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Ensemble Kalman Filtering for Inverse Optimal Control

Andrea Arnold and Hien Tran, *Member, SIAM*

***Abstract*—Solving the inverse optimal control problem for discrete-time nonlinear systems requires the construction of a stabilizing feedback control law based on a control Lyapunov function (CLF). However, there are few systematic approaches available for defining appropriate CLFs. We propose an approach that employs Bayesian filtering methodology to parameterize a quadratic CLF. In particular, we use the ensemble Kalman filter (EnKF) to estimate parameters used in defining the CLF within the control loop of the inverse optimal control problem formulation. Using the EnKF in this setting provides a natural link between uncertainty quantification and optimal design and control, as well as a novel and intuitive way to find the one control out of an ensemble that stabilizes the system the fastest. Results are demonstrated on both a linear and nonlinear test problem.**

***Index Terms*—inverse optimal control, Bayesian statistics, nonlinear filtering, ensemble Kalman filter (EnKF).**

I. Introduction

THE aim of nonlinear optimal control [1], [2] is to determine a control law for a given system that minimizes a cost functional relating the state and control variables. The solution to this problem relies on solving the Hamilton-Jacobi-Bellman (HJB) equation, which has been solved for linear systems [3] but is very difficult to solve for general nonlinear systems [4], [5]. An alternate approach is to find a stabilizing feedback control first, then establish that it optimizes a specified cost functional – this is known as the inverse optimal control problem.

Solving the inverse optimal control problem for discrete-time nonlinear systems requires the construction of a stabilizing feedback control law based on a control Lyapunov function (CLF). However, there are few systematic approaches available for defining appropriate CLFs. Available methods parameterize quadratic CLFs using a recursive speed-gradient algorithm [6], particle swarm optimization [7] or, more recently, the extended Kalman filter (EKF) [8].

This work develops a novel approach employing Bayesian filtering methodology to parameterize a quadratic CLF. In particular, we use the ensemble Kalman filter (EnKF) to estimate parameters used in defining the CLF within the control loop of the inverse optimal control problem. Using the EnKF in this setting provides a natural link between uncertainty quantification and optimal design and control, as well as an intuitive way to find the one control out of an ensemble that drives the system to zero the fastest.

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Andrea Arnold is with the Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA, e-mail: anarnold@wpi.edu.

Hien Tran is with the Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA, e-mail: tran@ncsu.edu.

In the Bayesian framework, unknown parameters are modeled as random variables with probability density functions representing distributions of possible values. The EnKF is a nonlinear Bayesian filter which uses ensemble statistics in combination with the classical Kalman filter equations for state and parameter estimation [9]–[11]. The EnKF has been employed in many settings, including weather prediction [12], [13] and mathematical biology [11]. To the authors' knowledge, this is the first proposed use of the EnKF in inverse optimal control problems. The novelty of using the EnKF in this setting allows us to generate an ensemble of control laws, from which we can then select the control law that drives the system to zero the fastest. While the nonlinear problem has no guarantee of a unique control, we use the control ensemble to find the best solution starting from a prior distribution of possible controls.

The paper is organized as follows. We review the main ideas behind optimal control and inverse optimal control in Section II and nonlinear Bayesian filtering and the EnKF in Section III. In Section IV, we describe the application of the EnKF to parametrizing the CLF for inverse optimal control problem. The results in Section V demonstrate the effectiveness of the EnKF CLF procedure on both a linear and nonlinear test example.

II. Optimal and Inverse Optimal Control

In this section we describe the optimal control problem and inverse optimal control problem for discrete-time nonlinear systems using similar notations as in [8]. For details on feedback control methodology for nonlinear dynamic systems, see, e.g., [14].

Consider the discrete-time affine nonlinear system

$$x_{k+1} = f(x_k) + g(x_k)u_k, \quad x_0 = x(0), \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system at time k, $u_k \in \mathbb{R}^m$ is the control input at time k, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth mappings with $f(0) = 0$ and $g(x_k) \neq 0$ for all $x_k \neq 0$. The nonlinear optimal control problem is to determine a control law u_k that minimizes the associated cost functional

$$V(x_k) = \sum_{n=k}^{\infty} \Big(L(x_n) + u_n^{\mathsf{T}} \mathsf{E} u_n \Big), \tag{2}$$

where $V : \mathbb{R}^n \to \mathbb{R}_+$ has $V(0) = 0$, $L : \mathbb{R}^n \to \mathbb{R}_+$ is positive semidefinite, and E is a real, symmetric positive definite $m \times m$ weighting matrix. The boundary condition $V(0) = 0$ is necessary so that $V(x_k)$ can be used as a CLF. The cost functional (2) can be rewritten as

$$V(x_k) = L(x_k) + u_k^{\mathsf{T}} \mathsf{E} u_k + V(x_{k+1}). \tag{3}$$

For an infinite horizon control problem, the time-invariant function $V^*(x_k)$ satisfies the discrete-time Bellman equation

$$V^*(x_k) = \min_{u_k} \left\{ L(x_k) + u_k^{\mathsf{T}} \mathsf{E} u_k + V^*(x_{k+1}) \right\}. \quad (4)$$

Taking the gradient of (4) with respect to u_k yields the optimal control

$$u_k^* = -\frac{1}{2} \mathsf{E}^{-1} g^{\mathsf{T}}(x_k) \frac{\partial V^*(x_{k+1})}{\partial x_{k+1}} \quad (5)$$

which, when substituting into (3), yields the discrete-time Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} V^*(x_k) &= L(x_k) + V^*(x_{k+1}) \\ &+ \frac{1}{4} \frac{\partial V^{*\mathsf{T}}(x_{k+1})}{\partial x_{k+1}} g(x_k) \mathsf{E}^{-1} g^{\mathsf{T}}(x_k) \frac{\partial V(x_{k+1})}{\partial x_{k+1}}. \end{aligned} \quad (6)$$

Since solving the discrete-time HJB equation (6) is very difficult for general nonlinear systems, an alternative approach is to consider the inverse optimal control problem. In inverse optimal control, the first step is to construct a stabilizing feedback control law, then to establish that the control law optimizes a given cost functional. By definition, the control law

$$u_k^* = -\frac{1}{2} \mathsf{E}^{-1} g^{\mathsf{T}}(x_k) \frac{\partial V^*(x_{k+1})}{\partial x_{k+1}} \quad (7)$$

is inverse optimal if it satisfies the following two criteria:

1) It achieves (global) exponential stability of the equilibrium point $x_k = 0$ for the system (1).
2) It minimizes the defined cost functional (2), for which $L(x_k) = -\overline{V}$ with

$$\overline{V} := V(x_{k+1}) - V(x_k) + u_k^{*\mathsf{T}} \mathsf{E} u_k^* \leq 0, \quad (8)$$

   where $V(x_k)$ is positive definite.

A control law satisfying the above definition can be defined using a quadratic control Lyapunov function (CLF) of the form

$$V(x_k) = \frac{1}{2} x_k^{\mathsf{T}} \mathsf{P} x_k, \quad (9)$$

where the matrix $\mathsf{P} \in \mathbb{R}^{n \times n}$ is symmetric positive definite (i.e., $\mathsf{P} = \mathsf{P}^{\mathsf{T}} > 0$). Once an appropriate CLF (9) has been selected, the state feedback control law (7) becomes

$$u_k^* = -\frac{1}{2} \Big(\mathsf{E} + \frac{1}{2} g^{\mathsf{T}}(x_k) \mathsf{P} g(x_k) \Big)^{-1} g^{\mathsf{T}}(x_k) \mathsf{P} f(x_k). \quad (10)$$

Therefore, the problem at hand is to select an appropriate matrix P to achieve stability and minimize a meaningful cost function.

As noted in the introduction, currently proposed methods to estimate the entries of the matrix P in the CLF (9) include a recursive speed-gradient algorithm [6], particle swarm optimization [15], and, more recently, use of the extended Kalman filter [8]. In this work, we propose use of Bayesian filtering techniques, in particular the ensemble Kalman filter (EnKF), to estimate the entries of the matrix P from a distribution of possible values, which allows us to find the best control out of an ensemble.

III. Nonlinear Bayesian Filtering and the EnKF

We approach the solution to the inverse optimal control problem from the Bayesian statistical framework, using nonlinear Bayesian filtering methodology to parameterize the quadratic CLF. In the Bayesian framework, the quantities of interest (such as the system states or parameters) are treated as random variables with probability distributions, and their joint posterior density is assembled using Bayes' theorem. In particular, if x denotes the states of a system and y some partial, noisy system observations, then Bayes' theorem gives

$$\pi(x \mid y) \propto \pi(y \mid x) \pi(x), \quad (11)$$

where the likelihood function $\pi(y \mid x)$ indicates how likely it is that the data y are observed if the state values were known and the prior distribution $\pi(x)$ encodes any known information on the states before taking the data into account.

Bayesian filtering methods rely on the use of discrete-time stochastic equations describing the model states and observations to sequentially update the joint posterior density. Assuming a time discretization t_k, $k = 0, 1, \ldots, T$, with the observations y_k occurring possibly in a subset of the discrete time instances (where $y_k = \emptyset$ if there is no observation at t_k), we can write an evolution-observation model for the stochastic state and parameter estimation problem using discrete-time Markov models. The state evolution equation

$$X_{k+1} = F(X_k) + V_{k+1}, \quad V_{k+1} \sim \mathcal{N}(0, \mathsf{Q}_{k+1}), \quad (12)$$

where F is a known propagation model and V_{k+1} is an innovation process, computes the forward time propagation of the state variables X_k given parameters θ, while the observation equation

$$Y_{k+1} = G(X_{k+1}) + W_{k+1}, \quad W_{k+1} \sim \mathcal{N}(0, \mathsf{R}_{k+1}), \quad (13)$$

where G is a known operator and W_{k+1} is the observation noise, predicts the observation at time t_{k+1} based on the current state and parameter values.

Letting $D_k = \{y_1, y_2, \ldots, y_k\}$ denote the set of observations up to time t_k, the stochastic evolution-observation model allows us to sequentially update the posterior distribution $\pi(x_k \mid D_k)$ using a two-step, predictor-corrector-type scheme:

$$\pi(x_k \mid D_k) \;\rightarrow\; \pi(x_{k+1} \mid D_k) \;\rightarrow\; \pi(x_{k+1} \mid D_{k+1}) \quad (14)$$

The first step (the prediction step) employs the state evolution equation (12) to predict the values of the states at time t_{k+1} without knowledge of the data. The second step (the analysis step or observation update) then uses the observation equation (13) to correct the prediction by taking into account the data at time t_{k+1}. If there is no data observed at t_{k+1}, then $D_{k+1} = D_k$ and the prediction density $\pi(x_{k+1} \mid D_k)$ is equivalent to the posterior $\pi(x_{k+1} \mid D_{k+1})$. Starting with a prior density $\pi(x_0 \mid D_0)$, $D_0 = \emptyset$, this updating scheme is repeated until the final posterior density is obtained when $k = T$.

A. Ensemble Kalman Filter

The ensemble Kalman filter [9], [10] is a Bayesian filter which uses ensemble statistics in combination with the classical Kalman filter equations to accommodate nonlinear models. While there are versions of the EnKF that perform

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joint state and parameter estimation [11], [16], for our purposes we need only consider the standard EnKF for state estimation, which will be adapted in the following section for the inverse optimal control problem. To avoid confusion with the states of the control system (1), here we denote the states in the filter as a_k, $k = 0, \dots, T$, as opposed to the typical x_k notation. The EnKF algorithm for state estimation is outlined as follows.

Assume the current density $\pi(a_k \mid D_k)$ at time t_k is represented in terms of a discrete ensemble of size N,

$$\mathcal{S}_{k|k} = \left\{(a^1_{k|k}), (a^2_{k|k}), \dots, (a^N_{k|k})\right\}. \tag{15}$$

In the prediction step, the states at time t_{k+1} are predicted using the state evolution equation (12) to form a state prediction ensemble,

$$a^j_{k+1|k} = F(a^j_{k|k}) + v^j_{k+1}, \quad j = 1, \dots, N, \tag{16}$$

where $v^j_{k+1} \sim \mathcal{N}(0, \mathsf{Q}_{k+1})$ represents error in the model prediction. Ensemble statistics yield the prediction ensemble mean

$$\overline{a}_{k+1|k} = \frac{1}{N}\sum_{j=1}^{N} a^j_{k+1|k} \tag{17}$$

and covariance matrix

$$\Gamma_{k+1|k} = \frac{1}{N-1}\sum_{j=1}^{N}(a^j_{k+1|k} - \overline{a}_{k+1|k})(a^j_{k+1|k} - \overline{a}_{k+1|k})^{\mathsf{T}}. \tag{18}$$

When an observation y_{k+1} arrives, an artificial observation ensemble is generated around the true observation, such that

$$y^j_{k+1} = y_{k+1} + w^j_{k+1}, \quad j = 1, \dots, N \tag{19}$$

where $w^j_{k+1} \sim \mathcal{N}(0, \mathsf{R}_{k+1})$ represents the observation error. The observation ensemble is compared to the observation model predictions,

$$\widehat{y}^j_{k+1} = G(a^j_{k+1|k}), \quad j = 1, \dots, N \tag{20}$$

which is computed using the observation function G as in (13). The posterior ensemble at time t_{k+1} is then computed by

$$a^j_{k+1|k+1} = a^j_{k+1|k} + \mathsf{K}_{k+1}\big(y^j_{k+1} - \widehat{y}^j_{k+1}\big) \tag{21}$$

for each $j = 1, \dots, N$, where the Kalman gain is defined as

$$\mathsf{K}_{k+1} = \Sigma^{a\widehat{y}}_{k+1}\big(\Sigma^{\widehat{y}\widehat{y}}_{k+1} + \mathsf{R}_{k+1}\big)^{-1} \tag{22}$$

with $\Sigma^{a\widehat{y}}_{k+1}$ denoting the cross covariance of the state predictions $a^j_{k+1|k}$ and observation predictions $\widehat{y}^j_{k+1}$, $\Sigma^{\widehat{y}\widehat{y}}_{k+1}$ the forecast error covariance of the observation prediction ensemble, and R_{k+1} the observation noise covariance. This formulation of the Kalman gain straightforwardly allows for nonlinear observations, as opposed to the more familiar formula for linear observation models [17]. Use of the artificial observation ensemble (19) ensures that the resulting posterior ensemble in (21) does not have too low a variance [10]. The posterior means and covariances for the states are then computed using posterior ensemble statistics, and the process repeats.

IV. EnKF for Inverse Optimal Control

To apply the EnKF to the inverse optimal control problem, we treat the entries of the symmetric positive definite P defining the quadratic CLF in (9) as the states of the filter and apply the following updating procedure to find the control that drives the system to zero the fastest. At time k, assume a discrete ensemble of P matrices

$$\mathsf{P}^j_{k|k} = \begin{bmatrix} \mathsf{P}^j_{1,1} & \cdots & \mathsf{P}^j_{1,n} \\ \vdots & \ddots & \vdots \\ \mathsf{P}^j_{1,n} & \cdots & \mathsf{P}^j_{n,n} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad j = 1, \dots, N, \tag{23}$$

where each $\mathsf{P}^j_{k|k}$ is symmetric positive definite. Using symmetry to our advantage, we need only update the upper triangular entries, which we place into vectors

$$p^j_{k|k} = \begin{bmatrix} \mathsf{P}^j_{1,1} \\ \vdots \\ \mathsf{P}^j_{1,n} \\ \vdots \\ \mathsf{P}^j_{n,n} \end{bmatrix} \in \mathbb{R}^{\widehat{n}}, \quad \widehat{n} = \frac{n(n+1)}{2}. \tag{24}$$

As in the prediction step of the filter, we generate a prediction ensemble

$$p^j_{k+1|k} = p^j_{k|k} + v^j_{k+1}, \quad v^j_{k+1} \sim \mathcal{N}(0, \mathsf{Q}), \tag{25}$$

where here the propagation function in equation (16) is the $\widehat{n} \times \widehat{n}$ identity matrix and the covariance of the innovation term v^j_{k+1} is some constant matrix Q. Prediction ensemble statistics can be computed as in (17)–(18), however they are not needed for the remaining computations.

Reformulating the prediction ensemble vectors $\{p^j_{k+1|k}\}_{j=1}^N$ into matrices $\{\mathsf{P}^j_{k+1|k}\}_{j=1}^N$, we can compute the corresponding predicted controls, states, and root mean square error (RMSE) values for each ensemble member using the following formulas. The predicted controls are given by

$$u^j_{k+1|k} = -\frac{1}{2}\Big(\mathsf{E} + \frac{1}{2}g^{\mathsf{T}}(x^j_{k|k})\mathsf{P}^j_{k+1|k}g(x^j_{k|k})\Big)^{-1} \cdot g^{\mathsf{T}}(x^j_{k|k})\mathsf{P}^j_{k+1|k}f(x^j_{k|k}) \tag{26}$$

as in (10) for each $j = 1, \dots, N$, which are then used to generate the state prediction ensemble

$$x^j_{k+1|k} = f(x^j_{k|k}) + g(x^j_{k|k})u^j_{k+1|k}, \quad j = 1, \dots, N \tag{27}$$

as in (1).

For the analysis step of the filter, we interpret as "observations" the RMSE values of the states as we drive them to zero. Since the aim is to find a control that drives the RMSE to zero, we treat RMSE = 0 as the true "observation" and generate an observation ensemble using the prescribed observation noise covariance matrix R as follows:

$$RMSE^j_{\text{obs}} = w^j_{k+1}, \quad w^j_{k+1} \sim \mathcal{N}(0, \mathsf{R}). \tag{28}$$

We then compare the "observed" RMSEs to the RMSEs of the predicted states, given by

$$\text{RMSE}^j_{k+1|k} = \sqrt{\frac{(x^j_{k+1|k})^2_1 + (x^j_{k+1|k})^2_2 + \cdots + (x^j_{k+1|k})^2_n}{n}} \tag{29}$$

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and compute the posterior ensemble as in (21) using

$$p^j_{k+1|k+1} = p^j_{k+1|k} + \mathsf{K}_{k+1}(\mathrm{RMSE}^j_{\mathrm{obs}} - \mathrm{RMSE}^j_{k+1|k}), \quad (30)$$

where the Kalman gain is defined as in (22) with $\Sigma^{a\widehat{y}}_{k+1}$ denoting the cross covariance of the predictions $p^j_{k+1|k}$ and RMSE predictions $\mathrm{RMSE}^j_{k+1|k}$, $\Sigma^{\widehat{y}\widehat{y}}_{k+1}$ the forecast error covariance of the RMSE prediction ensemble, and R the observation noise covariance. Posterior control law, state, and RMSE ensembles can be computed after reformulating the posterior ensemble of entry vectors $p^j_{k+1|k+1}$ into their corresponding matrices, and ensemble statistics can be computed.

This process is repeated for each successive time step until an appropriate control is found, based on some prescribed stopping criterion. In particular, if we want to find the control that drives the system to zero the fastest, we can stop when the minimum RMSE of all ensemble members is less than some prescribed tolerance.

V. Results

We demonstrate the effectiveness of the proposed methodology on two illustrated examples, one involving a linear system and the other a nonlinear system.

A. Example: Linear System

Consider the discrete-time linear system

$$x_{k+1} = \begin{bmatrix} 0.9974 & 0.0539 \\ -0.1078 & 1.1591 \end{bmatrix} x_k + \begin{bmatrix} 0.0013 \\ 0.0539 \end{bmatrix} u_k \quad (31)$$

with initial point $x_0 = [2, 1]$, where the goal is to minimize the performance measure

$$J = \frac{1}{2} \sum_{k=0}^{N-1} \left[0.25(x_1)^2_k + 0.05(x_2)^2_k + 0.05u^2_k \right]$$

as described in [18]. We set up the EnKF CLF estimator by letting

$$\mathsf{E} = 0.05, \qquad \mathsf{Q} = q_0 \mathsf{I}_2, \qquad \mathsf{R} = r_0, \quad (32)$$

where $q_0 = 1 \times 10^{-4}$ and $r_0 = 1 \times 10^{-3}$. We generate uniform prior of size $N = 1000$ ensemble members on the upper-triangular entries of the P matrix with minimum value 0.05 and maximum value 0.2. We set the stopping criterion such that the filter stops when $\min(\mathrm{RMSE}) < 1 \times 10^{-3}$ to find control that drives system to zero the fastest. After 103 steps, the procedure results in the following matrix P defining the CLF:

$$\mathsf{P} = \begin{bmatrix} 1.2429 & 1.7809 \\ 1.7809 & 2.7101 \end{bmatrix}. \quad (33)$$

Fig. 1 shows the corresponding states x_1 and x_2, control u, and RMSE. After an initial transient, the control defined by this resulting CLF matches the expected control for the linear system (31) [18].

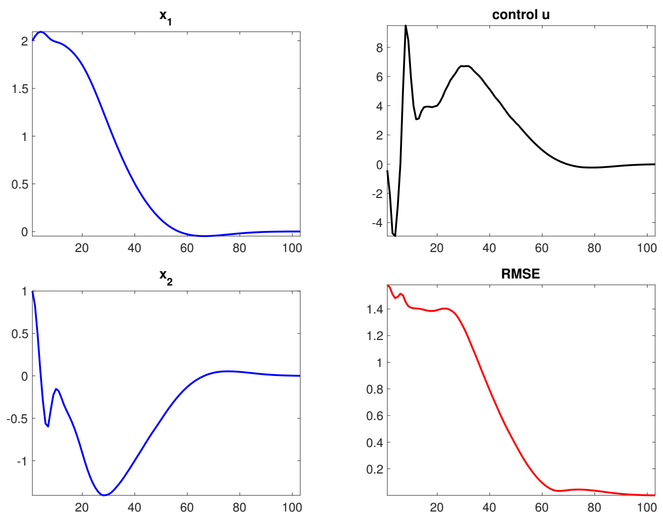

Fig. 1. The states x_1 and x_2, control u, and RMSE for the linear system (31) corresponding to the CLF with P matrix (33) estimated using the EnKF CLF procedure.

B. Example: Nonlinear System

Consider the discrete-time nonlinear system

$$x_{k+1} = f(x_k) + g(x_k)u_k, \quad x_0 = \begin{bmatrix} 2.5 \\ -1 \end{bmatrix} \in \mathbb{R}^2 \quad (34)$$

where

$$f(x_k) = \begin{bmatrix} 2x_{1,k} \sin(0.5x_{1,k}) + 0.1x^2_{2,k} \\ 0.1x^2_{1,k} + 1.8x_{2,k} \end{bmatrix} \quad (35)$$

and

$$g(x_k) = \begin{bmatrix} 0 \\ 2 + 0.1\cos(x_{2,k}) \end{bmatrix}. \quad (36)$$

This system is also considered in [8]. Here we take

$$\mathsf{E} = 1, \qquad \mathsf{Q} = q_0 \mathsf{I}_2, \qquad \mathsf{R} = r_0 \quad (37)$$

with $q_0 = 1 \times 10^{-2}$ and $r_0 = 1 \times 10^{-3}$, and we generate a uniform prior ensemble of size $N = 1,000$ on the upper-triangular entries of P with minimum value 0.001 and maximum value 0.1. We stop the algorithm when $\min(\mathrm{RMSE}) < 1 \times 10^{-3}$. After 9 steps, the resulting matrix P defining the CLF that drives the system to zero the fastest is given by

$$\mathsf{P} = \begin{bmatrix} 0.2089 & 0.2300 \\ 0.2300 & 0.2604 \end{bmatrix}. \quad (38)$$

Fig. 2 shows the corresponding states x_1 and x_2, control u, and RMSE.

VI. Conclusion

In this work we present a novel approach using nonlinear Bayesian filtering, in particular the EnKF, to parameterize a quadratic CLF for inverse optimal control. We demonstrate the effectiveness of using the EnKF to estimate the upper-triangular entries of the symmetric positive definite matrix P in (10) on both a linear and nonlinear example. Since the nonlinear problem does not guarantee a unique solution, the ensemble formulation allows us to find the control that stabilizes the system the fastest (i.e. using the least amount of steps).

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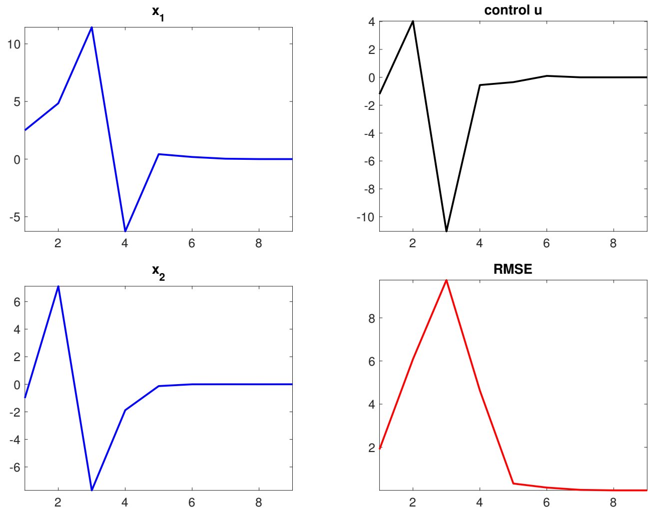

Fig. 2. The states x_1 and x_2, control u, and RMSE for the nonlinear system (34) corresponding to the CLF with P matrix (38) estimated using the EnKF CLF procedure.

While the EnKF was our filter of choice in this work, the algorithm can be straightforwardly modified to use other filtering schemes (such as particle filters) instead. Future work may study if the choice of filter affects the number of iterations needed to find a CLF matching the prescribed criteria. We also plan to apply the EnKF CLF method to an application relating to HIV drug therapy.

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