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Piecewise Multi-Linear Model Based Control for TORA System via Feedback Linearization

Tadanari Taniguchi and Michio Sugeno

***Abstract*—This paper deals with a piecewise model based controller design for nonlinear systems via feedback linearization. The model is a piecewise multi-linear system and a nonlinear approximation. The approximated model is fully parametric. Feedback linearization is applied to stabilize PML (Piecewise Multi-Linear) control system. We apply the piecewise model based controller to TORA (Translational Oscillator with Rotating Actuator) system. Although the controller is simpler than the conventional feedback linearization controller, the PML model based control can be applied to a wider region than the conventional one. Examples are shown to confirm the feasibility of our proposals by computer simulations.**

***Index Terms*—nonlinear control, feedback linearization, tora system, piecewise model.**

I. Introduction

We propose a piecewise model based controller design for nonlinear systems via feedback linearization. We apply the piecewise model based controller to TORA (Translational Oscillator with Rotating Actuator) system. The TORA system [1] has a cart of mass M connected to a wall with a linear spring (constant k). The cart can oscillate without friction in the horizontal plane. A rotating mass m in the cart is actuated by a motor. The mass is eccentric with a radius of eccentricity e and can be imagined to be a point mass mounted on a massless rotor. The rotating motion of the mass m controls the oscillation of the cart (see Fig. 2). TORA system is difficult to control because the system has a complex nonlinear dynamics.

Many methods have been studied for the stabilizing control of TORA system. The exact feedback linearization method is proposed in [1]. However the controller is limited in the angle of the rotor. The cascade and passivity based control designs is proposed in [2]. This method can be applied to the limited region of the state variable. The model-based fuzzy controls are proposed in [3], [4].

In this paper, we consider piecewise multi-linear (PML) model as a piecewise approximation model of TORA system. The model is built on hyper cubes partitioned in state space and is found to be bilinear (bi-affine) [5], so the model has simple nonlinearity. The model has the following features: 1) The PML model is derived from fuzzy if-then rules with singleton consequents. 2) It has a general approximation capability for nonlinear systems. 3) It is a piecewise nonlinear model and second simplest after the piecewise linear (PL) model. 4) It is continuous and fully parametric. The stabilizing conditions are represented by bilinear matrix inequalities (BMIs) [6], therefore, it takes long computing time to obtain a stabilizing controller. To overcome these difficulties, we have derived stabilizing conditions [7], [8], [9] based on feedback linearization, where [7] and [9] apply input-output linearization and [8] applies full-state linearization. The control system has the following features: 1) Only partial knowledge of vertices in piecewise regions is necessary, not overall knowledge of an objective plant. 2) Although the structure of the PML controller is very simple, the PML control system can be applied to a wider region than the feedback linearization [1].

This paper is organized as follows. Section II introduces the canonical form of PML models. Section III briefly presents TORA system. Section IV presents a stabilizing controller design using exact feedback linearization. Section V-VI propose a PML modeling and a PML model based control for TORA system via exact feedback linearization. Section VII shows some examples demonstrating the feasibility of the proposed methods. Finally, section VIII summarizes conclusions.

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T. Taniguchi is with IT Education Center, Tokai University, Hiratsuka, Kanagawa, 2591292 Japan email:taniguchi@tokai-u.jp
M. Sugeno is with Tokyo Institute of Technology.

II. Canonical Forms of Piecewise Multi-Linear Models

A. Open-Loop Systems

In this section, we introduce PML models suggested in [5]. We deal with the two-dimensional case without loss of generality. Define vector $d(\sigma,\tau)$ and rectangle $R_{\sigma\tau}$ in two-dimensional space as $d(\sigma,\tau) \equiv (d_1(\sigma), d_2(\tau))^T$,

$$R_{\sigma\tau} \equiv [d_1(\sigma), d_1(\sigma+1)] \times [d_2(\tau), d_2(\tau+1)]. \quad (1)$$

σ and τ are integers: $-\infty < \sigma, \tau < \infty$ where $d_1(\sigma) < d_1(\sigma+1), d_2(\tau) < d_2(\tau+1)$ and $d(0,0) \equiv (d_1(0), d_2(0))^T$. Superscript ^{T} denotes a *transpose* operation.

We consider a two-dimensional nonlinear system:

$$\dot{x} = f(x)$$

For $x = (x_1, x_2) \in R_{\sigma\tau}$, the PML system is expressed as

$$\begin{cases} \dot{x} = f_p(x) = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ x = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j), \end{cases} \quad (2)$$

where $f(i,j)$ is the vertex of nonlinear system $\dot{x} = f(x)$,

$$\begin{cases} \omega_1^\sigma(x_1) = \dfrac{(d_1(\sigma+1) - x_1)}{(d_1(\sigma+1) - d_1(\sigma))}, \\ \omega_1^{\sigma+1}(x_1) = \dfrac{(x_1 - d_1(\sigma))}{(d_1(\sigma+1) - d_1(\sigma))}, \\ \omega_2^\tau(x_2) = \dfrac{(d_2(\tau+1) - x_2)}{(d_2(\tau+1) - d_2(\tau))}, \\ \omega_2^{\tau+1}(x_2) = \dfrac{(x_2 - d_2(\tau))}{(d_2(\tau+1) - d_2(\tau))} \end{cases} \quad (3)$$

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and $\omega_1^i(x_1), \omega_2^j(x_2) \in [0,\ 1]$. In the above, we assume $f(0,0) = 0$ and $d(0,0) = 0$ to guarantee $\dot{x} = 0$ for $x = 0$.

A key point in the system is that state variable x is also expressed by a convex combination of $d(i,j)$ for $\omega_1^i(x_1)$ and $\omega_2^j(x_2)$, just as in the case of $\dot{x}$. As seen in equation (3), x is located inside $R_{\sigma\tau}$ which is a rectangle: a hypercube in general. That is, the expression of x is polytopic with four vertices $d(i,j)$. The model of $\dot{x} = f(x)$ is built on a rectangle including x in state space, it is also polytopic with four vertices $f(i,j)$. We call this form of the canonical model (2) parametric expression.

B. Closed-Loop Systems

We consider a two-dimensional nonlinear control system.

$$\begin{cases} \dot{x} = f(x) + g(x)u(x), \\ y = h(x). \end{cases} \tag{4}$$

For $x \in R_{\sigma\tau}$, the PML model (5) is constructed from a nonlinear system (4).

$$\begin{cases} \dot{x} = f_p(x) + g_p(x)u(x), \\ y = h_p(x), \end{cases} \tag{5}$$

where

$$\begin{cases} f_p(x) = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ g_p(x) = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) g(i,j), \\ h_p(x) = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) h(i,j), \\ x = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j), \end{cases} \tag{6}$$

and $f(i,j)$, $g(i,j)$, $h(i,j)$ and $d(i,j)$ are vertices of the nonlinear system (4). The modeling procedure in region $R_{\sigma\tau}$ is as follows:

1) Assign vertices $d(i,j)$ for $x_1 = d_1(\sigma)$, $d_1(\sigma+1)$, $x_2 = d_2(\tau)$, $d_2(\tau+1)$ of state vector x, then partition state space into piecewise regions (see Fig. 1).
2) Compute vertices $f(i,j)$, $g(i,j)$ and $h(i,j)$ in equation (6) by substituting values of $x_1 = d_1(\sigma)$, $d_1(\sigma+1)$ and $x_2 = d_2(\tau)$, $d_2(\tau+1)$ into original nonlinear functions $f(x)$, $g(x)$ and $h(x)$ in the system (4). Fig. 1 shows the expression of $f_1(x)$ and $x \in R_{\sigma\tau}$.

The overall PML model is obtained automatically when all vertices are assigned. Note that $f(x)$, $g(x)$ and $h(x)$ in the PML model coincide with those in the original system at vertices of all regions.

III. TORA SYSTEM

The TORA (Translational Oscillator with Rotating Actuator) system [1] has a cart of mass M connected to a wall with a linear spring (constant k). The cart can oscillate without friction in the horizontal plane. A rotating mass m in the cart is actuated by a motor. The mass is eccentric with a radius of eccentricity e and can be imagined to be a point mass mounted on a massless rotor. The rotating motion of the mass m controls the oscillation of the cart. The motor torque is the control variable. The dynamics of TORA system is

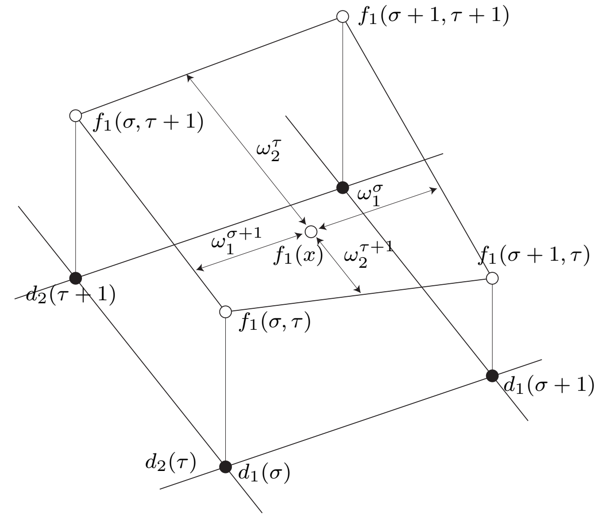

Fig. 1. Piecewise region ($f_{p_1}(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i\omega_2^j f_1(i,j)$, $x \in R_{\sigma\tau}$)

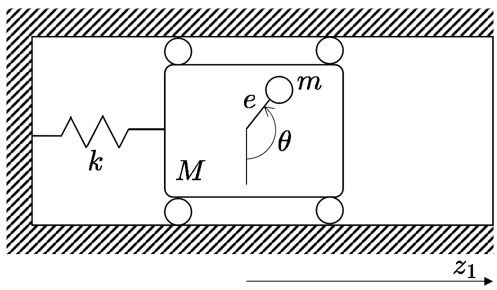

Fig. 2. Kinematic model of TORA system

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = \dfrac{-z_1 + \varepsilon z_4^2 \sin z_3}{1 - \varepsilon^2 \cos^2 z_3} - \dfrac{-\varepsilon \cos z_3}{1 - \varepsilon^2 \cos^2 z_3} v \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = \dfrac{1}{1 - \varepsilon^2 \cos^2 z_3} \left\{ \varepsilon \cos z_3 \left(z_1 - \varepsilon z_4^2 \sin z_3 \right) + v \right\} \\ y = z_1, \end{cases} \tag{7}$$

where z_1 and z_2 are the position and velocity of the cart. $z_3 = \theta$ and $z_4 = \dot{\theta}$ are the angle and angular velocity of the rotor. The parameter ε depends on the eccentricity e and the masses M and m. v and y are the control input and output.

The TORA system dynamics has many nonlinear terms. In the case of a coordinate transformation

$$\begin{aligned} x_1 &= z_1 + \varepsilon \sin z_3 \\ x_2 &= z_2 + \varepsilon z_4 \cos z_3 \\ x_3 &= z_3 \\ x_4 &= z_4 \\ u &= \frac{\varepsilon \cos z_3 (x_1 - \varepsilon \sin z_3 (1 + z_4^2)) + v}{1 - \varepsilon^2 \cos^2 z_3}, \end{aligned}$$

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we have

$$\begin{cases} \dot{x} = f + gu = \begin{pmatrix} x_2 \\ -x_1 + \varepsilon \sin x_3 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y = h = x_1, \end{cases} \tag{8}$$

where $x \in R^4$, $y \in R$.

IV. Controller design of TORA system via exact feedback linearization

We design the controller of TORA system (8) via the exact feedback linearization [10]. We calculate the time derivatives of the output y until the input u appears.

$$\begin{aligned} y &= h = x_1, \\ \dot{y} &= L_f h = x_2 \\ y^{(2)} &= L_f^2 h = -x_1 + \varepsilon \sin x_3 \\ y^{(3)} &= L_f^3 h = -x_2 + \varepsilon \cos x_3 \\ y^{(4)} &= L_f^4 h + L_g L_f^3 h u \\ &= x_1 - \varepsilon \sin x_3 - \varepsilon x_4^2 \sin x_3 + \varepsilon \cos x_3 u \end{aligned}$$

Then the controller is obtained as

$$\begin{aligned} u &= \frac{-L_f^4 h}{L_g L_f^3} + \frac{1}{L_g L_f^3} \mu \\ &= \frac{-x_1 + \varepsilon \sin x_3 + \varepsilon x_4^2 \sin x_3}{\varepsilon \cos x_3} + \frac{1}{\varepsilon \cos x_3} \mu, \end{aligned} \tag{9}$$

where μ is the linear controller for the linearized system (10).

$$\begin{cases} \dot{\xi} = A\xi + B\mu \\ y = C\xi, \end{cases} \tag{10}$$

where $\xi = (h, L_f h, L_f^2 h, L_f^3 h)^T$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T.$$

However the controller (9) is only well defined at $-\pi/2 < x_3 < \pi/2$ because the denominator of the controller is $\varepsilon \cos x_3$. Hence the rotor of TORA system can only be rotated at $-\pi/2 < \theta < \pi/2$.

V. PML Model of TORA system

We construct the PML model of TORA system (8). The nonlinear term $\sin x_3$ of TORA system is transformed into a PML model representation. The variable of x_3 is divided by m vertices $x_3 \in \{d_3(1), d_3(2), \ldots, d_3(m)\}$. For $x_3 \in \{d_3(\sigma), d_3(\sigma+1)\}$, the PML model is expressed as

$$\begin{cases} \dot{x} = f_p + g_p u = \begin{pmatrix} x_2 \\ -x_1 + \varepsilon f_{p_2}(x_3) \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y = h_p = x_1, \end{cases} \tag{11}$$

where

$$\begin{aligned} f_{p_2}(x_3) &= \sum_{i=\sigma}^{\sigma+1} w_3^i(x_3) f_s(i), \ \ f_s(i) = \sin d_3(i), \\ \omega_3^{\sigma}(x_3) &= \frac{(d_3(\sigma+1) - x_3)}{(d_3(\sigma+1) - d_3(\sigma))}, \\ \omega_3^{\sigma+1}(x_3) &= \frac{(x_3 - d_3(\sigma))}{(d_3(\sigma+1) - d_3(\sigma))}. \end{aligned}$$

σ is integer: $-\infty < \sigma < \infty$, $d_3(\sigma) < d_3(\sigma+1)$. The PML model is constructed with respect to $f_{p_2}(x_3) = \sin x_3$. The structure is independent of the state variables x_1, x_2 and x_4 since the variables are the linear terms.

Note that trigonometric functions of TORA system (7) are smooth functions and are of class C^{∞}. The PML models are not of class C^{∞}. In TORA system control, we have to calculate the fourth derivatives of the output y. Therefore the derivative PML models lose some dynamics. However the PML model based control for TORA system can be applied to a wider region than the conventional one.

VI. PML model based control for TORA system via exact feedback linearization

We define the output as $y = x_1$ in the same manner as the previous section, the time derivative of y is calculated as

$$\dot{y} = L_{f_p} h_p = x_2$$

The time derivative of y doesn't contain the control inputs u. We calculate the time derivative of $\dot{y}$. We get

$$\ddot{y} = L_{f_p}^2 h_p = f_{p_2}(x_1, x_3) = -x_1 + \varepsilon \sum_{i=\sigma}^{\sigma+1} w_3^i(x_3) f_s(i),$$

where $x_3 \in \{d_3(\sigma), d_3(\sigma+1)\}$. The time derivative of $\dot{y}$ also doesn't contain the control inputs u. We continue to calculate the time derivative of $\ddot{y}$. We get

$$y^{(3)} = L_{f_p}^3 h_p = -x_2 + \varepsilon \frac{f_s(\sigma+1) - f_s(\sigma)}{d_3(\sigma+1) - d_3(\sigma)} x_4$$

We continue to calculate the time derivative of $y^{(3)}$. We get

$$\begin{aligned} y^{(4)} &= L_{f_p}^4 h_p + L_{g_p} L_{f_p}^3 h_p u \\ &= x_1 - \varepsilon \sum_{i=\sigma}^{\sigma+1} w_3^i(x_3) f_s(i) + \varepsilon \frac{f_s(\sigma+1) - f_s(\sigma)}{d_3(\sigma+1) - d_3(\sigma)} u \end{aligned}$$

The stabilizing controller of (11) is designed as

$$\begin{aligned} u &= \frac{-L_{f_p}^4 h_p}{L_{g_p} L_{f_p}^3 h_p} + \frac{1}{L_{g_p} L_{f_p}^3 h_p} \mu \\ &= \frac{x_1 - \varepsilon \sum_{i=\sigma}^{\sigma+1} w_3^i(x_3) f_s(i)}{\varepsilon \frac{f_s(\sigma+1) - f_s(\sigma)}{d_3(\sigma+1) - d_3(\sigma)}} + \frac{1}{\varepsilon \frac{f_s(\sigma+1) - f_s(\sigma)}{d_3(\sigma+1) - d_3(\sigma)}} \mu \end{aligned} \tag{12}$$

where $x_3 \in \{d_3(\sigma), d_3(\sigma+1)\}$ and $\mu = -F\zeta$ is the linear controller of the linear system (13).

$$\begin{cases} \dot{\zeta} = A\zeta + B\mu, \\ y = C\zeta, \end{cases} \tag{13}$$

where $\zeta = (h_p,\ L_{f_p} h_p,\ L_{f_p}^2 h_p,\ L_{f_p}^3 h_p)^T$. The parameters A, B, C are the same as the linearized system (10).

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If $f_s(i) \neq f_s(i+1)$ and $d_3(i) \neq d_3(i+1)$, $i = 1, \ldots, m$, there exists a controller (12) u of TORA system (11) since $\det(L_{g_p} L_{f_p}^3 h_p) \neq 0$. Thus we have to construct the PML model of TORA system such that $f_s(i) \neq f_s(i+1)$ and $d_3(i) \neq d_3(i+1)$, where $i = 1, \ldots, m$ (see Fig. 3).

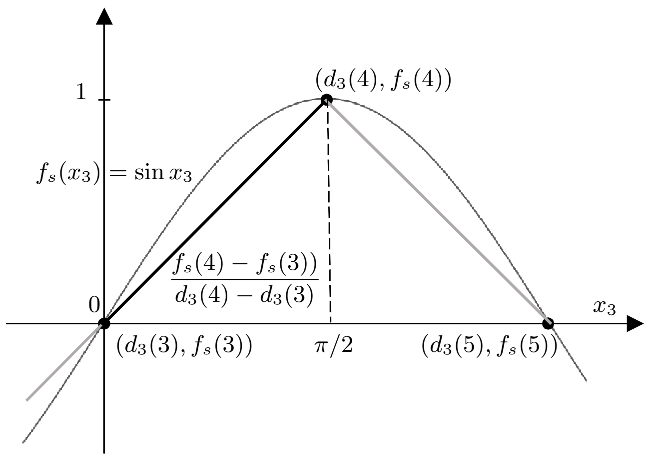

Fig. 3. PML modeling

Note that the PML model based controller (12) can be applied to a wider region than the conventional feedback linearized controller (9).

VII. Simulation results

We apply the feedback linearization controller (9) and the PML model based controller (12) to TORA system (7) in a computer simulation. We can select the arbitrary position and the arbitrary number of the vertices $d_3(i)$ in x_3. Although there are some modeling errors because the PML model is a nonlinear approximation, it is possible to adjust the approximated error. In the following simulations, the parameter ε is 0.5.

A. The difference between with the number of divided regions

The state variable x_3 of TORA system (8) is divided by four regions ($x_3 \in \{-\pi,\ -\pi/2,\ 0,\ \pi/2,\ \pi\}$) to construct the PML model. We consider the feedback gain $F = (1.000,\ 3.078,\ 4.236,\ 3.078)$ such that the linearized control system (13) is stable. The initial condition is $x(0) = (1,\ 0,\ 0,\ 0)^T$. Fig. 4 shows that the control system is unstable because of the model approximation error. Therefore the state variable x_3 is divided by eight regions ($x_3 \in \{-\pi,\ -3\pi/4,\ -\pi/2, \ldots, \pi\}$) to construct the PML model. Next the state variable x_3 is divided by 16 regions ($x_3 \in \{-\pi,\ -7\pi/8,\ -3\pi/4, \ldots, \pi\}$) to construct the PML model. It is enough to select the PML model divided by eight regions from the results of Figs. 5 and 6.

B. Exact feedback linearization and PML model based control

For an exact feedback linearized control and the PML model based control, we use the feedback gains $F = (1.000,\ 3.078,\ 4.236,\ 3.078)$ and the initial conditions $x(0) = (1.5, 0, 0, 0)^T$. Fig. 7 shows the exact feedback linearized control responses. The control and the input responses are stopped at a time when $x_3 > \pi/2$. Fig. 8 shows

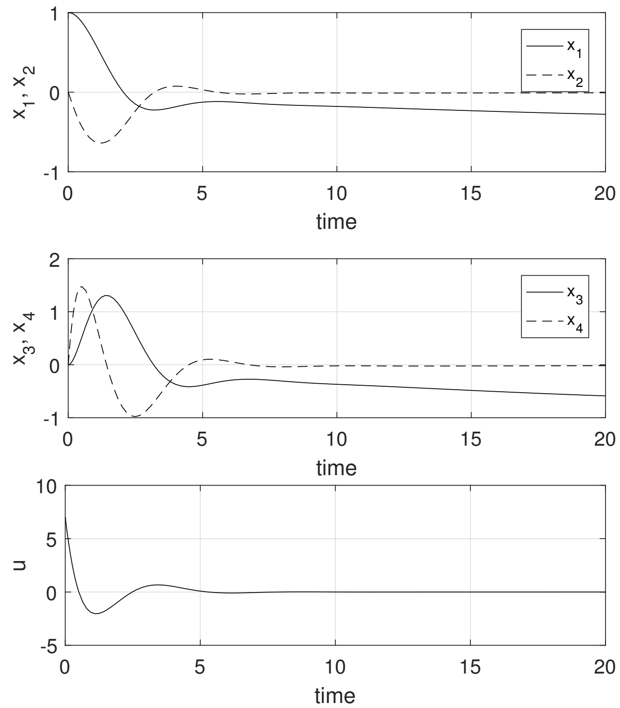

Fig. 4. Control and input responses of PML model based control with four regions

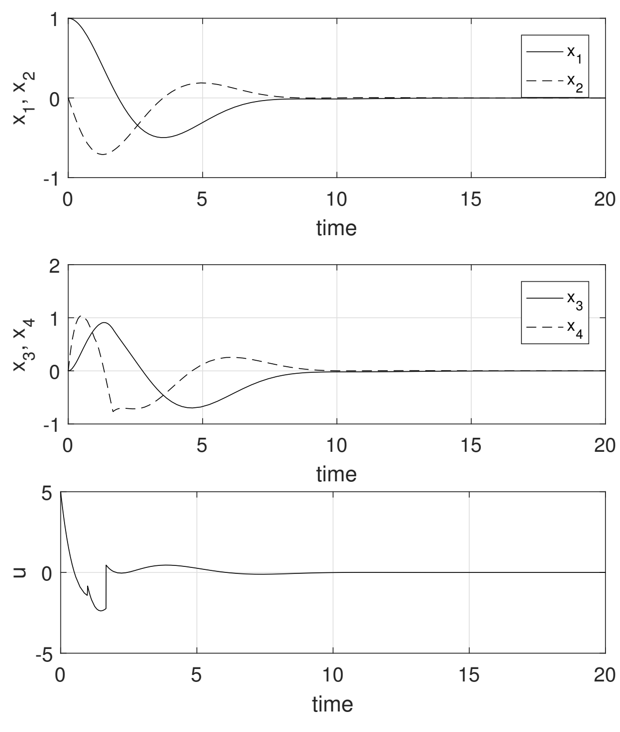

Fig. 5. Control and input responses of PML model based control with eight regions

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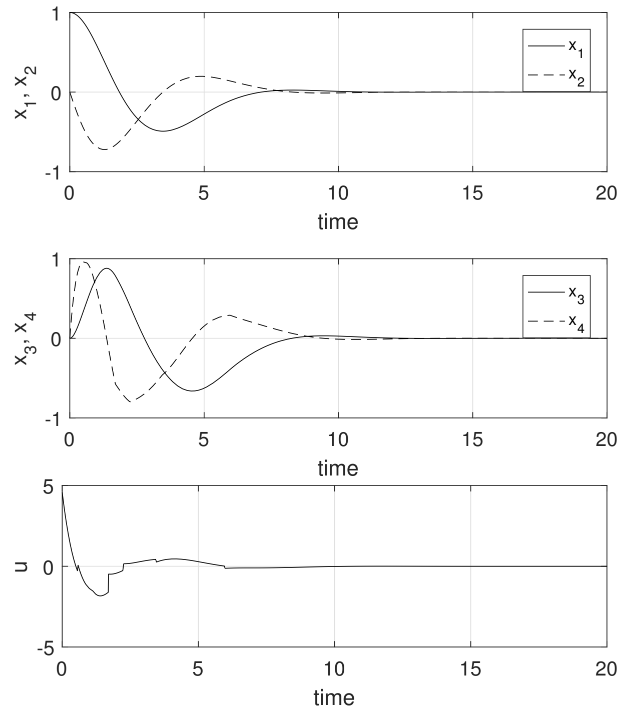

Fig. 6. Control and input responses of PML model based control with 16 regions

the control results when the state variable x_3 is divided by eight regions ($x_3 \in \{-\pi,\ 3\pi/4, \ldots, \pi\}$) to construct the PML model. The controller achieves stabilizing control at the external region of $\|x_3\| \leq \pi/2$.

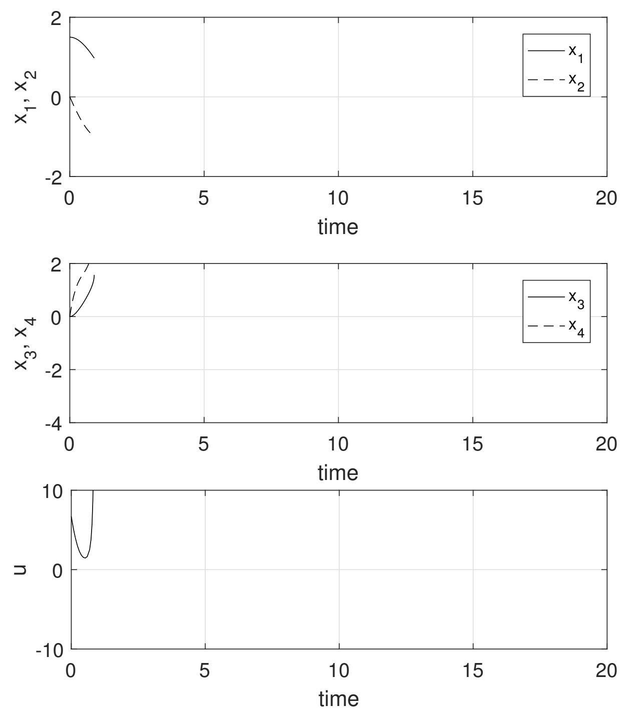

Fig. 7. Exact feedback linearization

VIII. Conclusions

This paper has proposed a piecewise model based controller design for nonlinear systems via feedback linearization. We have applied the piecewise model based controller

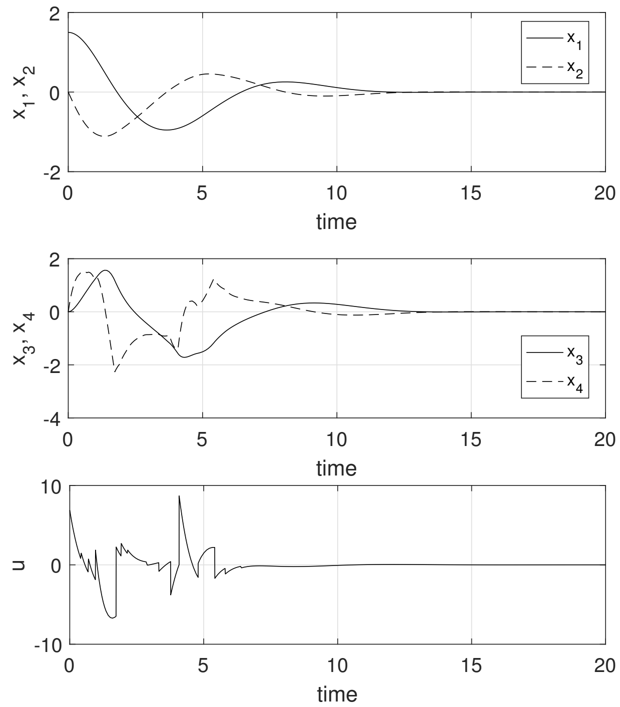

Fig. 8. PML model based control

to TORA system. Although the controller is simpler than the conventional feedback linearization controller, the PML model based control can be applied to a wider region than the conventional one. Examples have been shown to confirm the feasibility of our proposals by computer simulations. We will verify the robust performance and design an H_∞ controller in the future works.

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