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On the Contributions of Applied Pseudoanalytic Function Theory in Some Branches of Electrical Engineering

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***Abstract*—We examine how modern Applied Pseudoanalytic Function Theory has provided a better understanding of the Electrical Impedance Equation, and Dirac Equation; for they are of mayor importance in Electrical Impedance Tomography and Nuclear Medicine Dosimetry, for example. Also, we briefly review a contribution into Multispectral Photoacoustic Microscopy Theory.**

***Index Terms*—Pseudoanalytic Funtions, Electrical Impedance Equation, Dirac Equation.**

I. Introduction

There are only few cases of purely mathematical theories, as the one created by L. Bers and co-authors [1], which evolved during less of a century embracing many branches of both Theoretical and Experimental Physics. Moreover, when such branches, as separate as they could seem, converge on an applied discipline, as the Engineering is. This paper analyze how the modern Applied Pseudoanalytic Function Theory has strongly contributed at least into three fields of Electrical Engineering: Electrical Impedance Imaging, Nuclear Medicine, and briefly presented, in Multispectral Photoacoustic Microscopy Theory; remarking these fields were selected as examples, oh behalf of briefness, but the reader will easily notice the presented results can be immediately extended to other branches (see [2]).

We will show the main advances by studying the Electrical Impedance Equation, specifically the forward Dirichlet boundary value problem in the plane; and the Dirac Equation, for which a new class of solutions with unexpected behaviour has been found. But the work is not restrained to known results, since for the two first equations, preliminary proposals and brief descriptions of new results, and new techniques are presented, and as expected in an State of the Art treatise, we consider some of the most relevant questions to be answered, for we expect to achieve significant advances in Applied Mathematics to be applied Electrical Engineering.

II. Preliminaries

Let us discuss some concepts corresponding to the Applied Pseudoanalytic Function Theory, and to Applied Quaternionic Analysis. They have been slightly modified to better fit the main assets of this paper, since they are indispensable given the diversity of applications in Electrical Engineering, presented in onward paragraphs.

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A. Elements of pseudoanalytic functions

Let $\mathbb{C}$ be the set of complex-valued functions. Every element W belonging to $\mathbb{C}$ will be represented in the form $W = \mathbf{Re}\, W + i\mathbf{Im}\, W$, where $\mathbf{Re}\, W$ is the real part of W, whereas $\mathbf{Im}\, W$ denotes the imaginary part, and i is the standard imaginary unit $i^2 = -1$.

Definition 1: Let $(F, G) \in \mathbb{C}$, such that they satisfy the condition:

$$\mathbf{Im}\, \left(\overline{F} \cdot G\right) \neq 0; \tag{1}$$

where the overline-mark on a complex function indicates its complex conjugation: $\overline{F} := \mathbf{Re}\, F - i\mathbf{Im}\, F$. Hence (F, G) will be referred as a *Bers Generating Pair*, or on behalf of briefness, a *generating pair*.

Corollary 1: Let $(F, G) \in \mathbb{C}$ satisfy the condition (1). Thus any complex-valued function W accepts the representation:

$$W = \phi \cdot F + \psi \cdot G,$$

where ϕ and ψ are purely real-valued functions.

1) Derivative and integral in the sense of Bers, Taylor series in formal powers, and a special class of Vekua equation: Let the classical two-dimensional Cartesian axis be represented by the notations x_1, x_2. We can introduce the following notations:

$$\partial_z := \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}; \quad \partial_{\overline{z}} := \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}.$$

Usually, these complex differential operators (the Cuachy-Riemann operators, indeed), are presented with the factor $\frac{1}{2}$. Nonetheless, in the current study, it will be more convenient to omit it, without lose of generality. Also, on behalf of the space, we shall employ the abbreviate notation $\partial_s := \frac{\partial}{\partial x_s}$, where s represents the variable for which the partial derivative is applied, as well the subindex of its corresponding Cartesian axis. Therefore, the derivative in the sense of Bers of a complex-valued function W, represented as $\partial_{(F,G)}W$, is defined according to the expression:

$$\partial_{(F,G)}W := F\partial_z\phi + G\partial_z\psi;$$

yet, this will exist iff:

$$F\partial_{\overline{z}}\phi + G\partial_{\overline{z}}\psi = 0. \tag{2}$$

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The equation (2) was widely studied by professor Ilia Vekua in [5], that it is why (2) was named after him. Its great importance in this work will be clarified forward.

Remark 1: Let p be a purely real-valued function, non vanishing within a certain domain $\Omega \subset \mathbb{R}^2$, where, as usual, $\mathbb{R}^2$ represents the Cartesian two-dimensional plane. The pair of functions

$$F = p, \quad G = \frac{i}{p}, \tag{3}$$

satisfy the condition (1); then they conform a generating pair.

Definition 2: Let us introduce the notations:

$$B_{(F,G)} := \frac{\partial_z p}{p},\ b_{(F,G)} := \frac{\partial_{\overline{z}} p}{p}. \tag{4}$$

These pair of functions are known as the *characteristic coefficients* of the generating pair (F, G).

Remark 2: By virtue of the notations (4), the Vekua equation (2) can be rewritten in the form:

$$\partial_{\overline{z}} W - \frac{\partial_{\overline{z}} p}{p} \overline{W} = 0. \tag{5}$$

This significant form of the Vekua equation will be of extreme importance for the Biomedical Engineering applications discussed in this paper.

Definition 3: Let the pairs of functions (F_0, G_0) and (F_1, G_1) be two generating pairs, such that their characteristic coefficients fulfill the condition:

$$B_{(F_0,G_0)} = -b_{(F_1,G_1)};$$

Then (F_1, G_1) is called a *successor* pair of (F_0, G_0), as well (F_0, G_0) is caller a *predecessor* pair of (F_1, G_1).

Definition 4: Let the set of generating pairs

$$\{...\,(F_{-1}, G_{-1}), (F_0, G_0), (F_1, G_1)\,...\} \tag{6}$$

be such that every (F_{s+1}, G_{s+1}) is a successor pair of (F_s, G_s), where $\forall s \in \mathbb{P}$, and $\mathbb{P}$ is the set of Whole Numbers. Afterwards, this set will be called a *generating sequence*. Particularly, if it happens that $(F_s, G_s) = (F_{s+k}, G_{s+k})$, where $k \in \mathbb{P}$, the generating sequence will be called *periodic*, with period k. Moreover, if a generating pair $(F, G) = (F_s, G_s)$, we will say that (F, G) is *embedded* into the generating sequence of the form (6).

Remark 3: Suppose the function p depends upon only x_2 (the vertical axis). Then, the generating pair defined according to (3), will be embedded within a periodic generating sequence, with period $k = 1$.

Definition 5: Let (F, G) be a generating pair. Its *adjoin* pair, denoted as (F^*, G^*) is defined as follows:

$$F^* = -iF,\ G^* = -iG.$$

Definition 6: The integral in the sense of Bers, when it exists (see [1] for details), has the form

$$\int_\Lambda W d_{(F,G)} := F\,\mathbf{Re} \int_\Lambda W \cdot G^* dz + G\,\mathbf{Re} \int_\Lambda W \cdot G^* dz, \tag{7}$$

where Λ is a rectifiable curve going from 0 till z, and $z = x_1 + ix_2$.

Definition 7: The *formal power* $Z^0(a_0, 0; z)$ with a constant coefficient $a_0 \in \mathbb{C}$, center at 0, and depending upon z, is defined by the expression

$$Z^0(a_0, 0; z) = \lambda F + \mu G;$$

where λ and μ are two constant numbers fulfilling the relation

$$\lambda F(0) + \mu G(0) = a_0. \tag{8}$$

Higher formal powers are approached by the formulae

$$Z^n(a_n, 0; z) = n \int_\Lambda Z^{n-1}(a_{n-1}, 0; z)\, d_{(F,G)}.$$

Notice the integral at the right-hand side of the equation, is an integral in the sense of Bers (7).

Proposition 1: Any pseudoanalytic function W accepts the expansion

$$W = \Sigma_{n=0}^{\infty} Z^n(a_n, 0; z). \tag{9}$$

Hence, since by definition W fulfills the Vekua equation (5), this expansion is a representation of the general solution for (5), being every formal power $Z^n(a_n, 0; z)$ a particular solution of (5) itself. As a matter of fact, the expansion (9) of W is often called *Taylor series in Formal Powers* of the complex-valued pseudoanalytic function W. We shall use this articulation very often in the upcoming paragraphs.

2) One method to fully employ the Applied Pseudoanalytic Function Theory to certain Electrical Engineering problems: The following proposition was first posed in [6], but it had already been used in a variety of works, *e.g.* [7] or [8], first as a conjecture [9], thereafter as a proposition. We present here the proposition only, recommending the reader to review the previously cited works for the complete proof.

Proposition 2: Let p be a a non-vanishing function, defined within a bounded domain $\Omega \subset \mathbb{R}^2$. We can always introduce an *infinitesimally-piece-wise separable-variables function*, according to the expression

$$p_{x_2,\infty} := \{p_{x_2}(x_1) = p(x_1, x_2)\};$$

where x_2 represents every fixed point at the x_2-axis, reflection of the pair of coordinates (x_1, x_2) where p is defined, such that $p_{x_2,\infty}$ is uniquely related to p at each point (x_1, x_2), whence $p_{x_2,\infty}$ will preserve every property of p, at least from the numerical point of view (a mayor concern in Engineering). Moreover, from this perspective, $p_{x_2,\infty}$ can be considered a function depending only on x_2; a property that will allow the numerical construction of a periodical generating sequence, with period $k = 1$.

B. Elements of Quaternionic Analysis

For a complete explanation and proper proofs of the subsequent concepts, we recommend to consult the book [10], since the volume is entirely focused into physical applications. Let the set of quaternionic complex-valued functions q be denoted by $\mathbb{H}(\mathbb{C})$. Subsequently, each element $q \in \mathbb{H}(\mathbb{C})$ will be written as

$$q = q_0 + q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3 =: q_0 + \overrightarrow{q}; \tag{10}$$

where $q_0, q_1, q_2, q_3 \in \mathbb{C}$, whereas $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the standard quaternionic units:

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1;\ \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1. \tag{11}$$

By definition, the standard imaginary unit i commutes with the quaternionic units: $i\mathbf{e}_s = \mathbf{e}_s i$; where $s = 1, 2, 3$.

Remark 4: Hereafter, the notation introduced in (10) will be evoked as we show now: q_0, often called the *scalar part*

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of the quaternion q, will be denoted as $\mathbf{Sc}\, q := q_0$; whereas $q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$, called the *vectorial* part of q, will be denoted as both $\mathbf{Vec}\, q = \overrightarrow{q} := q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3$.

As these properties ensued, the multiplication of two quaternionic complex-valued functions q and f is non-commutative, thus we shall introduce the notation

$$M^f q := q \cdot f,$$

to indicate the right hand-side multiplication of q by f.

The Moisil-Theodoresco derivative operator D is defined as follows:

$$D = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3, \tag{12}$$

where $\partial_s := \frac{\partial}{\partial x_s}$, for $s = 1, 2, 3$; and when applied on $q \in \mathbb{H}(\mathbb{C})$, the result can be written down employing the classical notation of vector calculus:

$$Dq = \text{grad}\, q_0 - \text{div}\, \overrightarrow{q} + \text{rot}\, \overrightarrow{q}; \tag{13}$$

where "grad", "div" and "rot" are the classical derivative operators employed in vector calculus. These isomorphic relation between the quaternionic derivative D and the classical vector-derivative operators, is the key to rewrite a set of equations of Mathematical Physics, specially important in Biomedical Engineering, in both special complex Vekua equations, and special bicomplex Vekua equations.

III. Review of equations of Mathematical Physics with special relevance in Electrical Engineering

A. The Electrical Impedance Equation

Let us consider the Electrical Impedance Equation

$$\text{div}\ (\sigma\, \text{grad}\, u) = 0; \tag{14}$$

where σ represents the electrical conductivity, and u is the electric potential. Also known as the Generalized Ohms Law, among other names, it describes a significant number of electrical phenomena, both for the static and dynamical cases. Nevertheless, the mathematical entanglement embedded within this equation, specially when σ turns a complex-valued function (called then impedance), has provoked a tendency to restrict its study to the static case, or to such frequencies where the wavelength is considerably bigger than the dimensions of the body under examination.

More precisely, when focusing into Dirichlet boundary value problems, the inverse problem is of special interest for Medical Imaging, since it is known as *Electrical Impedance Tomography* (EIT). Many important treatises has been written on this topic, among which it is worth of mention the one posed by Webster [11], perhaps one of the first tomes fully dedicated to enclosure all relevant advances in EIT, at the time of its first edition. Correctly posed by P. Calderón in mathematical form [12], the problem can be summarized as follows. Let $\Omega \in \mathbb{R}^3$ be a bounded domain, and let Γ be its boundary. Suppose the electric potential u is known at each point of the boundary Γ. Thus, as Calderón proved, there can be one and only one conductivity function σ inside the domain Ω, related to such electric potential u at the boundary. The challenge: How to determine such function σ within Ω.

The relevance for Electrical Impedance Imaging becomes obvious. Lets focus into Medical Imaging. Every interior tissue of an organic system possesses a specific conductivity. If we can conveniently allocate a finite set of electrodes around a body, whose electric potential values are known, and then we are able to obtain the conductivity within, we will possess a map of the location, kind, shape, and perhaps even dynamics, of all tissues inward, with a minimal risk of damaging the tissues. In other words, we would obtain a full image of the inner body by applying extremely small electrical currents.

Yet, the advances on this direction seem to have been slowed down for the complexity explained above. They were V. Kravchenko [3], and K. Astala and L. Päivarintä [13] who, independently, first noticed the close relation of the two-dimensional case of (14) and the special Vekua equation (5) (as a matter of fact, Astala and Päivarintä studied the Beltrami equation, but is is well known that it is fully equivalent to the Vekua equation (5)). The relation comes along these lines. Let us introduce the notations

$$\overrightarrow{\mathcal{E}} := \sqrt{\sigma}\, \text{grad} u,\ \overrightarrow{\sigma} := \frac{D\sqrt{\sigma}}{\sqrt{\sigma}}; \tag{15}$$

then it is possible to rewrite the equation (14) into the quaternionic equation

$$\left(D + M^{\overrightarrow{\sigma}}\right)\overrightarrow{\mathcal{E}} = 0, \tag{16}$$

indicating that $\overrightarrow{\mathcal{E}}$ can be expanded as $\overrightarrow{\mathcal{E}} := \mathbf{e}_1\mathcal{E}_1 + \mathbf{e}_2\mathcal{E}_2 + \mathbf{e}_3\mathcal{E}_3$; where $\mathcal{E}_s := \sqrt{\sigma}\partial_s u$; $s = 1, 2, 3$. Suppose now that σ depends upon only x_2 (this would seem a very strong restriction, nevertheless, by virtue or Proposition 2, for numerical analysis purposes, there is no lose of generality), and let us analyze the equation in the plane, by dismissing x_3. Introducing another set of notations

$$\partial_{\overline{z}} := \partial_1 + i\partial_2,\ W = \mathcal{E}_1 - i\mathcal{E}_2,$$

the quaternionic equation will turn into the Vekua equation

$$\partial_{\overline{z}} W - \frac{\partial_{\overline{z}}\sqrt{\sigma}}{\sqrt{\sigma}}\overline{W} = 0;$$

which possesses exactly the same structure that (5), therefore it can be studied employing the full set of mathematical principles introduced in the previous Section.

Kravchenko focused into the forward problem (a fundamental matter for many algorithms that recursively solve it, in order to approach a solution for the inverse problem [11]), finally proving that the real parts of the formal powers (9), are indeed a complete set to approach solutions of the forward problem, obtaining impressively high accuracy when numerical calculations were performed. Astala and Päivarintä did likewise, concentrating their efforts on the inverse problem, obtaining also very important results in subsequent works (see *e.g.* [14]).

Yet again, the important contributions of V. Kravchenko had perhaps a slight mathematical limitation before fully applying his results in Electrical Impedance Imaging: It is required that the electrical conductivity σ can be represented by means of a separable-variables function.

When analyzing the work [14], one can immediately notice that there is no separable-variables function capable to approach the non-smooth conductivities posed there. Therefore, the bridge among these important advances is still to be made, and it is a central question that needs to be carefully studied.

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1) A technique to overpass some of the pure-mathematical restrictions: This was of the essence in the works [7] and [8], because, from the numerical point of view, and by means of Proposition 2, any function (analytically expressed or not) depends solely on one single variable (this is, σ is a infinitesimally-piece-wise separable-variables function for numerical purposes). Thus all mathematical elements analyzed before can be perfectly applied to approach numerical solutions of the forward Dirichlet boundary value problem, and therefore to be employed into the algorithms that approach solutions of the inverse problem.

2) A special case for novel further research: But there is a special case that has not be properly studied in this direction, perhaps because the Proposition 2 contributes, at the present time, only for numerical purposes.

But when it comes to numerical analysis, in the light of the preceding paragraphs, we have already considered the non-static case of (14). After the proper and well known calculations, derived from the the Maxwell equations for isotropic inhomogeneous media, we shall obtain an equation of the form

$$\text{div}\ (\gamma\ \text{grad}\ u) = 0; \tag{17}$$

where $\gamma := \sigma + i\omega\epsilon$ is the electrical impedance, ω is the frequency of the electromagnetic wave, and ϵ is the electrical permittivity. It is well known in Electromagnetic Theory that, in general, ϵ is a function depending at least upon x_1, x_2, x_3; and on the frequency ω. Still, the influence of ω turns considerable only for certain high frequencies. Since our concern in these paragraphs is the Electrical Impedance Tomography, such frequencies would not be employed, or even detected, by the measurement equipments. But the dependence of the spacial axis is highly important. This directed our attention into the conditions imposed by the mathematical analysis for continuous variables. It would only be possible to use the theory in its pure form, when σ and ω depended both on the same one single variable, meaning they compose a separable variable function; which immediately would retrieve any possibility to directly apply the theory into Electrical Imaging.

For these reasons, two steps leaded us into Engineering applications in the plane. The first one has already been mentioned for the case of a purely real conductivity: γ shall be considered an infinitesimally-piece-wise separable-variables function, by the utilization of the Proposition 2. Hence we can already numerically analyze arbitrary isotropic functions ϵ and ω. The second one is to examine again the quaternionic equation (16), but introducing a slightly variation for a new set of notations.

$$\partial_{\overline{z}(\mathbb{H})} = \partial_1 + \mathbf{e}_1\partial_2;\ W_{(\mathbb{H})} := \mathcal{E}_1 - \mathbf{e}_1\mathcal{E}_2; \tag{18}$$

where $\mathcal{E}_s := \sqrt{\gamma}\,\partial_s u$ for $s = 1, 2$. Then, equation (17) will turn into the bicomplex Vekua equation

$$\partial_{\overline{z}(\mathbb{H})} W_{(\mathbb{H})} - \frac{\partial_{\overline{z}(\mathbb{H})}\sqrt{\gamma}}{\sqrt{\gamma}}\overline{W}_{(\mathbb{H})} = 0; \tag{19}$$

where $\overline{W}_{(\mathbb{H})}$ denotes the *quaternionic conjugation* of $W_{(\mathbb{H})}$: $\overline{W}_{(\mathbb{H})} := \mathbf{Sc}\ W_{(\mathbb{H})} - \mathbf{Vec}\ W_{(\mathbb{H})}$.

Of course, this is not the first time that a bicomplex Vekua equation arise from Mathematical Physics. In several works they have been detected and analyzed. A quite advisable reference about such bicomplex equations is [3], since the included authors list will guide the reader thereafter.

Still, it is necessary to remark that, perhaps, this paper is one of the first works fully dedicated to Engineering applications, whatever method could have been employed before. We dare to appoint such because, as it was already mentioned in [3], most of the properties and main assets of Pseudoanalytic Theory can be generalized for the bicomplex Vekua equation (19), and also because we just preliminary posed a method for numerically solving the forward Dirichlet boundary valued problem of a bicomplex Vekua equation, where arbitrary impedance complex-valued functions are considered.

As a conjecture, it had already been noticed in [15] that the scalar parts of the formal powers, constitute a complete set for approaching solutions of the forward Dirichlet boundary value problem, when separable variables complex-valued functions are studied.

From the numerical point of view, bias the employment of Proposition 2, we rediscovered the conjecture, considering not only separable variables functions, but arbitrary conductivity complex-valued functions too, approaching solutions for the forward Dirichlet boundary value problem with high accuracy. Of course, these are preliminary results, but a full work about it is onward.

In other words, the new achievement for Electrical Engineering is the capability of approaching solutions for the forward Dirichlet boundary value problem in the plane, considering arbitrary electrical permittivities ϵ, and arbitrary conductivities σ. This implies that we can immediately use the numerical method into the algorithms that recursively solve the forward problem (see [11]) to approach solutions for the inverse problem.

B. The Dirac Equation

The analysis of the Dirac Equation in, *e.g.*, Biomedical Engineering, does not require extended justifications. It shall be enough to mention its relevance in Nuclear Medicine Radiation Dosimetry, an advanced branch of Nuclear Medicine with mayor importance [17]. Let us examine a special class of the Dirac Equation: The case of a massive particles, with spin $1/2$, under the influence of an arbitrary electric potential, just as it is the case of the electron. We shall point out that the basis of this technique were early presented in [16]. The technique does not provide general solutions for the Dirac Equation, but we would like to underline that, in Engineering procedures, quite often we restrain the available technology to the cases that are mostly understood, so we can have certain control of them. Thus, let us study the Dirac Equation of the form

$$\left[\gamma_0\partial_t - \sum_{n=1}^{3}\gamma_n\partial_n + im + \gamma_0 u(x_1)\right]\Phi(t, \mathbf{x}) = 0, \tag{20}$$

where m is the mass of a particle with spin $\frac{1}{2}$, $u(x_1)$ represents the electric potential, $\partial_t := \frac{\partial}{\partial t}$, t is the time variable, $\mathbf{x}$ denotes any point (x_1, x_2, x_3) belonging to $\mathbb{R}^3$, and γ_s, $s = 0, 1, 2, 3$; are the classical Pauli-Dirac matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

As mentioned before, this equation was previously studied in a variety of works. One of them is [18], where numerical calculations were performed, and unexpected behaviors were detected for the very first time, related to different kind of electric potentials, when new sets of solutions were approached. We now review the techniques posed there, showing the clear contribution of the Pseudoanalytic function Theory to this area. Lets introduce a pair of matrix transformations, first discovered in [3]:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \\ -1 & 0 & 0 & -1 \\ 0 & i & i & 0 \end{pmatrix}, \quad (21)$$

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & -i & -1 & 0 \\ -1 & 0 & 0 & -i \\ 1 & 0 & 0 & -i \\ 0 & i & -1 & 0 \end{pmatrix}.$$

Considering the particle-wave de Broglie duality principle, we can write $\Phi(t, x) = \phi e^{i\omega t}$, where ω is the energy of the particle, and applying the operators $\mathbf{A}$ and $\mathbf{A}^{-1}$ as follows:

$$\mathbf{A}\gamma_1\gamma_2\gamma_3 \left[\gamma_0\partial_t - \sum_{n=1}^{3} \gamma_n\partial_n + im + \gamma_0 u(x_1) \right] \mathbf{A}^{-1},$$

we will obtain the biquaternionic equation:

$$\left(D - M^{g\mathbf{e}_1 + m\mathbf{e}_2} \right) f = 0; \quad (22)$$

where $g := iu(x_1) + i\omega$, and $f := \mathbf{A}\phi$. Suppose $f = \alpha Q$, where α is purely scalar, and $Q = q_1\mathbf{e}_1 + q_3\mathbf{e}_3$. We can obtain special solutions splitting the main quaternionic Dirac equation into a pair of decoupled equations:

$$DQ - Qg\mathbf{e}_1 = 0, \ \partial_1\alpha + m\alpha = 0.$$

The second equation is solved immediately: $\alpha = Ke^{-mx_2}$, where K is a scalar constant. Consequently, the first equation will turn into

$$\partial_{\overline{z}(\mathbb{H})} Q_{(\mathbb{H})} - \frac{\partial_{\overline{z}(\mathbb{H})} p}{p} \overline{Q}_{(\mathbb{H})} = 0,$$

where $\partial_{\overline{z}(\mathbb{H})} := \partial_1 + \mathbf{e}_3\partial_3$, $Q_{(\mathbb{H})} := q_1 - q_3\mathbf{e}_1$ and $p = e^{\int g dx_1}$. Once again, we find a biquaternionic Vekua equation, for which we can go all the way around to numerically approach the Taylor series in formal powers for $Q_{(\mathbb{H})}$. The very contribution of this special technique for obtaining a new class of solutions for the massive Dirac Equation, raised at the moment of approaching the probability density functions obtained when the solutions were rewritten in classical form. Surprisingly, the functions corresponding to higher formal powers, shown a very similar behavior, as if they were independent of the electric potential influencing them. No other experiments have been performed in this direction, and the physical interpretation remains unknown.

Because of this, we enhance the importance of continuing exploring the new results, given their high importance in Nuclear Medicine. Also, it is very important to remark that such theoretical results had never been noticed, or at least formally described, before employing the modern elements of pseudoanalytic functions.

C. Other area of Engineering where Applied Pseudoanalytic Function Theory has contributed

1) Multispectral Photoacoustic Microscopy: On behalf of space, and since it would perhaps require a separated work (even the same Mathematical principles are employed) worth of mention in this work are some advances into the Photoacoustic Microscopy Theory (see *e.g.* [19]). Many works bare witnesses of the complexity for Mathematical Physics in this area; shall it serve as an example [20], entitled "*Near-infrared multispectral photoacoustic microscopy using a Graded-Index Fiber Amplifier*."

Let us examine some results to analyze Graded-Index Optical Fibers. In order to reach this, we provide an explanation without employing single details of mathematical modelling. Also, we will cite and follow the paper [21], for avoiding an overflow of specific bibliography, given the extensive evolving of the full techniques.

Kravchenko had already analyzed the Sturm-Liouville equation in [4]:

$$\frac{d}{dx}\left(p\frac{du}{dx} \right) + qu = 0,$$

where p, q and u are complex-valued functions of the real variable x; by means of a factorization of its differential operator, where a Vekua equation with the form (5) appeared. After several important works published in the midtime, he and Porter studied the spectral parameter power series for the Sturm-Liouville problems in 2010

$$\frac{d}{dx}\left(p\frac{du}{dx} \right) + qu = \lambda u,$$

where λ is an arbitrary complex constant; based on the results upcoming from the Taylor series in formal powers. As we all know, possessing the spectral parameters of any differential equation is the backbone to fully understand it. This becomes clearer when analyzing the obtained results of Castillo, Khmelnytskaya, Kravchenko and Oviedo, referring the reflectance and transmittance of finite inhomogeneous layers [22]:

$$\frac{d}{dx}\left(p\frac{du}{dx} \right) + qu = \beta^2 ru, \quad (23)$$

where r is a complex-valued function of the real variable x, and β is an arbitrary complex constant; such that the coefficients p, q, r, and u are supposed to allow the existence of a solution u_0 of:

$$\frac{d}{dx}\left(p\frac{du_0}{dx} \right) + qu_0 = 0.$$

More details about the properties of u_0 are provided in [22]. Then, again, the general solution of (23) can be expressed by means of the linear combination of two Taylor series in formal powers. Finally, Castillo, Kravchenko and Torba, in

2013, used the new tools to give a quite novel perspective of the Bessel equations, reaching our topic of interest, since the main differential equation of the Graded-Index Optical Fibers is one of them (in [21] they wrote cylindrical waveguides in lieu of Graded-Index Optical Fibers). They achieved this by analyzing the an equation of the form:

$$-\frac{d^2u}{dx^2}+\left(\frac{l\,(l+1)}{x^2}+q\right)u=\Upsilon\left(r_1\frac{du}{dx}+r_2u\right);$$

where l is a real number such that $l\geq-\frac{1}{2}$, q is a complex-valued continuous function within a certain interval, depending upon a real variable $x\in(0,a]$ (see [21] for details), satisfying a growth bound $|q|\leq Cx^{\varpi}$ at the origin for some $\varpi\geq-2$; $r_0,\ r_1\in C\,[0,a]$ are complex-valued functions, $0<a\in\mathbb{R}$, and Υ is a complex spectral parameter.

Thereafter, employing this last result, in [21] Castillo, Kravchenko and Torba examined the specific case of Graded-Index Optical Fibers, without mayor lose (or not anyone) of generality. In short terms, by means of the expanded knowledge of the special Vekua equation (5), and of its general solution, written in terms of Taylor series in formal powers, they obtained an infinite set of solutions for the equation

$$\frac{d^2}{dt^2}\xi+\frac{1}{2}\frac{d}{dr}\xi+\left(\kappa^2\eta^2\,(r)-\beta^2-\frac{\varphi^2}{r^2}\right)\xi=0;\qquad(24)$$

where, as explained in [21], the obtention of ξ will allow the approaching of the corresponding electromagnetic field, $\eta\,(r)$ represents the refractive index profile, $\kappa=\frac{2\pi}{\tau}$ is the vacuum wave number, β is the propagation constant, and φ is a mode parameter.

Even more, they were able to give full explanations of the physical meaning for most of the results, and showed the improvement of accuracy obtained by using this novel numerical method, in comparison to other classical numerical methods, fully dedicated to approach solutions of the equation (24).

A specific question comes after these paragraphs. Clearly, the technology at hand can manufacture a wide class of Graded-Index Optical Fibers, warranting to follow, with high precision and accuracy, the specifications imposed for such fibers. Would the results presented in [20] be improved if it were considered some Graded-Index Fiber, deeply-enough studied with the results posed in [21]? In our opinion, it is an important question for Electrical Engineering.

IV. Disclosure

Given the diversity of applications examined in this work, to write a single section for conclusions turned inconvenient. Nonetheless, the reader can notice that the closing paragraphs of each Section are precisely the corresponding conclusions.

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