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Families of Gaussian Integer Sequences From Sidon Sequences

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***Abstract*—A sequence consisting of complex numbers whose real and image parts are both integers is called Gaussian integer sequence. Gaussian integer sequences with auto-correlation functions have a lot of attentions due to their applications in wireless communications. This paper proposes three families of Gaussian integer sequences. The first two families are perfect sequences. The latter is non-perfect sequences. Experimental results show that the ratio of numbers for two distinct Gaussian integers in a proposed sequence is significantly larger than that in the previously known sequences. Moreover, the difference between perfect and non-perfect Gaussian integer sequences is also described.**

***Index Terms*—Auto-correlation, perfect Gaussian integers sequences, Sidon sequences.**

I. INTRODUCTION

GAUSSIAN integers sequences are complex sequences, where a complex number having real and image parts are both integers is called Gaussian integer. Gaussian integer sequences with auto-correlation functions have been widely applied to wireless communications. For example, sequences can apply to a low-complexity selected mapping scheme for peak-to-average power ratio (PAPR) reduction in orthogonal frequency division multiplexing (OFDM) systems [1]. In order to improve the computational complexity of a precoded OFDM system, sequences can be constructed as a new transform matrix [2]. Sequences employed as the frequency-domain comb-spectrum codes in comb-spectrum code division multiple access (CDMA) system are a practical method to achieve the optimal bit error rate [3]. Frequency-domain spreading codes generated from sequences can avoid serious multiple access interference for a multi-carrier CDMA system [4].

The complex sequence $\mathcal{S} = \{s(t)\}_{t=0}^{P-1}$ of length P, where $s(t) = u(t) + v(t)j$ for $u(t), v(t) \in \mathbb{Z}$, and $j = \sqrt{-1}$, is said to be a *perfect Gaussian integer sequence* if

$$R_{\mathcal{S}}(\tau) = \sum_{t=0}^{P-1} s(t)\overline{s(t+\tau)} \qquad (1)$$

is nonzero for $\tau = 0$ and is zero for any $1 \leq \tau \leq P-1$, where $\overline{s}$ denotes the conjugate of the complex number s. The construction methods of perfect Gaussian integer sequences with different lengths have been recently and widely studied in the literature [5]-[13]. Using two groups of base sequences whose four elements are $1, -1, i$, and $-i$ to linear combinations of sequences yields perfect Gaussian integer sequences of even length [5]. Prime-length sequences [6] were obtained by cyclotomic classes of order 2 and 4 over a finite field. Later, the Whiteman's generalized cyclotomy of order 2 can also be made use of generating perfect Gaussian integer sequences of twin-prime length [7]. Two approaches, complex transformation and interleaving techniques [8], enable one to derive the even-length desired sequences. As described in [9], the unsample techniques were newly proposed methods for constructing perfect Gaussian integer sequences of composite length. Several commonly used methods, such as zero padding, convolution, and Gaussian sum, produce a number of sequences with arbitrary length [10]. Recently, perfect Gaussian integer sequences of odd period can be constructed via trace representation of binary sequences with balanced property, which were shown in [11] and extended in [12]. The long perfect Gaussian integer sequences [13] were also obtained from the short sequences together with polynomial computation over an extension field.

In this paper, three families of Gaussian integer sequences are presented and constructed by the Sidon sequences. Each sequence is composed of two distinct complex numbers. The two former families are perfect sequences with ideal auto-correlation functions. The latter is non-perfect sequences, which are nearly ideal auto-correlation functions. Simulation result reveals that the ratio of two numbers for two distinct Gaussian integers is large when compared to the existing sequences as described in [14]. Finally, the perfect and non-perfect Gaussian integer sequences developed here are illustrated as well.

The paper is organized as follows. Section II provides the mathematical results of the Sidon sequences. The main results of this paper are in Section III. Three families of Gaussian integer sequences are proposed and demonstrated. Section IV illustrates a comparison between the distinct sequences. Section V concludes this paper.

II. SIDON SEQUENCES

Let q be a prime power. Let $P = q^2 + q + 1$ be an odd number. Let $S_P = \{a(t) \mid t = 0, 1, ..., q\}$ be a Sidon sequence over Z_P, which is a sequence of $q+1$ integers satisfying $(a(i) + a(j))(\text{mod } P)$ that are distinct, where $i \leq j$ and $0 \leq a(t) < P$.

Example 1: Let $q = 3$ and $P = 13$. The Sidon sequence of 4 integers can be the following sequence: $S_{13} = \{a(0), a(1), a(2), a(3)\} = \{0, 1, 4, 6\}$. Let $q = 4$ and $P = 21$. Then, $S_{21} = \{0, 2, 7, 8, 11\}$.

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TABLE I
SIDON SEQUENCES OF LENGTH $7 \leq P \leq 133$

P	B
7	0110100
13	1100101000000
21	101000011001000000000
31	1001000000101000010000000000001
57	101000001001000000100000000000000000000010001100000000000
73	1100010000001000001001000000000000000000000000000101000000010000000000000
91	11000010001000000000000100100000001000000100000000000101000000000000000000000000000000000 00
133	10000000011000000100000000000010000000000010010000000000100001000000000000000000100010100 0 00000000000000000000000000000000000000000000

Next, every Sidon sequence can be further expressed as a binary sequence of length P, denoted by $B = \{b(0), b(1), ..., b(P-1)\}$, where

$$b(t) = \begin{cases} 1, & \text{for } t \in S_P \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Example 2: Let $P = 13$. Since $S_{13} = \{0, 1, 4, 6\}$, it follows directly from (2) that $B = \{1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0\}$. Similarly, for $P = 21$, there is a binary sequence $B = \{1, 0, 1, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$.

The binary sequences B of length $7 \leq P \leq 133$ are listed in Table 1. These sequences will be used in the construction method of Gaussian integer sequences as shown in the next section.

III. PROPOSED GAUSSIAN INTEGER SEQUENCES

The purpose of this section is to construct two families of perfect Gaussian integer sequences and a family of Gaussian integer sequences as described in the following three theorems, respectively. Examples are provided as well.

Theorem 1: Let $q = p^2$, where p is a prime and $q \neq 3 \pmod 4$. Let $\phi(q) = p(p-1)$ be the Euler function. Let S_P be a Sidon sequence of $q+1$ integers. A perfect Gaussian integer sequence $\mathcal{S} = \{s(t)\}_{t=0}^{P-1}$ of length $P = q^2 + q + 1$ over two Gaussian integers $G_0 = n + mj$ and $G_1 = -\phi(q)(n + mj)$ can be constructed by

$$s(t) = \begin{cases} G_0, & \text{for } t \notin S_P \\ G_1, & \text{for } t \in S_P, \end{cases} \tag{3}$$

where n and m are arbitrary integers.

Proof: For $1 \leq \tau < P$, the number of $s(t) = G_0$ and $\overline{s(t+\tau)} = \overline{G_0}$ is $q^2 - q$, the number of $s(t) = G_1$ and $\overline{s(t+\tau)} = \overline{G_1}$ is 1, the number of $s(t) = G_0$ and $\overline{s(t+\tau)} = \overline{G_1}$ is q, and the number of $s(t) = G_1$ and $\overline{s(t+\tau)} = \overline{G_0}$ is q. Then, for $\tau = 1, 2, ..., P-1$, it follows from (1) that

$$\begin{aligned} R_{\mathcal{S}}(\tau) &= \sum_{t=0}^{P-1} s(t)\overline{s(t+\tau)} \\ &= (q^2 - q) \times (-\phi(q)(n + mj)) \\ &\qquad \times (-\phi(q)(n - mj)) \\ &+ 1 \times (n + mj) \times (n - mj) \\ &+ q \times (-\phi(q)(n + mj)) \times (n - mj) \\ &+ q \times (n + mj) \times (-\phi(q)(n - mj)) \\ &= 0. \end{aligned} \tag{4}$$

If $\tau = 0$, then

$$\begin{aligned} R_{\mathcal{S}}(\tau) &= \sum_{t=0}^{P-1} s(t)\overline{s(t+\tau)} \\ &= q^2 \times (-\phi(q)(n + mj)) \times (-\phi(q)(n - mj)) \\ &+ (q + 1) \times (n + mj) \times (n - mj) \\ &= (q^2\phi(q)^2 + q + 1)(n^2 + m^2) \\ &\neq 0, \end{aligned} \tag{5}$$

which completes the proof.

Example 3: Consider $q = 4$ and $P = 4^2 + 4 + 1 = 21$. It follows from Theorem 1 that using Sidon sequence $S_{21} = \{0, 2, 7, 8, 11\}$ and two Gaussian integers $G_0 = 5 + 1j$ and $G_1 = -\phi(4)(5 + 1j) = -10 - 2j$ into (3) yields the perfect Gaussian integer sequence

$$\begin{aligned} \mathcal{S} = (&\underbrace{-10-2j}_{0}, 5+1j, \underbrace{-10-2j}_{2}, 5+1j, 5+1j, \\ &5+1j, 5+1j, \underbrace{-10-2j}_{7}, \underbrace{-10-2j}_{8}, 5+1j, \\ &5+1j, \underbrace{-10-2j}_{11}, 5+1j, 5+1j, 5+1j, \\ &5+1j, 5+1j, 5+1j, 5+1j, 5+1j, 5+1j) \end{aligned} \tag{6}$$

Theorem 2: Let $q = u^2 + w^2$, where q is a prime power and $q \neq 3 \pmod 4$. Let both u and w be positive integers, where $u \leq w$. Let S_P be a Sidon sequence of $q + 1$ integers. A perfect Gaussian integer sequence $\mathcal{S} = \{s(t)\}_{t=0}^{P-1}$ of length

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P over two Gaussian integers $G_0 = -j$ and $G_1 = u+(q+w)j$ can be generated by

$$s(t) = \begin{cases} -j, & \text{for } t \notin S_P \\ u+(q+w)j, & \text{for } t \in S_P. \end{cases} \quad (7)$$

Proof: This proof is similar to that of Theorem 1. For $1 \leq \tau < P$, one has

$$\begin{aligned} R_{\mathcal{S}}(\tau) &= (q^2-q) \times (-j) \times (j) \\ &+ 1 \times (u+(q+w)j) \times (u-(q+w)j) \\ &+ q \times (-j) \times (u-(q+w)j) \\ &+ q \times (u+(q+w)j) \times (j) \\ &= 0 \end{aligned} \quad (8)$$

and

$$\begin{aligned} R_{\mathcal{S}}(0) &= q^2 \times (-j) \times (j) \\ &+ (q+1)(u+(q+w)j)(u-(q+w)j) \\ &\neq 0. \end{aligned} \quad (9)$$

This proof is thus complete.

Example 4: Let $q = 4 = 0^2 + 2^2$ and $P = 4^2 + 4 + 1 = 21$. There exist two Gaussian integers $G_0 = -j$, $G_1 = 6j$ and the sequence $S_{21} = \{0, 2, 7, 8, 11\}$ such that the perfect Gaussian integer sequence of length 21 is determined from Theorem 2 as

$$\begin{aligned} \mathcal{S} = (&\underbrace{6j}_{0}, -j, \underbrace{6j}_{2}, -j, -j, -j, -j, \underbrace{6j}_{7}, \\ &\underbrace{6j}_{8}, -j, -j, \underbrace{6j}_{11}, -j, -j, \\ &-j, -j, -j, -j, -j, -j, -j) \end{aligned} \quad (10)$$

Theorem 3: Let $q \equiv 3 \pmod 4$ and q be a prime power. Let S_P be a Sidon sequence of $q+1$ integers. A Gaussian integer sequence $\mathcal{S} = \{s(t)\}_{t=0}^{P-1}$ of length $P = q^2+q+1$ over two Gaussian integers $G_0 = 0$ and $G_1 = j$ can be determined by

$$s(t) = \begin{cases} 0, & \text{for } t \notin S_P \\ j, & \text{for } t \in S_P. \end{cases} \quad (11)$$

Proof: The sequence $\mathcal{S}$ has auto-correlation values

$$\begin{aligned} R_{\mathcal{S}}(0) &= q^2 \times 0 \times 0 + (q+1) \times (j)(-j) \\ &= q+1 \end{aligned} \quad (12)$$

and

$$\begin{aligned} R_{\mathcal{S}}(\tau) &= (q^2-q) \times 0 \times 0 + 1 \times (j)(-j) \\ &+ q \times 0 \times (-j) + q \times (j) \times 0 \\ &= 1 \neq 0 \end{aligned} \quad (13)$$

for $1 \leq \tau < P$. As a result, $\mathcal{S}$ is not a perfect sequence.

Example 5: Consider $q = 3$ and $P = 3^2 + 3 + 1 = 13$. Theorem 3 means that two Gaussian integers $G_0 = 0$, $G_1 = j$, and the Sidon sequence $S_{13} = \{0, 1, 4, 6\}$ give the Gaussian integer sequence

$$\mathcal{S} = (\underbrace{j}_{0}, \underbrace{j}_{1}, 0, 0, \underbrace{j}_{4}, 0, \underbrace{j}_{6}, 0, 0, 0, 0, 0, 0) \quad (14)$$

whose auto-correlation values are $\{R_{\mathcal{S}}(\tau)\}_{\tau=0}^{12} = \{4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.

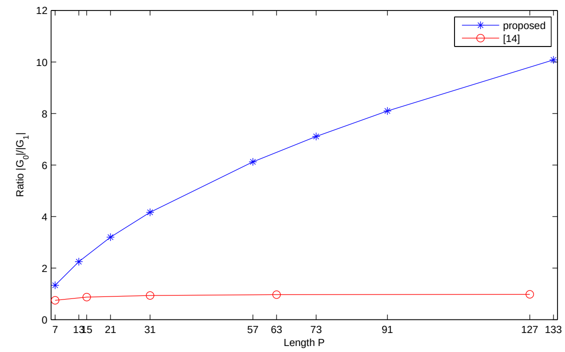

Fig. 1. Ratio for different $\mathcal{S}$

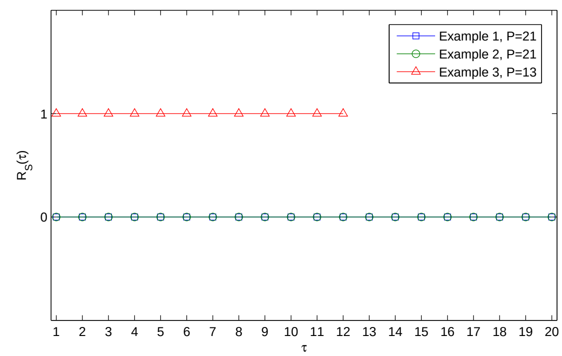

Fig. 2. Auto-correlation values of $\mathcal{S}$

IV. Comparison

Each sequence presented in Section III consists of two Gaussian integers. Let $|G_i|$ be the number of the complex number G_i in the sequence $\mathcal{S}$. Figure 1 shows the ratio of $|G_0|$ and $|G_1|$ in the perfect Gaussian integer sequences as stated in the proposed paper and the previous paper [14]. An observation of this figure is that the numbers of two Gaussian integers in a sequence obtained from [14] almost equal, i.e., the ratio is close to one. However, for the sequences proposed in this paper, the ratio is increasing as a sequence length becomes large.

Figure 2 depicts the auto-correlation values of the perfect Gaussian integer sequences as shown in the three examples of Section III. Clearly, for any nonzero τ, $R_{\mathcal{S}}(\tau) = 1$ in Example 3 is larger than $R_{\mathcal{S}}(\tau) = 0$ in Examples 1 and 2. Although the sequence of length 13 is not perfect sequence, its auto-correlation values are parallel to those of the perfect sequences of length 21.

V. Conclusions

This paper has presented three families of Gaussian integer sequences which can be constructed from Sidon sequences. These sequences consist of two Gaussian integers. This work has verified that a family of the presented Gaussian integer sequences is nearly perfect sequences. The difference between the proposed and existing sequences [14], [15] is the ratio of numbers for two Gaussian integers.

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