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Convergence of Powers of a Nonnegative Interval Matrix in Max Algebra

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Abstract—**In this paper, we proposed the notion of max algebra of nonnegative interval matrices. Some properties of nonnegative interval matrices in max algebra are derived. Necessary and sufficient conditions for the powers of a nonnegative interval matrices in max algebra to be nilpotent, asymptotically p-periodic and convergent are proposed as well.**

Index Terms—**Interval matrix; Max algebra; Convergence, Asymptotical p-period, Nilpotence.**

I. Introduction

INTERVAL matrix have been extensive studied in the literature (see, e.g. [1],[5-8], [10]). The max-algebra of interval matrices are established in [11]. In this paper, we shall follow this concept to study the limit behavior of max-product powers of a nonnegative interval matrix. We refer to Alefeld and Herzberger [1] for the background materials of interval matrices. Real numbers are denoted by lowercase letters a, b. The $\overline{a}$ and $\underline{a}$ denote the upper and lower bounds of a real closed interval $[\underline{a}, \overline{a}]$, respectively. The set of all these closed intervals is denoted by $I(\mathcal{R})$. We may denote an interval $[\underline{a}, \overline{a}]$ by $[a] = [\underline{a}, \overline{a}]$. Let $* \in \{+, -, \times, \div\}$ be one of the usual binary operations on the set of real numbers. For $[a] = [\underline{a}, \overline{a}]$ $[b] = [\underline{b}, \overline{b}] \in I(\mathcal{R})$ the binary operation $[a] * [b] = \{a * b : a \in [a], b \in [b]\}$, is assumed that $0 \neq [b]$ in the case of division. For a nonnegative interval $[a] = [\underline{a}, \overline{a}]$, the width $d([\underline{a}, \overline{a}])$ and the absolute value $|[\underline{a}, \overline{a}]|$ are defined by

$$d([\underline{a}, \overline{a}]) = \overline{a} - \underline{a},$$

$$|[\underline{a}, \overline{a}]| = \max\{|\underline{a}|, |\overline{a}|\}, \text{ respectively.}$$

We called $[a] = [\underline{a}, \overline{a}]$ a point interval if $\underline{a} = \overline{a}$. In this case, we say $[a] = [\underline{a}, \overline{a}]$ is degenerated to a point interval.

A matrix with entries belonging to $I(\mathcal{R})$ is called an interval matrix. The set of all real $n \times n$ interval matrices is denoted by $I(R^{n\times n})$. We denote an interval matrix $\mathcal{A} \in (R^{n\times n})$ by $\mathcal{A} = [\underline{A}, \overline{A}]$. We may denote $\mathcal{A}_{ij} = [\underline{A}_{ij}, \overline{A}_{ij}] = [\underline{a}_{ij}, \overline{a}_{ij}]$. Two interval matrices $\mathcal{A}$ and $\mathcal{B}$ are equal if and only if $\mathcal{A}_{ij} = \mathcal{B}_{ij}$ for all $i, j = 1, 2, \ldots, n$. That is $\underline{a}_{ij} = \underline{b}_{ij}$ and $\overline{a}_{ij} = \overline{b}_{ij}$ for all $i, j = 1, 2, \ldots, n$. For interval matrices $\mathcal{A}, \mathcal{B} \in I(R^{n\times n})$ and an interval $[x] = [\underline{x}, \overline{x}] \in I(\mathcal{R})$, the matrix operations $+, -, \times$ are formally defined as

$\mathcal{A} \pm \mathcal{B} = ([a]_{ij} \pm [b]_{ij})$,
$\mathcal{A} \times \mathcal{B} = (\Sigma_{k=1}^{n} [a]_{ik} \times [b]_{kj})$,
$[x] \cdot \mathcal{A} = ([x] \times [a]_{ij})$.

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Let I be an $n \times n$ identity matrix. The powers of interval matrix $\mathcal{A}$ are defined as

$\mathcal{A}^0 = I$,
$\mathcal{A}^k = \mathcal{A}^{k-1} \times \mathcal{A}, \quad k = 1, 2, \ldots.$

As noted by Mayer [6], the product of the interval matrices is not associative in general. Therefore, $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C}$ may not be equal to $\mathcal{A} \times (\mathcal{B} \times \mathcal{C})$. An interval $[a] = [\underline{a}, \overline{a}]$ is said to be nonnegative if $\underline{a} \geq 0$. The set of all nonnegative interval is denoted by $I(\mathcal{R}^+)$.

II. Maximum circuit geometric mean

Let A be an $n \times n$ nonnegative matrix. The weighted directed graph $\mathcal{D}(A)$ associated with A has vertex set $\{1, 2, \ldots, n\}$ and an edge (i, j) from vertex i to vertex j with weight a_{ij} if and only if $a_{ij} > 0$. A path $L(i_1, i_2, \ldots, i_k, i_{k+1})$ of length k is a sequence of k edges $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$. The *weight* of a path $L(i_1, i_2, \ldots, i_{k+1})$, as denoted by $w(L(i_1, i_2, \ldots, i_{k+1}))$ or simply by $w(L)$, is defined by

$$w(L(i_1, i_2, \ldots, i_{k+1})) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_{k+1}}.$$

A *circuit* C of length $k \geq 2$ is a path $L(i_1, i_2 \ldots, i_{k+1})$ with $i_{k+1} = i_1$, and $i_1, i_2, \ldots, i_k$ are distinct. The class of circuits includes *loops*, ie., circuits of length 1. Associated with this circuit C is the *circuit geometric mean* known as $\hat{w}(C) = (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1})^{1/k}$. The *maximum circuit geometric mean* in $\mathcal{D}(A)$ is denoted by $\mu(A)$. Note that we also consider *empty* circuits, namely, circuits that consist of only one vertex and have length 0. For empty circuits, the associated circuit geometric mean is zero. A circuit C with $\hat{w}(C) = \mu(A)$ is called a critical circuit. Vertices on critical circuits are called critical vertices and edges on critical circuits are called critical edges.

Definition 1 [11]. Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix. The maximum circuit geometric mean of $\mathcal{A}$ denoted by $\mu(\mathcal{A})$, is $\mu(\mathcal{A}) = \max\{\mu(A) : A \in \mathcal{A}\}$.

As $\mu(A) \geq \mu(B)$ for all nonnegative matrices $A \geq B$, we see that $\mu(\mathcal{A}) = \mu(\overline{A})$. Let $\mathcal{A} = [\underline{A}, \overline{A}] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}]_{ij})$ be given. Recall that $r \cdot \mathcal{A} = (r \cdot \mathcal{A}_{ij}) = ([r\underline{a}_{ij}, r\overline{a}_{ij}])$, for all real number r. Suppose that $\mu(\mathcal{A}) \neq 0$. Set $k = \frac{1}{\mu(\mathcal{A})}$. It is easy to see that $\mu(k \cdot \mathcal{A}) = 1$.

Theorem 1 [11]. Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix with $\mu(\mathcal{A}) \leq 1$. Then there exists a diagonal real matrix D such that $|D^{-1} \times \mathcal{A} \times D| \leq [J_n]$, i.e., $|D^{-1} \times \mathcal{A} \times D|_{ij} \leq 1$, for all $1 \leq i, j \leq n$.

III. Max product of powers of a nonnegative interval matrix

We refer to [2-4] and [9] for the study of nonnegative interval matrices in max algebra. Let $[a] = [\underline{a}, \overline{a}]$, $[b] = [\underline{b}, \overline{b}]$

be two nonnegative intervals. Define the maximum of $[a]$ and $[b]$ by

$$[a] \vee [b] = \{a \vee b : a \in [a], b \in [b]\},$$

here $a \vee b = \max\{a, b\}$. It was shown in [11] that $[a] \vee [b] = [\max\{\underline{a}, \underline{b}\}, \max\{\overline{a}, \overline{b}\}]$ is also an interval for all $[a], [b] \in I(\mathcal{R})$.

Define $\max\{[a], [b]\} = [a] \vee [b]$. The max algebra interval system is defined as follow: Let $I(\mathcal{R}^+_{\max,\times}) = (I(\mathcal{R}^+), \oplus, \otimes)$ be consisted of the set of nonnegative interval numbers with sum $[a] \oplus [b] = [a] \vee [b]$ and the product of $[a] \otimes [b]$ is defined by $[a] \otimes [b] = [a] \times [b]$. The following theorem shows that the max algebra on interval is a semiring with identity element $[0] = [0, 0]$. Moreover, $\oplus$ is idempotent, ie., $[a] \oplus [a] = [a]$. Let $\{[a_1], [a_2], \ldots, [a_k]\}$ be a finite set of nonnegative intervals. Define $\bigvee_{j=1}^{k} [a_j] = [a_1] \vee [a_2] \vee \cdots \vee [a_k]$. The order $\leq$ is defined by $[a] \leq [b]$ if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two nonnegative interval matrices. The max-product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})_{ij} = \bigvee_{k=1}^{n} \mathcal{A}_{ij} \otimes \mathcal{B}_{ij}.$$

Let I be an $n \times n$ identity matrix. The powers of nonnegative interval matrix $\mathcal{A}$ in the max algebra interval system are defined as

$$\mathcal{A}^0_\otimes = I,$$

$$\mathcal{A}^k_\otimes = \mathcal{A}^{k-1}_\otimes \times \mathcal{A}, \quad k = 1, 2, \ldots.$$

Note that $\mathcal{A} \otimes \mathcal{B}$ may not be equal to $\mathcal{B} \otimes \mathcal{A}$. However, $(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$.

It is easy to see that if $\mu(\mathcal{A}) > 1$ then $\lim_{k\to\infty} \mathcal{A}^k$ diverges. In this section, we shall study the convergence of power of an interval matrix $\mathcal{A}$ with $\mu(\mathcal{A}) \leq 1$.

Definition 2. Let $\mathcal{A}$ be an $n \times n$ nonnegative interval matrix. $\mathcal{A}$ is said to be *nilpotent* if $\mathcal{A}^k_\otimes = 0$, for some positive integer k.

Theorem 2. Let $\mathcal{A}$ be an $n \times n$ nonnegative interval matrix. Then
(1). $\mathcal{A}$ is nilpotent.
(2). $\mathcal{A}^n_\otimes = 0$
(3). There is a real permutation matrix P such that $P \otimes \mathcal{A} \otimes P^T$ is a strictly lower triangular interval matrix.

Proof. (1) $\Rightarrow$ (2). Let k be a positive integer such that $\mathcal{A}^k_\otimes = 0$. Suppose, by contradiction, that $\mathcal{A}^n_\otimes \neq 0$. Then there exist $1 \leq i_1, i_2, \ldots, i_{n+1} \leq n$ such that

$$\mathcal{A}_{i_1 i_2} \otimes \mathcal{A}_{i_2 i_3} \otimes \cdots \otimes \mathcal{A}_{i_n i_{n+1}} \neq 0.$$

As $\{i_1, i_2, \ldots, i_{n+1}\} \subset \{1, 2, \ldots, n\}$, there are $1 \leq r < s \leq n$ such that $i_r = i_s$. Therefore,

$$\mathcal{A}_{i_r i_{r+1}} \otimes \cdots \otimes \mathcal{A}_{i_{s-1} i_s} \neq 0.$$

Thus, $(\mathcal{A}^{s-r}_\otimes)_{i_r i_r} \neq 0$, which implies that $(\mathcal{A}^{(s-r)k}_\otimes)_{i_r i_r} \neq 0$. This contradicts to $\mathcal{A}^k_\otimes = 0$.

(2) $\Rightarrow$ (3). We prove the assertion by induction on dimension n. The case of $n = 1$ is trivial. Assume that $n > 1$ and the assertion is true for all cases of m less than n.

Claim: There exists $1 \leq i \leq n$ such that $\mathcal{A}_{ij} = 0$ for all $j = 1, 2, \ldots, n$.

Suppose to the contrary that for each i there exists j such that $\mathcal{A}_{ij} \neq 0$. Let $1 \leq i_1 \leq n$ be given. Then there exists i_2 such that $\mathcal{A}_{i_1 i_2} > 0$. Corresponding to this i_2, there exists i_3 such that $\mathcal{A}_{i_2 i_3} > 0$. By continuing this process, we obtain $\{i_1, i_2, \ldots, i_{n+1}\} \subset \{1, 2, \ldots, n\}$ such that $\mathcal{A}_{i_j i_{j+1}} > 0$ for all $j = 1, 2, \ldots, n$. Thus there exist $1 \leq r < s \leq n+1$ such that $i_r = i_s$. Then $(\mathcal{A}^{s-r}_\otimes)_{i_r i_s} \geq \mathcal{A}_{i_r i_{r+1}} \otimes \cdots \otimes \mathcal{A}_{i_{s-1} i_s} > 0$. Hence , $(\mathcal{A}^{(s-r)n}_\otimes)_{i_r i_s} > 0$ a contradiction. This proves the claim. Let P_1 be a permutation matrix such that

$$P_1 \otimes \mathcal{A} \otimes P_1^T = \begin{bmatrix} 0 & \mathbf{0} \\ b^* & \tilde{\mathcal{A}} \end{bmatrix},$$

where $\tilde{\mathcal{A}}$ is an $(n-1) \times (n-1)$ nonnegative interval matrix. Observe that $\tilde{\mathcal{A}}^n_\otimes = 0$, this implies $\tilde{\mathcal{A}}$ is nilpotent. According to (1) $\Rightarrow$ (2), we see that $\tilde{\mathcal{A}}^{n-1}_\otimes = 0$. By induction assumption, there exists an $(n-1) \times (n-1)$ permutation P_2 such that $P_2 \otimes \tilde{\mathcal{A}} \otimes P_2^T$ is a strictly lower triangular matrix. Set

$$P = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & P_2 \end{bmatrix} \otimes P_1.$$

Then

$$P \otimes \mathcal{A} \otimes P^T = \begin{bmatrix} 0 & \mathbf{0} \\ 0 & P_2 \otimes \tilde{\mathcal{A}} \otimes P_2^T \end{bmatrix}$$

is strictly lower triangular. This completes the proof.

(3)$\Rightarrow$ (1). It is trivial.

Theorem 3. Let $\mathcal{A}$ be an $n \times n$ nonnegative interval matrix. Then $\mu(\mathcal{A}) = 0$ if and only if $\mathcal{A}$ is nilpotent.

Proof. ($\Rightarrow$) Assume $\mathcal{A}^n_\otimes \neq 0$. Then there exists $1 \leq i_1, i_2, \ldots, i_{n+1}, \leq n$ such that $\mathcal{A}_{i_t i_{t+1}} \neq 0$ for all $1 \leq t \leq n$. Since $1 \leq i_1, i_2, \ldots, i_{n+1}, \leq n$, there exist $1 \leq r < s \leq n$ such that $i_r = i_s$. Thus $\mathcal{A}_{i_t i_{t+1}} \neq 0$ for all $r \leq t \leq s-1$. Hence $\overline{A}_{i_t i_{t+1}} \neq 0$ for all $r \leq t \leq s-1$. Thus $\mu(\overline{A}) \neq 0$. This implies that $\mu(\mathcal{A}) \neq 0$, a contradiction.

($\Leftarrow$) Assume $\mu(\mathcal{A}) \neq 0$. Then $\mu(\overline{A}) \neq 0$. Then there exists $i_1, i_2, \ldots, i_k, i_{k+1}$ with $i_1 = i_{k+1}$ such that $\overline{A}_{i_t i_{t+1}} > 0$ for all $t = 1, 2, \ldots, k$. This implies that $(\overline{A}^k_\otimes)_{i_1 i_1} > 0$. Hence, $(\overline{A}^{kn}_\otimes)_{i_1 i_1} > 0$. Therefor, $(\mathcal{A}^{kn}_\otimes)_{i_1 i_1} > 0$, a contradiction.

Definition 3. Let $\mathcal{A}$ be an $n \times n$ nonnegative interval matrix. Then $\mathcal{A}$ is called to be *asymptotically stable* if $\lim_{k\to\infty} \mathcal{A}^k_\otimes = 0$.

Denition 4. Let $\mathcal{A}$ be an $n \times n$ nonnegative interval matrix. Then $\mathcal{A}$ is called to be *asymptotically p periodic* if $\lim_{k\to\infty} \mathcal{A}^{j+kp}_\otimes = \tilde{\mathcal{A}}^j$ exists for $j = 1, 2, \ldots, p$. The minimal such p is called asymptotic period p. If $p = 1$, then $\mathcal{A}$ is called to be convergent , ie., $\lim_{k\to\infty} \mathcal{A}^k_\otimes = \tilde{\mathcal{A}}$ exists.

Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix with $\mu(\mathcal{A}) \leq 1$. By Theorem 2, there exists a diagonal real matrix D such that $|D^{-1} \times \mathcal{A} \times D| \leq [J_n]$, i.e., $|D^{-1} \times \mathcal{A} \times D|_{ij} \leq 1$, for all $1 \leq i, j \leq n$. Thus, we may assume that $|\mathcal{A}_{ij}| \leq 1$ for all $1 \leq i, j \leq n$.

Theorem 4 [11]. Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix. Then $\mathcal{A}^k_\otimes = [\underline{A}^k_\otimes, \overline{A}^k_\otimes]$, for all $k \geq 1$.

Theorem 5. Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix with $\mathcal{A}_{ij} \leq 1$ for all $1 \leq i, j \leq n$. Then $\mathcal{A}$ is asymptotically stable if and only if $\lim_{k\to\infty} \overline{A}^k_\otimes = 0$.

Proof. By Theorem 4, we have $\mathcal{A}^k_\otimes = [\underline{A}^k_\otimes, \overline{A}^k_\otimes]$. Observe that $0 \leq (\underline{A}^k_\otimes)_{ij} \leq (\overline{A}^k_\otimes)_{ij}$ for all i, j. If $\lim_{k\to\infty} \overline{A}^k_\otimes = 0$ then we see that $\lim_{k\to\infty} \underline{A}^k_\otimes = 0$. This implies that

$\lim_{k\to\infty} \mathcal{A}_{\otimes}^{k} = 0$. It is clear that if $\lim_{k\to\infty} \mathcal{A}_{\otimes}^{k} = 0$ then $\lim_{k\to\infty} \overline{A}_{\otimes}^{k} = 0$.

Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix with $\mathcal{A}_{ij} \leq 1$ for all $1 \leq i, j \leq n$. Corresponding to $\mathcal{A}$ we define an $n \times n$ Boolean matrix $\hat{A}$ by

$$\hat{A}_{ij} := \begin{cases} 1 & \text{if } |\mathcal{A}|_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6. Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix with $\mathcal{A}_{ij} \leq 1$ for all $1 \leq i, j \leq n$. Then the following statements are equivalent

(1) $\mathcal{A}$ is asymptotically stable.

(2) The directed graph $\mathcal{D}(\hat{A})$ contains no cycles.

(3) $\mu(\mathcal{A}) < 1$.

Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix with $\mathcal{A}_{ij} \leq 1$ for all $1 \leq i, j \leq n$. Corresponding to $\mathcal{A}$ we define two $n \times n$ Boolean matrix $\hat{\underline{A}}$ and $\hat{\overline{A}}$ by

$$\hat{\underline{A}}_{ij} := \begin{cases} 1 & \text{if } \underline{A}_{ij} = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{\overline{A}}_{ij} := \begin{cases} 1 & \text{if } \overline{A}_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each $k \geq 1$, we define an $n \times n$ Boolean matrix by $\Gamma^k(\mathcal{A}) = [\hat{\underline{A}}_{\otimes}^{k}, \hat{\overline{A}}_{\otimes}^{k}]$.

Theorem 7. Let $\mathcal{A} = [\underline{A}, \overline{A}]$ be an $n \times n$ nonnegative interval matrix. Then $\mathcal{A}$ is asymptotically p periodic if and only if the sequence of $\{\Gamma^k(\mathcal{A})\}$ is p periodic.

IV. Conclusion

In the literature, there are many authors studied the max algebra system of nonnegative real numbers. In this paper, we extend the notion of max algebra system of nonnegative matrices to the notion of max algebra system of nonnegative interval matrices. Properties of max-product powers of an interval matrix are established.

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