Multiobjective Fuzzy Random Bimatrix Games and An Equilibrium Solution Concept

Hitoshi Yano

***Abstract*—In this paper, we formulate multiobjective bimatrix games with fuzzy random payoffs, and introduce an equilibrium solution concept based on the fuzzy decision. By applying possibility measure and an expectation model to such games, the corresponding equilibrium solution is defined. To circumvent the computational difficulties to obtain an equilibrium solution, the algorithm based on the bisection method is proposed, in which equilibrium conditions in the membership function space are replaced into equilibrium conditions in the expected payoff space.**

***Index Terms*—multiobjective bimatrix games, fuzzy random variables, expectation model, possibility measure, fuzzy decision.**

I. INTRODUCTION

To deal with bimatrix games with triangular fuzzy numbers, Maeda [9] defined an equilibrium solution concept using possibility measure and the threshold values for the level sets [2]. He formulated the corresponding mathematical programming problem to obtain such parametric equilibrium solutions. Using the expected value concept for possibility measure and necessity measure, Li et.al. [6], [7] formulated quadratic programming problems to obtain the corresponding Nash equilibrium solutions for bimatrix games with triangular fuzzy numbers. Mako et al. [10] focused on bimatrix games with LR fuzzy numbers. Corresponding to the fuzzy Nash-equilibrium solution concept, they proposed the fuzzy correlated equilibrium solution concept, which is based on a joint distribution for mixed strategies of both players. Gao [3] introduced three kinds of uncertain equilibrium solution concepts based on uncertainty theory [8], which depend on the values of confidence levels. From a similar point of view based on uncertainty theory, Tang et al. [16] proposed an uncertain equilibrium solution concept based on the Hurwicz criterion.

For multiobjective bimatrix games, Corley [1] first defined a Pareto equilibrium solution concept, and formulated quadratic programming problems to obtain Pareto equilibrium solutions through the Karush-Kuhn-Tucker conditions, in which multiobjective functions are scalarized by the weighting coefficients. Nishizaki et al. [12] formulated multiobjective bimatrix games incorporating fuzzy goals. They transformed multiobjective bimatrix games into usual bimatrix games by applying the weighting methods or the minimum operator [14], [22], and defined the corresponding equilibrium solution concepts. They formulated the nonlinear programming problems to obtain such equilibrium solutions. Using dominance cones proposed by Yu [21], Nishizaki et al.[11] defined a nondominated equilibrium solution concept which is a generalization of Nash-equilibrium solution concept, and formulate nonlinear programming problem to obtain nondominated equilibrium solutions by applying the Karush-Kuhn-Tucker conditions.

On the other hand, the concept of fuzzy random variable was first introduced by Kwakernaak [4], and its definition in an n-dimensional Euclidean space were given by Puri and Ralescu [13]. Roughly speaking, fuzzy random variables defined by Wang and Zhang [17] can be interpreted as random variables whose realized values are not real values, but rather are fuzzy sets. From the perspective that both randomness and fuzziness are often involved simultaneously in real-world decision making problems, we have already formulated several kinds of multiobjective fuzzy random Stackerberg games with simple recourses, introduced the equilibrium solution concepts, and proposed the interactive algorithms to obtain a satisfactory solution of the player from among an equilibrium solution set [18], [19], [20].

In this paper, we focus on multiobjective bimatrix games with triangular-type fuzzy random variables. After such multiobjective bimatrix games are transformed into usual bimatrix games by applying possibility measure [2] and the expectation model [15] for stochastic programming problems, an equilibrium solution concept based on the fuzzy decision [14], [22] is introduced. In section II, multiobjective fuzzy random bimatrix games are formulated. In section III, by applying possibility measure [2] and an expectation model [15] for stochastic programming problems, the corresponding equilibrium solution based on the fuzzy decision [14], [22] is defined. To circumvent the computational difficulties to deal with each objective function based on possibility measure directly, the algorithm based on the bisection method is proposed, in which equilibrium conditions in the membership function space are replaced into equilibrium conditions in the expected payoff space. In section IV, a numerical example of two-objective bimatrix games with fuzzy random payoffs illustrates interactive processes under a hypothetical player to show the efficiency of the proposed method.

II. MULTIOBJECTIVE FUZZY RANDOM BIMATRIX GAMES

In this section, we consider multiobjective bimatrix games with fuzzy payoffs. Let $i \in \{1, 2, \cdots, m\}$ be a pure strategy of Player 1 and $j \in \{1, 2, \cdots, n\}$ be a pure strategy of Player 2. $\tilde{\bar{A}}_k \stackrel{\rm def}{=} (\tilde{\bar{a}}_{kij}), k = 1, \ldots, K$ are Player 1's $(m \times n)$-payoff matrices, and $\tilde{\bar{B}}_l \stackrel{\rm def}{=} (\tilde{\bar{b}}_{lij}), l = 1, \cdots, L$ are Player 2's $(m \times n)$-payoff matrices, whose elements $\tilde{\bar{a}}_{kij}$ and $\tilde{\bar{b}}_{lij}$ are fuzzy random variables [4] (The symbols "–" and "~" mean randomness and fuzziness respectively). Throughout this paper, we assume that under the occurrence of scenarios $s_k \in \{1, \cdots, S_k\}$ and $t_l \in \{1, \cdots, T_l\}$, $\tilde{a}_{ks_kij}$ and $\tilde{b}_{lt_lij}$ are realizations of fuzzy random variables $\tilde{\bar{a}}_{kij}$ and $\tilde{\bar{b}}_{lij}$, which are fuzzy numbers whose membership functions are defined

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as follows.

$$\mu_{\tilde{a}_{ks_kij}}(u) = \begin{cases} \max\left\{1 - \frac{a_{ks_kij}-u}{\alpha_{kij}}, 0\right\}, & u \le a_{ks_kij} \\ \max\left\{1 - \frac{u-a_{ks_kij}}{\beta_{kij}}, 0\right\}, & u > a_{ks_kij} \end{cases}$$
$$i = 1, \cdots, m, j = 1, \cdots, n, s_k = 1, \cdots, S_k, \quad k = 1, \cdots, K \tag{1}$$

$$\mu_{\tilde{b}_{lt_lij}}(v) = \begin{cases} \max\left\{1 - \frac{b_{lt_lij}-v}{\gamma_{lij}}, 0\right\}, & v \le b_{lt_lij} \\ \max\left\{1 - \frac{v-b_{lt_lij}}{\delta_{lij}}, 0\right\}, & v > b_{lt_lij} \end{cases}$$
$$i = 1, \cdots, m, j = 1, \cdots, n, t_l = 1, \cdots, T_l, \quad l = 1, \cdots, L \tag{2}$$

where the spread parameters $\alpha_{kij} > 0, \beta_{kij} > 0$, $\gamma_{lij} > 0$ and $\delta_{lij} > 0$ are constants and the mean value a_{ks_kij} and b_{lt_lij} vary depending on the scenarios s_k and t_l. Moreover, we assume that a scenario s_k occurs with a probability p_{1ks_k}, where $\sum_{s_k=1}^{S_k} p_{1ks_k} = 1$, and a scenario t_l occurs with a probability p_{2lt_l}, where $\sum_{t_l=1}^{T_l} p_{2lt_l} = 1$.

Then, a multiobjective bimatrix game with fuzzy random payoffs can be formulated as follows, where T means transportation.

P1

$$\underset{\boldsymbol{x}\in X}{\text{maximize}}\ (\boldsymbol{x}^T\tilde{\bar{A}}_1\boldsymbol{y}, \cdots, \boldsymbol{x}^T\tilde{\bar{A}}_K\boldsymbol{y})$$
$$\underset{\boldsymbol{y}\in Y}{\text{maximize}}\ (\boldsymbol{x}^T\tilde{\bar{B}}_1\boldsymbol{y}, \cdots, \boldsymbol{x}^T\tilde{\bar{B}}_L\boldsymbol{y})$$

where

$$X \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathrm{R}^m \mid \sum_{i=1}^m x_i = 1, x_i \ge 0, i = 1, \cdots, m\},$$
$$Y \stackrel{\text{def}}{=} \{\boldsymbol{y} \in \mathrm{R}^n \mid \sum_{i=1}^n y_j = 1, y_j \ge 0, j = 1, \cdots, n\},$$

are mixed strategies for Player 1 and Player 2. It should be noted here that, the expected payoffs for the scenarios $s_k \in \{1, \cdots, S_k\}$ and $t_l \in \{1, \cdots, T_l\}$ can be expressed as fuzzy numbers whose membership functions can be defined as follows [2].

$$\mu_{\boldsymbol{x}^T\tilde{A}_{ks_k}\boldsymbol{y}}(u) = \begin{cases} \max\left\{1 - \frac{\boldsymbol{x}^T A_{ks_k}\boldsymbol{y}-u}{\boldsymbol{x}^T\boldsymbol{\alpha}_k\boldsymbol{y}}, 0\right\}, \\ \qquad u \le \boldsymbol{x}^T A_{ks_k}\boldsymbol{y} \\ \max\left\{1 - \frac{u-\boldsymbol{x}^T A_{ks_k}\boldsymbol{y}}{\boldsymbol{x}^T\boldsymbol{\beta}_k\boldsymbol{y}}, 0\right\}, \\ \qquad u > \boldsymbol{x}^T A_{ks_k}\boldsymbol{y} \end{cases}$$
$$s_k = 1, \cdots, S_k, k = 1, \cdots, K \tag{3}$$

$$\mu_{\boldsymbol{x}^T\tilde{B}_{lt_l}\boldsymbol{y}}(v) = \begin{cases} \max\left\{1 - \frac{\boldsymbol{x}^T B_{lt_l}\boldsymbol{y}-v}{\boldsymbol{x}^T\boldsymbol{\gamma}_l\boldsymbol{y}}, 0\right\}, \\ \qquad v \le \boldsymbol{x}^T B_{lt_l}\boldsymbol{y} \\ \max\left\{1 - \frac{v-\boldsymbol{x}^T B_{lt_l}\boldsymbol{y}}{\boldsymbol{x}^T\boldsymbol{\delta}_l\boldsymbol{y}}, 0\right\}, \\ \qquad v > \boldsymbol{x}^T B_{lt_l}\boldsymbol{y} \end{cases}$$
$$t_l = 1, \cdots, T_l, l = 1, \cdots, L \tag{4}$$

where $\tilde{A}_{ks_k} \stackrel{\text{def}}{=} (\tilde{a}_{ks_kij})$, $\tilde{B}_{lt_l} \stackrel{\text{def}}{=} (\tilde{b}_{lt_lij})$, $A_{ks_k} \stackrel{\text{def}}{=} (a_{ks_kij})$, $B_{lt_l} \stackrel{\text{def}}{=} (b_{lt_lij})$, $\boldsymbol{\alpha}_k \stackrel{\text{def}}{=} (\alpha_{kij})$, $\boldsymbol{\beta}_k \stackrel{\text{def}}{=} (\beta_{kij})$, $\boldsymbol{\gamma}_l \stackrel{\text{def}}{=} (\gamma_{lij})$, $\boldsymbol{\delta}_l \stackrel{\text{def}}{=} (\delta_{lij})$.

Considering the imprecise nature of each player's judgment, it is natural to assume that Players 1 and 2 have fuzzy goals $\tilde{G}_{1k}, k = 1, \cdots, K$ and $\tilde{G}_{2l}, l = 1, \cdots, L$ for the expected payoffs. In this paper, it is assumed that such fuzzy goals can be quantified by eliciting the corresponding membership function defined as follows.

$$\mu_{\tilde{G}_{1k}}(u) \stackrel{\text{def}}{=} \frac{u - E_{k10}}{E_{k11} - E_{k10}}, k = 1, \cdots, K \tag{5}$$
$$\mu_{\tilde{G}_{2l}}(v) \stackrel{\text{def}}{=} \frac{v - E_{l20}}{E_{l21} - E_{l20}}, l = 1, \cdots, L \tag{6}$$

where E_{k10}, E_{l20} represent the maximum value of an unacceptable level of the expected payoffs, and E_{k11}, E_{l21} represent the minimum value of a sufficiently satisfactory level of the payoffs. Throughout this section, we make the following assumption.

Assumption 1: The membership functions $\mu_{\tilde{G}_{1k}}(u), k = 1, \cdots, K$ and $\mu_{\tilde{G}_{2l}}(v), l = 1, \cdots, L$ are continuous and strictly monotone increasing on the corresponding supports for the membership functions of $\boldsymbol{x}^T\tilde{A}_{ks_k}\boldsymbol{y}, s_k = 1, \cdots, S_k$ and $\boldsymbol{x}^T\tilde{B}_{lt_l}\boldsymbol{y}, t_l = 1, \cdots, T_l$, respectively.

$$[E_{k10}, E_{k11}] \supset \bigcup_{s_k=1,\cdots,S_k} \{u \in \mathrm{R}^1 \mid \mu_{\boldsymbol{x}^T\tilde{A}_{ks_k}\boldsymbol{y}}(u) > 0, \quad \forall \boldsymbol{x} \in X, \forall \boldsymbol{y} \in Y\}, k = 1, \cdots, K \tag{7}$$
$$[E_{l20}, E_{l21}] \supset \bigcup_{t_l=1,\cdots,T_l} \{v \in \mathrm{R}^1 \mid \mu_{\boldsymbol{x}^T\tilde{B}_{lt_l}\boldsymbol{y}}(v) > 0, \quad \forall \boldsymbol{x} \in X, \forall \boldsymbol{y} \in Y\}, l = 1, \cdots, L \tag{8}$$

III. An equilibrium solution concept based on possibility measure

To deal with P1, we first apply a concept of possibility measure [2] to each objective function in P1.

P2

$$\underset{\boldsymbol{x}\in X}{\text{maximize}}\ \left(\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_1\boldsymbol{y}}(\tilde{G}_{11}), \ldots, \Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_K\boldsymbol{y}}(\tilde{G}_{1K})\right)$$
$$\underset{\boldsymbol{y}\in Y}{\text{maximize}}\ \left(\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_1\boldsymbol{y}}(\tilde{G}_{21}), \ldots, \Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_L\boldsymbol{y}}(\tilde{G}_{2L})\right)$$

By applying an expectation model [15] to each objective function in P2, P2 can be transformed into a usual multiobjective bimatrix game as follows.

P3

$$\underset{\boldsymbol{x}\in X}{\text{maximize}}(E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_1\boldsymbol{y}}(\tilde{G}_{11})], \cdots, E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_K\boldsymbol{y}}(\tilde{G}_{1K})])$$
$$\underset{\boldsymbol{y}\in Y}{\text{maximize}}(E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_1\boldsymbol{y}}(\tilde{G}_{21})], \cdots, E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_L\boldsymbol{y}}(\tilde{G}_{2L})])$$

From Assumption 1, the following relations always hold.

$$0 < \Pi_{\boldsymbol{x}^T\tilde{A}_{ks_k}\boldsymbol{y}}(\tilde{G}_{1k}) < 1,\ s_k = 1, \ldots, S_k, \quad \forall \boldsymbol{x} \in X, \forall \boldsymbol{y} \in Y \tag{11}$$
$$0 < \Pi_{\boldsymbol{x}^T\tilde{B}_{lt_l}\boldsymbol{y}}(\tilde{G}_{2l}) < 1,\ t_l = 1, \ldots, T_l, \quad \forall \boldsymbol{x} \in X, \forall \boldsymbol{y} \in Y \tag{12}$$

To define an equilibrium solution concept to P3, We assume that both players adopt the fuzzy decision [14], [22] to integrate multiple objectives in P3. Then, P3 can be reduced to the following bimatrix game.

P4

$$\underset{\boldsymbol{x}\in X}{\text{maximize}} \quad \min_{k=1,\ldots,K} E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_k\boldsymbol{y}}(\tilde{G}_{1k})] \tag{13a}$$
$$\underset{\boldsymbol{y}\in Y}{\text{maximize}} \quad \min_{l=1,\ldots,L} E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})] \tag{13b}$$

Now, we can introduce an equilibrium solution concept to P4.

Definition 1: $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in X \times Y$ is an equilibrium solution to P4, if the following inequalities hold.

$$\min_{k=1,\ldots,K} E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})] \geq \min_{k=1,\ldots,K} E[\Pi_{\boldsymbol{x}^{T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})],\ \forall \boldsymbol{x} \in X \quad (14a)$$

$$\min_{l=1,\ldots,L} E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{B}}_l\boldsymbol{y}^*}(\tilde{G}_{2l})] \geq \min_{l=1,\ldots,L} E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})],\ \forall \boldsymbol{y} \in Y \quad (14b)$$

From the definition of the membership functions (5), (6) and Assumption 1, $E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_k\boldsymbol{y}}(\tilde{G}_{1k})]$ and $E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})]$ can be expressed as the following forms.

$$\begin{aligned}
&E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_k\boldsymbol{y}}(\tilde{G}_{1k})] \\
&= \sum_{s_k=1}^{S_k} p_{1ks_k} \cdot \Pi_{\boldsymbol{x}^T\tilde{A}_{ks_k}\boldsymbol{y}}(\tilde{G}_{1k}) \\
&= \sum_{s_k=1}^{S_k} p_{1ks_k} \cdot \left(\frac{\sum_{i=1}^m \sum_{j=1}^n (a_{ks_kij} + \beta_{kij}) x_i y_j - E_{k10}}{E_{k11} - E_{k10} + \sum_{i=1}^m \sum_{j=1}^n \beta_{kij} x_i y_j} \right) \\
&= \frac{\sum_{i=1}^m \sum_{j=1}^n (\sum_{s_k=1}^{S_k} p_{1ks_k} \cdot (a_{ks_kij} + \beta_{kij})) x_i y_j - E_{k10}}{E_{k11} - E_{k10} + \sum_{i=1}^m \sum_{j=1}^n \beta_{kij} x_i y_j} \\
&\stackrel{\rm def}{=} \Pi_{\boldsymbol{x}^T\tilde{A}_k(p_{1k})\boldsymbol{y}}(\tilde{G}_{1k})
\end{aligned} \quad (15)$$

$$\begin{aligned}
&E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})] \\
&= \sum_{t_l=1}^{T_l} p_{2lt_l} \cdot \Pi_{\boldsymbol{x}^T\tilde{B}_{lt_l}\boldsymbol{y}}(\tilde{G}_{2l}) \\
&= \sum_{t_l=1}^{T_l} p_{2lt_l} \cdot \left(\frac{\sum_{i=1}^m \sum_{j=1}^n (b_{lt_lij} + \delta_{lij}) x_i y_j - E_{l20}}{E_{l21} - E_{l20} + \sum_{i=1}^m \sum_{j=1}^n \delta_{lij} x_i y_j} \right) \\
&= \frac{\sum_{i=1}^m \sum_{j=1}^n (\sum_{t_l=1}^{T_l} p_{2lt_l} \cdot (b_{lt_lij} + \delta_{lij})) x_i y_j - E_{l20}}{E_{l21} - E_{l20} + \sum_{i=1}^m \sum_{j=1}^n \delta_{lij} x_i y_j} \\
&\stackrel{\rm def}{=} \Pi_{\boldsymbol{x}^T\tilde{B}_l(p_{2l})\boldsymbol{y}}(\tilde{G}_{2l})
\end{aligned} \quad (16)$$

where

$$\tilde{A}_k(p_{1k}) \stackrel{\rm def}{=} (\sum_{s_k=1}^{S_k} p_{1ks_k} \cdot \tilde{a}_{ks_kij}),$$

$$\tilde{B}_l(p_{2l}) \stackrel{\rm def}{=} (\sum_{t_l=1}^{T_l} p_{2lt_l} \cdot \tilde{b}_{lt_lij}),$$

are $(m \times n)$-fuzzy payoff matrices, respectively, which depends on the probability vectors $p_{1k} \stackrel{\rm def}{=} (p_{1k1}, \cdots, p_{1kS_k})$ and $p_{2l} \stackrel{\rm def}{=} (p_{2l1}, \cdots, p_{1lT_l})$.

It is very difficult to obtain the equilibrium solution to P4 in the computational aspect, since (15) and (16) are bilinear fractional functions. To circumvent such a difficulty, at first, we consider the following bimatrix game, which is equivalent to P4.

P5

$$\underset{\boldsymbol{x}\in X,\ v_1\in \mathrm{R}^1}{\text{maximize}} \quad v_1$$

subject to

$$E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_k\boldsymbol{y}}(\tilde{G}_{1k})] \geq v_1,\ k = 1, \ldots, K \quad (17a)$$

$$\underset{\boldsymbol{y}\in Y,\ v_2\in \mathrm{R}^1}{\text{maximize}} \quad v_2$$

subject to

$$E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})] \geq v_2,\ l = 1, \ldots, L \quad (17b)$$

Assume that $(\boldsymbol{x}^*, \boldsymbol{y}^*, v_1^*, v_2^*)$ is an equilibrium solution to P5. Then, the following equalities hold at $(\boldsymbol{x}^*, \boldsymbol{y}^*, v_1^*, v_2^*)$.

$$\min_{k=1,\ldots,K} E[\Pi_{\boldsymbol{x}^T\tilde{\bar{A}}_k\boldsymbol{y}}(\tilde{G}_{1k})] - v_1^* = 0 \quad (18a)$$

$$\min_{l=1,\ldots,L} E[\Pi_{\boldsymbol{x}^T\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})] - v_2^* = 0 \quad (18b)$$

From (15) and (16), (18a) and (18b) are equivalent to the following equalities.

$$\min_{k=1,\ldots,K} \Pi_{\boldsymbol{x}^T\tilde{A}_k(p_{1k})\boldsymbol{y}}(\tilde{G}_{1k}) - v_1^* = 0 \quad (19a)$$

$$\min_{l=1,\ldots,L} \Pi_{\boldsymbol{x}^T\tilde{B}_l(p_{2l})\boldsymbol{y}}(\tilde{G}_{2l}) - v_2^* = 0 \quad (19b)$$

Consider the v_1^*-level set for the fuzzy numbers $\boldsymbol{x}^{*T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*$ and v_2^*-level set for the fuzzy numbers $\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}^*$ as follows.

$$\begin{aligned}
&(\boldsymbol{x}^{*T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*)_{v_1^*} \\
\stackrel{\rm def}{=}\ &[\boldsymbol{x}^{*T}A^L_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^*, \boldsymbol{x}^{*T}A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^*] \\
&(\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}^*)_{v_2^*} \\
\stackrel{\rm def}{=}\ &[\boldsymbol{x}^{*T}B^L_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^*, \boldsymbol{x}^{*T}B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^*]
\end{aligned}$$

where

$$A^L_{k,\,v_1^*}(p_{1k}) \stackrel{\rm def}{=} (\sum_{s_k=1}^{S_k} p_{1ks_k} \cdot a^L_{ks_kij,\,v_1^*})$$

$$A^R_{k,\,v_1^*}(p_{1k}) \stackrel{\rm def}{=} (\sum_{s_k=1}^{S_k} p_{1ks_k} \cdot a^R_{ks_kij,\,v_1^*})$$

$$B^L_{l,\,v_2^*}(p_{2l}) \stackrel{\rm def}{=} (\sum_{t_l=1}^{T_l} p_{2lt_l} \cdot b^L_{lt_lij,\,v_2^*})$$

$$B^R_{l,\,v_2^*}(p_{2l}) \stackrel{\rm def}{=} (\sum_{t_l=1}^{T_l} p_{2lt_l} \cdot b^R_{lt_lij,\,v_2^*})$$

$A^L_{k,\,v_1^*}(p_{1k})$, $A^R_{k,\,v_1^*}(p_{1k})$, $B^L_{l,\,v_2^*}(p_{2l})$, and $B^R_{l,\,v_2^*}(p_{2l})$ are $(m \times n)$-matrices. $a^L_{ks_kij,\,v_1^*}$, $a^R_{ks_kij,\,v_1^*}$, $b^L_{lt_lij,\,v_2^*}$ $b^R_{lt_lij,\,v_2^*}$ mean the extreme points of the v_1^*-level set for $\tilde{a}_{ks_kij}$ and the extreme points of the v_2^*-level set for $\tilde{b}_{lt_lij}$.

It is obvious that (19a) is equivalent to the following equalities, since $\mu_{\tilde{G}_{1k}}(\cdot)$ is strictly monotone increasing and the right hand side function of the membership function of $\boldsymbol{x}^T\tilde{A}_k(p_{1k})\boldsymbol{y}$ is strictly monotone decreasing.

$$\min_{k=1,\ldots,K} \left(\boldsymbol{x}^{*T}A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*) \right) = 0 \quad (20)$$

Similarly, (19b) is equivalent to the following equalities.

$$\min_{l=1,\ldots,L} \left(\boldsymbol{x}^{*T}B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*) \right) = 0 \quad (21)$$

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Corresponding to (20) and (21), we consider the following bimatrix game, in which (v_1^*, v_2^*) are given as parameters in advance.

P6(v_1^*, v_2^*)

$$\underset{\boldsymbol{x}\in X}{\text{maximize}} \quad \min_{k=1,\ldots,K}\{\boldsymbol{x}^T A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\}$$
$$\underset{\boldsymbol{y}\in Y}{\text{maximize}} \quad \min_{l=1,\ldots,L}\{\boldsymbol{x}^T B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\}$$

For P6(v_1^*, v_2^*), we introduce an equilibrium solution concept.

Definition 2: $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium solution to P6(v_1^*, v_2^*), if the following inequalities hold.

$$\min_{k=1,\ldots,K}\{\boldsymbol{x}^{*T} A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\} \geq \min_{k=1,\ldots,K}\{\boldsymbol{x}^T A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\},\ \forall \boldsymbol{x}\in X \tag{23a}$$

$$\min_{l=1,\ldots,L}\{\boldsymbol{x}^{*T} B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\} \geq \min_{l=1,\ldots,L}\{\boldsymbol{x}^{*T} B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\},\ \forall \boldsymbol{y}\in Y \tag{23b}$$

Then, the following relationships between equilibrium solutions to P6(v_1^*, v_2^*) and equilibrium solutions to P5 hold.

Theorem 1: If $(\boldsymbol{x}^*, \boldsymbol{y}^*, v_1^*, v_2^*)$ is an equilibrium solution to P5, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium solution to P6(v_1^*, v_2^*).

(Proof) : Assume that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is not an equilibrium solution to P6(v_1^*, v_2^*). Then, there exists some $\boldsymbol{x}\in X$ such that

$$\min_{k=1,\ldots,K}\left\{\boldsymbol{x}^{*T} A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right\} < \min_{k=1,\ldots,K}\left\{\boldsymbol{x}^T A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right\}, \tag{24}$$

or, there exists some $\boldsymbol{y}\in Y$ such that

$$\min_{l=1,\ldots,L}\left\{\boldsymbol{x}^{*T} B^R_{l,v_2^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\right\} < \min_{l=1,\ldots,L}\left\{\boldsymbol{x}^{*T} B^R_{l,v_2^*}(p_{1k})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\right\}. \tag{25}$$

Assume that there exists some $\boldsymbol{x}\in X$ such that the inequality (24) is satisfied. Then, from (20), the following relation holds.

$$\begin{aligned} 0 &= \min_{k=1,\cdots,K}\left(\boldsymbol{x}^{*T} A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right) \\ &< \min_{k=1,\cdots,K}\left(\boldsymbol{x}^T A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right) \end{aligned}$$

Since $\mu_{\tilde{G}_{1k}}(\cdot)$ is strictly monotone increasing and the right hand side function of the membership function of $\boldsymbol{x}^T\tilde{A}_k(p_{1k})\boldsymbol{y}$ is strictly monotone decreasing, the above relation is equivalent to the following inequality.

$$\begin{aligned} v_1^* &= \min_{k=1,\cdots,K}\Pi_{\boldsymbol{x}^{*T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k}) \\ &= \min_{k=1,\ldots,K}E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})] \\ &< \min_{k=1,\cdots,K}\Pi_{\boldsymbol{x}^{T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k}) \\ &= \min_{k=1,\ldots,K}E[\Pi_{\boldsymbol{x}^{T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})]. \end{aligned}$$

This contradicts the fact that $(\boldsymbol{x}^*, \boldsymbol{y}^*, v_1^*, v_2^*)$ is an equilibrium solution to P5. Similarly, we can prove for the case that there exists $\boldsymbol{y}\in Y$ such that (25) is satisfied.

Theorem 2: If $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium solution to P6(v_1^*, v_2^*), where the following relations hold,

$$\min_{k=1,\cdots,K}\left(\boldsymbol{x}^{*T} A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right) = 0 \tag{26}$$

$$\min_{l=1,\cdots,L}\left(\boldsymbol{x}^{*T} B^R_{l,v_2^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\right) = 0, \tag{27}$$

then, $(\boldsymbol{x}^*, \boldsymbol{y}^*, v_1^*, v_2^*)$ is an equilibrium solution to P5.

(Proof) : Assume that $(\boldsymbol{x}^*, \boldsymbol{y}^*, v_1^*, v_2^*)$ is not an equilibrium solution to P5. Then, there exists some $\boldsymbol{x}\in X$ such that

$$\begin{aligned} v_1^* &= \min_{k=1,\ldots,K}E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})] \\ &< \min_{k=1,\ldots,K}E[\Pi_{\boldsymbol{x}^{T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})] \end{aligned}$$

or, there exists some $\boldsymbol{y}\in Y$ such that

$$\begin{aligned} v_2^* &= \min_{l=1,\ldots,L}E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{B}}_l\boldsymbol{y}^*}(\tilde{G}_{2l})] \\ &< \min_{l=1,\ldots,L}E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})]. \end{aligned}$$

From (15) and (16), this means that there exists some $\boldsymbol{x}\in X$ such that

$$\begin{aligned} v_1^* &= \min_{k=1,\ldots,K}\Pi_{\boldsymbol{x}^{*T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k}) \\ &< \min_{k=1,\ldots,K}\Pi_{\boldsymbol{x}^{T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k}), \end{aligned} \tag{28}$$

or, there exists some $\boldsymbol{y}\in Y$ such that

$$\begin{aligned} v_2^* &= \min_{l=1,\ldots,L}\Pi_{\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}^*}(\tilde{G}_{2l}) \\ &< \min_{l=1,\ldots,L}\Pi_{\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}}(\tilde{G}_{2l}). \end{aligned} \tag{29}$$

Assume that there exists some $\boldsymbol{x}\in X$ such that (28) is satisfied. Then, the following relation holds.

$$\begin{aligned} 0 &= \min_{k=1,\cdots,K}\left(\boldsymbol{x}^{*T} A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right) \\ &< \min_{k=1,\cdots,K}\left(\boldsymbol{x}^{T} A^R_{k,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right) \end{aligned}$$

This contradicts the fact that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium solution to P6(v_1^*, v_2^*). Similarly, we can prove for the case that there exists $\boldsymbol{y}\in Y$ such that (29) is satisfied.

From the above theorems, instead of solving P5 directly, we can obtain an equilibrium solution to P5 by solving P6(v_1^*, v_2^*), where (v_1^*, v_2^*) satisfies the equality conditions (26) and (27). On the other hand, an equilibrium solution to P6(v_1^*, v_2^*) is obtained by solving the following nonlinear programming problem [12].

P7(v_1^*, v_2^*)

$$\underset{\boldsymbol{x}\in X,\ \boldsymbol{y}\in Y,\ p,\ q,\ \sigma_1,\ \sigma_2}{\text{maximize}} \quad \sigma_1 + \sigma_2 - p - q \tag{30a}$$

subject to

$$A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\boldsymbol{e}_1 \leq p\boldsymbol{e}_1,\ k=1,\ldots,K \tag{30b}$$

$$\boldsymbol{x}^T B^R_{l,\,v_2^*}(p_{2l}) - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\boldsymbol{e}_2 \leq q\boldsymbol{e}_2,\ l=1,\ldots,L \tag{30c}$$

$$\boldsymbol{x}^T A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*) \geq \sigma_1,\ k=1,\ldots,K \tag{30d}$$

$$\boldsymbol{x}^T B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*) \geq \sigma_2,\ l=1,\ldots,L, \tag{30e}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are $(m\times 1)$ and $(n\times 1)$ column vectors whose elements are all ones. It should be noted here that $p \geq \sigma_1$, $q \geq \sigma_2$, and $\sigma_1 + \sigma_2 - p - q \leq 0$ always hold, because of the constraints in P7(v_1^*, v_2^*).

The following theorem shows the relationship between an optimal solution to P7(v_1^*, v_2^*) and an equilibrium solution to P5.

Theorem 3: Let $(\boldsymbol{x}^*, \boldsymbol{y}^*, p^*, q^*, \sigma_1^*, \sigma_2^*)$ be an optimal solution to P7(v_1^*, v_2^*). If $\sigma_1^* = p^* = 0$, $\sigma_2^* = q^* = 0$, then $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an equilibrium solution for P5.

(Proof) : Since $(\boldsymbol{x}^*, \ \boldsymbol{y}^*)$, $p^* = q^* = \sigma_1^* = \sigma_2^* = 0$ is a feasible solution to P7(v_1^*, v_2^*), the following inequalities hold.

$$A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\boldsymbol{e}_1 \le \boldsymbol{0},\ k = 1, \dots, K \quad (31a)$$

$$\boldsymbol{x}^{*T} B^R_{l,\,v_2^*}(p_{2l}) - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\boldsymbol{e}_2 \le \boldsymbol{0},\ l = 1, \dots, L \quad (31b)$$

$$\boldsymbol{x}^{*T} A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*) \ge 0,\ k = 1, \dots, K \quad (31c)$$

$$\boldsymbol{x}^{*T} B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*) \ge 0,\ l = 1, \dots, L, \quad (31d)$$

From (31c) and (31d), it holds that

$$\min_{k=1,\cdots,K} \left(\boldsymbol{x}^{*T} A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right) = 0,$$

$$\min_{l=1,\cdots,L} \left(\boldsymbol{x}^{*T} B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\right) = 0.$$

This means that the following equalities hold.

$$v_1^* = \min_{k=1,\dots,K} \Pi_{\boldsymbol{x}^{*T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k}) \quad (32a)$$

$$v_2^* = \min_{l=1,\dots,L} \Pi_{\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}^*}(\tilde{G}_{2l}) \quad (32b)$$

On the other hand, from (31a) and (31c), the following inequality holds.

$$\min_{k=1,\cdots,K} \left(\boldsymbol{x}^{*} A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right)$$
$$\ge \min_{k=1,\dots,K} \left(\boldsymbol{x}^{T} A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)\right),$$
$$\forall x \in X \quad (33a)$$

From (31b) and (31d), the following inequality holds.

$$\min_{l=1,\cdots,L} \left(\boldsymbol{x}^{*} B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y}^* - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\right)$$
$$\ge \min_{l=1,\dots,L} \left(\boldsymbol{x}^{*T} B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y} - \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)\right),$$
$$\forall y \in Y \quad (34a)$$

The above inequalities (32a), (32b), (33a) and (34a) can be equivalently expressed as follows.

$$v_1^* = \min_{k=1,\dots,K} \Pi_{\boldsymbol{x}^{*T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k})$$
$$\ge \min_{k=1,\dots,K} \Pi_{\boldsymbol{x}^{T}\tilde{A}_k(p_{1k})\boldsymbol{y}^*}(\tilde{G}_{1k}), \forall x \in X$$

$$v_2^* = \min_{l=1,\dots,L} \Pi_{\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}^*}(\tilde{G}_{2l})$$
$$\ge \min_{l=1,\dots,L} \Pi_{\boldsymbol{x}^{*T}\tilde{B}_l(p_{2l})\boldsymbol{y}}(\tilde{G}_{2l}), \forall y \in Y$$

From (15) and (16), it holds that

$$\min_{k=1,\dots,K} E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})]$$
$$\ge \min_{k=1,\dots,K} E[\Pi_{\boldsymbol{x}^{T}\tilde{\bar{A}}_k\boldsymbol{y}^*}(\tilde{G}_{1k})],\ \forall \boldsymbol{x} \in X,$$
$$\min_{l=1,\dots,L} E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{B}}_l\boldsymbol{y}^*}(\tilde{G}_{2l})]$$
$$\ge \min_{l=1,\dots,L} E[\Pi_{\boldsymbol{x}^{*T}\tilde{\bar{B}}_l\boldsymbol{y}}(\tilde{G}_{2l})],\ \forall \boldsymbol{y} \in Y.$$

This means that an optimal solution to P7(v_1^*, v_2^*) is an equilibrium solution to P5.

Unfortunately, we cannot obtain an equilibrium solution to P5 by solving P7(v_1^*, v_2^*), because the parameters (v_1^*, v_2^*) are unknown. However, since the first term $\boldsymbol{x}^T A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y}$ in the left hand of the inequality constraint (30d) is strictly monotone decreasing with respect to v_1^*, and the second term $\mu^{-1}_{\tilde{G}_{1k}}(v_1^*)$ in the left hand of the inequality constraint (30d) is strictly monotone increasing with respect to v_1^*, there exists some value of v_1^* such that $\boldsymbol{x}^T A^R_{k,\,v_1^*}(p_{1k})\boldsymbol{y} = \mu^{-1}_{\tilde{G}_{1k}}(v_1^*)$. In a similar way, we can fined v_2^* such that $\boldsymbol{x}^T B^R_{l,\,v_2^*}(p_{2l})\boldsymbol{y} = \mu^{-1}_{\tilde{G}_{2l}}(v_2^*)$.

From such a point of view, we can develop the algorithm to find the values of (v_1^*, v_2^*) such that $\sigma_1^* = 0$,$\sigma_2^* = 0$ by updating (v_1^*, v_2^*) sequentially, in which the conditions (26), (27) are satisfied. Using the bisection method with respect to (v_1^*, v_2^*), we can find the values of (v_1^*, v_2^*) such that $\sigma_1^* = \sigma_2^* = 0$.

Algorithm 1

Step 1 Set the initial values of the parameter (v_1^*, v_2^*) as $(0.5, 0.5)$.

Step 2 Solve P7(v_1^*, v_2^*), and obtain the optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*, p^*, q^*, \sigma_1^*, \sigma_2^*)$.

Step 3 If $\sigma_1^* > 0$, then $v_1^* \leftarrow v_1^* + \Delta v_1$, else if $\sigma_1^* < 0$, then $v_1^* \leftarrow v_1^* - \Delta v_1$. If $\sigma_2^* > 0$, then $v_2^* \leftarrow v_2^* + \Delta v_2$, else if $\sigma_2^* < 0$, $v_2^* \leftarrow v_2^* - \Delta v_2$, where Δv_1 and Δv_2 are sufficiently small positive constants, and return to Step 2. If $\mid \sigma_1^* \mid \le \epsilon$ and $\mid \sigma_2^* \mid \le \epsilon$, then stop, where ϵ is a sufficiently small positive constant.

IV. A NUMERICAL EXAMPLE

To show the efficiency of the proposed algorithm, consider the following numerical example, in which each player has two kinds of fuzzy random payoff matrices $\tilde{\bar{A}}_1, \tilde{\bar{A}}_2, \tilde{\bar{B}}_1, \tilde{\bar{B}}_2$. Assume that under the occurrence of scenarios $s_k \in \{1, 2, 3\}, k = 1, 2$ and $t_l \in \{1, 2, 3\}, l = 1, 2$, realizations of fuzzy random payoff matrices are expressed as the following fuzzy payoff matrices $\tilde{A}_{1s_1}, \tilde{A}_{2s_2}, \tilde{B}_{1t_1}, \tilde{B}_{2t_2}$.

$$\tilde{A}_{11} = \begin{bmatrix} (100, 40, 40) & (180, 50, 50) \\ (170, 42, 42) & (70, 21, 21) \end{bmatrix}$$

$$\tilde{A}_{12} = \begin{bmatrix} (110, 40, 40) & (228, 50, 50) \\ (176, 42, 42) & (88, 21, 21) \end{bmatrix}$$

$$\tilde{A}_{13} = \begin{bmatrix} (150, 40, 40) & (240, 50, 50) \\ (230, 42, 42) & (130, 21, 21) \end{bmatrix}$$

$$\tilde{A}_{21} = \begin{bmatrix} (40, 20, 20) & (70, 30, 30) \\ (40, 15, 15) & (120, 40, 40) \end{bmatrix}$$

$$\tilde{A}_{22} = \begin{bmatrix} (60, 20, 20) & (100, 30, 30) \\ (30, 15, 15) & (100, 40, 40) \end{bmatrix}$$

$$\tilde{A}_{23} = \begin{bmatrix} (50, 20, 20) & (100, 30, 30) \\ (26, 15, 15) & (80, 40, 40) \end{bmatrix}$$

$$\tilde{B}_{11} = \begin{bmatrix} (100, 30, 30) & (16, 10, 10) \\ (37, 20, 20) & (75, 25, 25) \end{bmatrix}$$

$$\tilde{B}_{12} = \begin{bmatrix} (110, 30, 30) & (21, 10, 10) \\ (47, 20, 20) & (93, 25, 25) \end{bmatrix}$$

$$\tilde{B}_{13} = \begin{bmatrix} (150,30,30) & (35,10,10) \\ (60,20,20) & (120,25,25) \end{bmatrix}$$
$$\tilde{B}_{21} = \begin{bmatrix} (40,20,20) & (85,25,25) \\ (25,10,10) & (13,5,5) \end{bmatrix}$$
$$\tilde{B}_{22} = \begin{bmatrix} (65,20,20) & (70,25,25) \\ (35,10,10) & (15,5,5) \end{bmatrix}$$
$$\tilde{B}_{23} = \begin{bmatrix} (45,20,20) & (76,25,25) \\ (30,10,10) & (17,5,5) \end{bmatrix}$$

In the above matrices, each element is a triangular-type fuzzy number denoted as $(a_{ks_kij}, \alpha_{kij}, \beta_{kij})$ and $(b_{lt_lij}, \gamma_{lij}, \delta_{lij})$, respectively. The corresponding probabilities are set as $p_{1ks_k} = 1/3, k = 1,2, s_k = 1,2,3$ and $p_{2lt_l} = 1/3, l = 1,2, t_l = 1,2,3$, respectively. Assume that hypothetical players set their membership functions as follows.

$$\mu_{\tilde{G}_{11}}(u) = \frac{u-0}{230-0}, \quad \mu_{\tilde{G}_{12}}(u) = \frac{u-0}{110-0}$$
$$\mu_{\tilde{G}_{21}}(v) = \frac{v-0}{150-0}, \quad \mu_{\tilde{G}_{22}}(v) = \frac{v-0}{90-0}$$

The step sizes and the terminal condition at Step 3 of the proposed algorithm are set as $\Delta v_1 = \Delta v_2 = 0.001$ and $\epsilon = 0.1$. Then, applying Algorithm 1 proposed in the previous section, the equilibrium solution based on the fuzzy decision is obtained as follows.

$$(x_1^*, x_2^*) = (0.322157, 0.677843)$$
$$(y_1^*, y_2^*) = (0.643082, 0.356918)$$
$$(E[\Pi_{\boldsymbol{x}^T \tilde{\bar{A}}_1 \boldsymbol{y}}(\tilde{G}_{11})], E[\Pi_{\boldsymbol{x}^T \tilde{\bar{A}}_2 \boldsymbol{y}}(\tilde{G}_{12})])$$
$$= (0.725596, 0.617737)$$
$$(E[\Pi_{\boldsymbol{x}^T \tilde{\bar{B}}_1 \boldsymbol{y}}(\tilde{G}_{21})], E[\Pi_{\boldsymbol{x}^T \tilde{\bar{B}}_2 \boldsymbol{y}}(\tilde{G}_{22})])$$
$$= (0.54552, 0.472824)$$

V. Conclusion

In this paper, we have formulated multiobjective fuzzy random bimatrix games, and introduced an equilibrium solution concept based on the fuzzy decision, in which the expectation model and possibility measure are applied to deal with fuzzy random payoffs. By applying Algorithm 1, we can efficiently obtain such equilibrium solutions to multiobjective fuzzy random bimatrix games, in which the equilibrium conditions in the membership function space are transformed into the equilibrium conditions in the expected payoff space to circumvent the computational difficulties. In the proposed method, it is assumed that a realization of each element of fuzzy random payoffs is a triangular-type fuzzy number, and fuzzy goals for the objective functions of two player are expressed as linear membership functions. In the near future, we would like to deal with more generalized models of multiobjective fuzzy random bimatrix games, in which nonlinear membership functions are involved.

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