M_z^0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{3,j,n+1} \left(\frac{M_0 - M_z^{h,j}}{T_1} \right), \tag{29}$$

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where, for $l = 1, 2, 3,$

$$a_{l,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ -2(n-j+1)^{\alpha+1}, & \text{if } 1 \leq j \leq n, \\ 1, & \text{if } j = n+1, \end{cases} \quad (30)$$

and

$$b_{l,j,n+1} = \frac{h^{\alpha}}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right), \; 0 \leq j \leq n. \quad (31)$$

We will use the discretized formulas (24)–(31) to obtain the numerical solutions for the IVP (1)-(2) in the following section.

C. Simulation results

Given the parameter values described in Eq. (20) and the starting point $a = 0$, we will exhibit the simulation results of the IVP (1)-(2) obtained using the formulas of the exact solution in Eq. (19), the RVIM in Eqns. (21)-(23), and the PECE method in Eqns. (24)-(31) for $\alpha = 1,\ 0.9,\ 0.8$. In particular, the absolute errors of the numerical results, generated by the RVIM and the PECE method compared to those obtained using the exact solution, are calculated for $\alpha = 1$. Moreover, the absolute differences of the numerical results, calculated by the RVIM and the PECE method, are measured for $\alpha = 0.9,\ 0.8$.

The following results are for $\alpha = 1$. Using the iteration formulas (21)-(23), we obtain the 80th terms of the approximations $M_x^{80}(t)$, $M_y^{80}(t)$ and $M_z^{80}(t)$ as follows.

$$\begin{aligned} M_x^{80}(t) =& 18849.6t - 942478t^2 - 8.80607 \times 10^7 t^3 + \dots \\ & - 7.11155 \times 10^{82} t^{158} - 2.0116 \times 10^{81} t^{159}, \\ M_y^{80}(t) =& 100 - 5000t - 1.65153 \times 10^6 t^2 + \dots \\ & + 8.48415 \times 10^{82} t^{159} + 2.36986 \times 10^{81} t^{160}, \\ M_z^{80}(t) =& 100t - 50t^2 + 16.6667t^3 + \dots \\ & + 1.1178 \times 10^{-115} t^{79} - 1.39724 \times 10^{-117} t^{80}. \end{aligned} \quad (32)$$

The simulation results of the problem for $\alpha = 1$ using all of the methods, i.e., the exact formulas (19), the RVIM in Eq. (32), and the PECE method in Eqns. (24)-(31) with the step size $h = 10^{-4}$ are shown in Fig. 1. It can be easily observed that the numerical simulations obtained by the RVIM and the PECE method are in very good agreement with the exact solutions. The absolute errors between numerical solutions, which are calculated using the RVIM and the PECE method, and the exact solutions are shown in Table I and II, respectively. The following numerical conclusions obtained from Table I, II are as below. The approximate solution component $M_z(t)$, obtained by the two methods, is the most accurate when compared with its corresponding exact solution. When t is larger, the approximate solution components $M_x(t)$, $M_y(t)$, achieved by the RVIM, are significantly less accurate compared with their corresponding exact solutions. However, the fluctuation of the absolute errors in the PECE method is appreciably lower than that of the RVIM when t is increasing. From the simulation results of the RVIM and the PECE method compared with the exact solutions, the two methods are reliable and efficient tools for obtaining approximate solutions of the problem for $\alpha = 0.9,\ 0.8$ as well.

Next, we will simulate numerical results of the problem for $\alpha = 0.9,\ 0.8$ as follows. Applying the RVIM to the problem via the iteration formulas (21)-(23), the 80th term

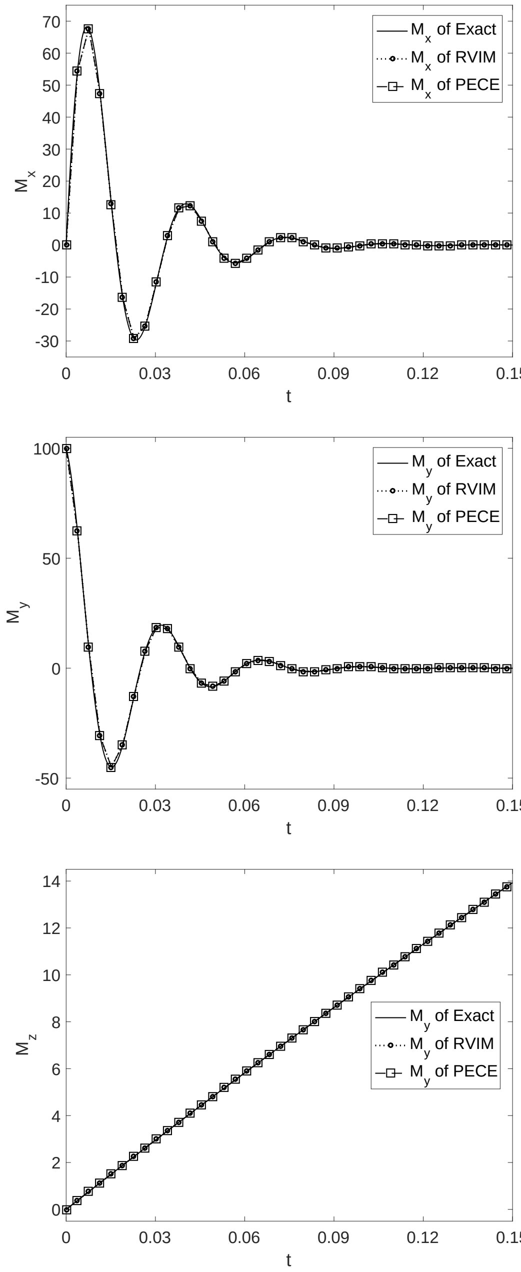

Fig. 1. Simulation comparisons of the solutions $M_x(t)$, $M_y(t)$, $M_z(t)$ for the IVP (1)-(2) using the exact solutions formulas the RVIM, and the PECE method for $\alpha = 1$.

approximations of the solutions for $\alpha = 0.9$ are expressed as

$$\begin{aligned} M_x^{80}(t) =& 18849.6t - 942478t^2 - 8.80607 \times 10^7 t^3 + \dots \\ & - 7.11155 \times 10^{82} t^{158} - 2.0116 \times 10^{81} t^{159}, \\ M_y^{80}(t) =& 100 - 5000t - 1.65153 \times 10^6 t^2 + \dots \\ & + 8.48415 \times 10^{82} t^{159} + 2.36986 \times 10^{81} t^{160}, \\ M_z^{80}(t) =& 100t - 50t^2 + 16.6667t^3 + \dots \\ & + 1.1178 \times 10^{-115} t^{79} - 1.39724 \times 10^{-117} t^{80}. \end{aligned} \quad (33)$$

The obtained 80th term approximations using the RVIM for

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TABLE I
THE ABSOLUTE ERRORS OF NUMERICAL RESULTS OBTAINED BY THE RVIM WITH $N = 80$ COMPARED WITH THE EXACT SOLUTIONS FOR THE IVP (1)-(2) WHEN $\alpha = 1$

t	| Exact-RVIM |		
	$\|\Delta M_x\|$	$\|\Delta M_y\|$	$\|\Delta M_z\|$
0	0	0	0
0.03	1.33E-12	4.36E-13	2.22E-15
0.06	1.97E-10	1.21E-10	2.66E-15
0.09	1.08E-07	3.92E-08	0
0.12	3.52E-06	3.03E-05	5.32E-15
0.15	1.65E-03	1.46E-02	1.77E-15

TABLE II
THE ABSOLUTE ERRORS OF NUMERICAL RESULTS OBTAINED USING THE PECE METHOD WITH $h = 10^{-4}$ COMPARED WITH THE EXACT SOLUTIONS FOR THE IVP (1)-(2) WHEN $\alpha = 1$

t	| Exact-PECE |		
	$\|\Delta M_x\|$	$\|\Delta M_y\|$	$\|\Delta M_z\|$
0	0	0	0
0.03	8.22E-03	1.12E-03	4.85E-09
0.06	3.26E-03	1.75E-03	9.41E-09
0.09	5.39E-04	1.11E-03	1.37E-08
0.12	6.53E-05	3.63E-04	1.77E-08
0.15	7.42E-05	7.11E-05	2.15E-08

$\alpha = 0.8$ are

$$\begin{aligned} M_x^{80}(t) =& 20238.2t^{0.8} - 1.3185 \times 10^6 t^{1.6} - 1.77232 \times 10^8 t^{2.4} \\ &+ \cdots - 8.01025 \times 10^{150} t^{126.4} \\ &- 7.46033 \times 10^{149} t^{127.2}, \\ M_y^{80}(t) =& 100 - 5368.36t^{0.8} - 2.31044 \times 10^6 t^{1.6} + \dots \quad (34) \\ &+ 3.14647 \times 10^{151} t^{127.2} + 2.89747 \times 10^{150} t^{128}, \\ M_z^{80}(t) =& 107.367t^{0.8} - 69.9484t^{1.6} + 33.5435t^{2.4} + \dots \\ &+ 2.19822 \times 10^{-86} t^{63.2} - 7.88103 \times 10^{-88} t^{64}. \end{aligned}$$

In a similar fashion, the numerical simulations results of the problem for $\alpha = 0.9, 0.8$ utilizing the RVIM in Eq. (33)-(34), and the PECE method in Eqns. (24)-(31) with the step size $h = 10^{-4}$ are graphically portrayed in Figs. 2, 3, respectively. It is not difficult to observe from such figures that the numerical results from the RVIM and the PECE method are still in good agreement for $\alpha = 0.9, 0.8$. The absolute differences of the results obtained by the two methods are numerically shown in Table III and IV for $\alpha = 0.9$ and $\alpha = 0.8$, respectively. We can conclude from the last two tables that the two methods provide the numerical data which are quite close to each other for the specified values of t. Especially, the solution component $M_z(t)$, obtained via the two methods, has the lowest discrepancy.

TABLE III
THE ABSOLUTE DIFFERENCES AT THE SPECIFIED VALUES OF t COMPARED USING THE RESULTS OF THE RVIM ($N = 80$) AND THE PECE METHOD ($h = 10^{-4}$) FOR THE IVP (1)-(2) WHEN $\alpha = 0.9$

t	| RVIM-PECE |		
	$\|\Delta M_x\|$	$\|\Delta M_y\|$	$\|\Delta M_z\|$
0	0	0	0
0.02	7.14E-03	1.08E-02	2.52E-06
0.04	6.19E-04	1.02E-03	7.25E-07
0.06	4.13E-05	4.64E-05	1.86E-07
0.08	1.58E-03	6.41E-03	5.45E-08

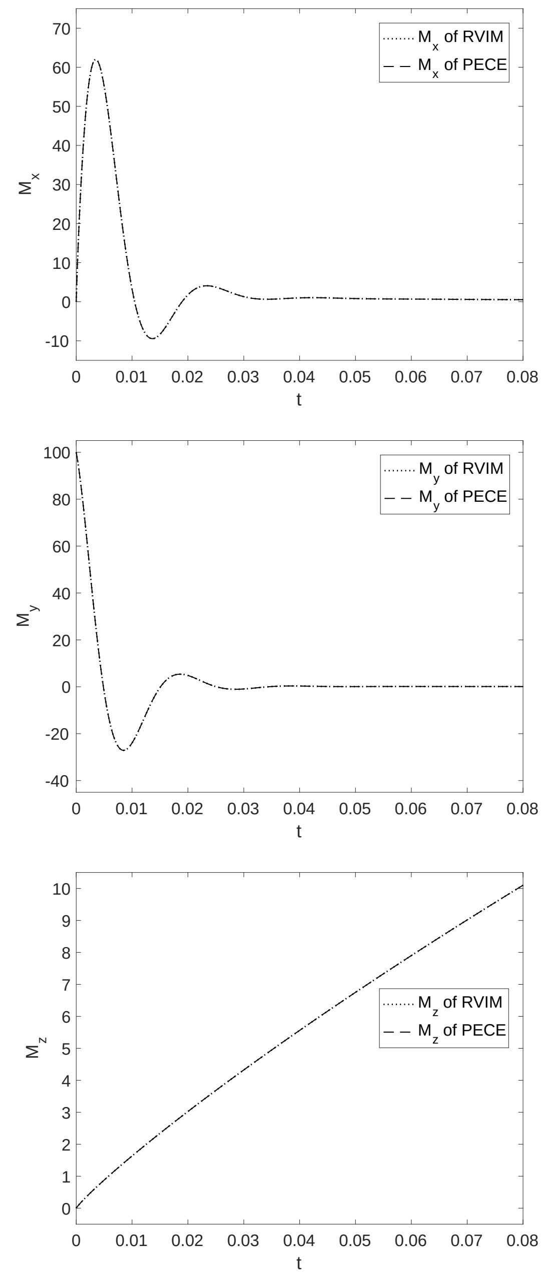

Fig. 2. Simulation comparisons of the solutions $M_x(t)$, $M_y(t)$, $M_z(t)$ for the IVP (1)-(2) using the RVIM, and the PECE method for $\alpha = 0.9$.

V. CONCLUSION

In this paper, we have obtained approximate solutions of the fractional-order initial value problem (1)–(2) for the NMR Bloch equations. Approximate analytical solutions have been computed via the RVIM and approximate numerical solutions have been computed via the PECE method. The simulations of the solutions, calculated by the two methods, have been compared for integer order $\alpha = 1$ and fractional orders $\alpha = 0.9, 0.8$. The exact solutions of the problem for $\alpha = 1$ have been used to measure the accuracy of the solutions obtained by the two approximate methods. A comparison of the simulation results for the two approximate methods with the exact solution for $\alpha = 1$ show that the approximate methods give quite accurate solutions. Comparisons of the approximate analytical and numerical results have shown that the approximate solutions are in

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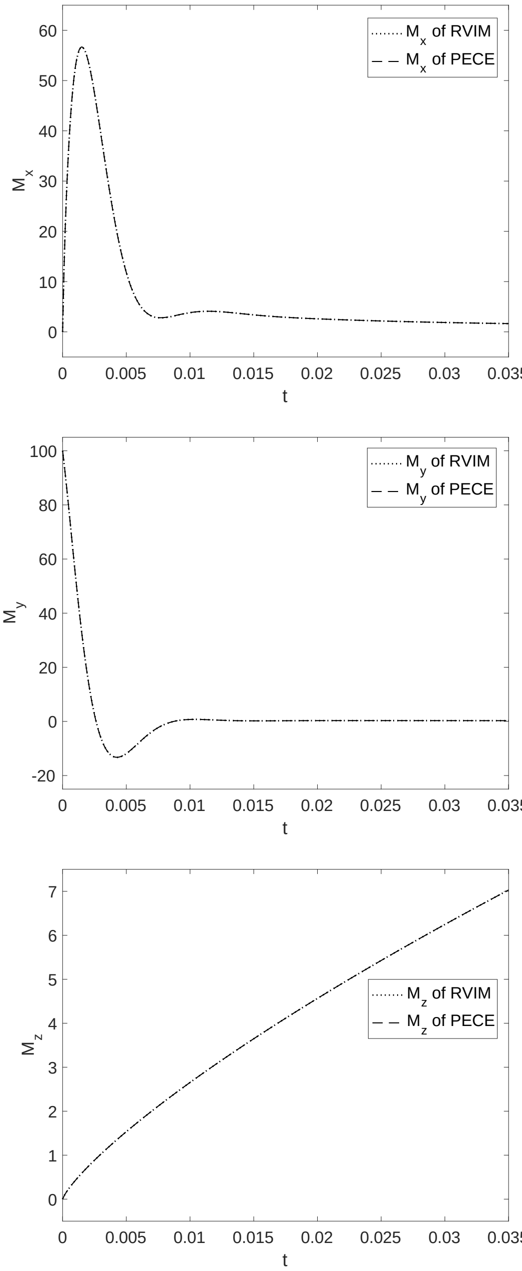

Fig. 3. Simulation comparisons of the solutions $M_x(t)$, $M_y(t)$, $M_z(t)$ for the IVP (1)-(2) using the RVIM, and the PECE method for $\alpha = 0.8$.

good agreement for $\alpha = 0.9$, 0.8. Finally, we believe that the RVIM and PECE methods can be reliably and efficiently applied to solve fractional-order differential equation systems from other engineering and applied science problems.

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TABLE IV
THE ABSOLUTE DIFFERENCES AT THE SPECIFIED VALUES OF t COMPARED USING THE RESULTS OF THE RVIM ($N = 80$) AND THE PECE METHOD ($h = 10^{-4}$) FOR THE IVP (1)-(2) WHEN $\alpha = 0.8$

t	RVIM-PECE		
	$\lvert\Delta M_x\rvert$	$\lvert\Delta M_y\rvert$	$\lvert\Delta M_z\rvert$
0	0	0	0
0.005	5.10E-02	6.80E-02	4.33E-05
0.010	1.57E-02	1.23E-02	1.60E-05
0.015	2.28E-03	7.21E-04	8.08E-06
0.020	4.88E-04	1.43E-05	4.48E-06
0.025	4.86E-05	3.21E-05	2.50E-06
0.030	7.95E-05	2.15E-05	1.28E-06
0.035	5.18E-03	2.001E-03	4.68E-07

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