Multiobjective Fuzzy Random Bimatrix Games and An Equilibrium Solution Concept

Hitoshi Yano

Abstract—In this paper, we formulate multiobjective bimatrix games with fuzzy random payoffs, and introduce an equilibrium solution concept based on the fuzzy decision. By applying possibility measure and an expectation model to such games, the corresponding equilibrium solution is defined. To circumvent the computational difficulties to obtain an equilibrium solution, the algorithm based on the bisection method is proposed, in which equilibrium conditions in the membership function space are replaced into equilibrium conditions in the expected payoff space.

Index Terms—multiobjective bimatrix games, fuzzy random variables, expectation model, possibility measure, fuzzy decision.

I. INTRODUCTION

To deal with bimatrix games with triangular fuzzy numbers, Maeda [9] defined an equilibrium solution concept using possibility measure and the threshold values for the level sets [2]. He formulated the corresponding mathematical programming problem to obtain such parametric equilibrium solutions. Using the expected value concept for possibility measure and necessity measure, Li et al. [6], [7] formulated quadratic programming problems to obtain the corresponding Nash equilibrium solutions for bimatrix games with triangular fuzzy numbers. Mako et al. [10] focused on bimatrix games with LR fuzzy numbers. Corresponding to the fuzzy Nash-equilibrium solution concept, they proposed the fuzzy correlated equilibrium solution concept, which is based on a joint distribution for mixed strategies of both players. Gao [3] introduced three kinds of uncertain equilibrium solution concepts based on uncertainty theory [8], which depend on the values of confidence levels. From a similar point of view based on uncertainty theory, Tang et al. [16] proposed an uncertain equilibrium solution concept based on the Hurwicz criterion.

For multiobjective bimatrix games, Corley [1] first defined a Pareto equilibrium solution concept, and formulated quadratic programming problems to obtain Pareto equilibrium solutions through the Karush-Kuhn-Tucker conditions, in which multiobjective functions are scalarized by the weighting coefficients. Nishizaki et al. [12] formulated multiobjective bimatrix games incorporating fuzzy goals. They transformed multiobjective bimatrix games into usual bimatrix games by applying the weighting methods or the minimum operator [14], [22], and defined the corresponding equilibrium solution concepts. They formulated the nonlinear programming problems to obtain such equilibrium solutions. Using dominance cones proposed by Yu [21], Nishizaki et al. [11] defined a nondominated equilibrium solution concept which is a generalization of Nash-equilibrium solution concept, and formulate nonlinear programming problem to obtain nondominated equilibrium solutions by applying the Karush-Kuhn-Tucker conditions.

On the other hand, the concept of fuzzy random variable was first introduced by Kwakernaak [4], and its definition in an $n$-dimensional Euclidean space were given by Puri and Ralescu [13]. Roughly speaking, fuzzy random variables defined by Wang and Zhang [17] can be interpreted as random variables whose realized values are not real values, but rather are fuzzy sets. From the perspective that both randomness and fuzziness are often involved simultaneously in real-world decision making problems, we have already formulated several kinds of multiobjective fuzzy random Stackerberg games with simple recourses, introduced the equilibrium solution concepts, and proposed the interactive algorithms to obtain a satisfactory solution of the player from among an equilibrium solution set [18], [19], [20].

In this paper, we focus on multiobjective bimatrix games with triangular-type fuzzy random variables. After such multiobjective bimatrix games are transformed into usual bimatrix games by applying possibility measure [2] and the expectation model [15] for stochastic programming problems, the corresponding equilibrium solution based on the fuzzy decision [14], [22] is introduced. In section II, multiobjective fuzzy random bimatrix games are formulated. In section III, by applying possibility measure [2] and an expectation model [15] for stochastic programming problems, the corresponding equilibrium solution based on the fuzzy decision [14], [22] is defined. To circumvent the computational difficulties to deal with each objective function based on possibility measure directly, the algorithm based on the bisection method is proposed, in which which equilibrium conditions in the membership function space are replaced into equilibrium conditions in the expected payoff space. In section IV, a numerical example of two-objective bimatrix games with fuzzy random payoffs illustrates interactive processes under a hypothetical player to show the efficiency of the proposed method.

II. MULTIOBJECTIVE FUZZY RANDOM BIMATRIX GAMES

In this section, we consider multiobjective bimatrix games with fuzzy payoffs. Let $i \in \{1, 2, \cdots, m\}$ be a pure strategy of Player 1 and $j \in \{1, 2, \cdots, n\}$ be a pure strategy of Player 2. $\tilde{A}_k \triangleq (\tilde{a}_{kij})$, $k = 1, \cdots, K$ are Player 1’s $(m \times n)$-payoff matrices, and $\tilde{B}_l \triangleq (\tilde{b}_{lij})$, $l = 1, \cdots, L$ are Player 2’s $(m \times n)$-payoff matrices, whose elements $\tilde{a}_{kij}$ and $\tilde{b}_{lij}$ are fuzzy random variables [4] (The symbols $\cdots$ and $\cdots$ mean randomness and fuzziness respectively). Throughout this paper, we assume that under the occurrence of scenarios $s_k \in \{1, \cdots, S_k\}$ and $t_l \in \{1, \cdots, T_l\}$, $\tilde{a}_{kst_{ij}}$ and $\tilde{b}_{lt_{ij}}$ are realizations of fuzzy random variables $\tilde{a}_{kij}$ and $\tilde{b}_{l_{ij}}$, which are fuzzy numbers whose membership functions are defined

H. Yano is with Department of Social Sciences, Graduate School of Humanities and Social Sciences, Nagoya City University, Nagoya, 467-8501, Japan, e-mail: yano@hum.nagoya-cu.ac.jp
as follows.

\[
\mu_{\tilde{a}_{ki}}(u) = \begin{cases} 
\frac{1}{\alpha_{kij}} & \text{if } u \leq a_{kij} \\
\frac{1}{\beta_{ij}} & \text{if } u > a_{kij}
\end{cases}
\]

\[i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, K,
\]

\[k = 1, \ldots, K
\]

where the spread parameters \(\alpha_{kij} > 0, \beta_{ij} > 0, \gamma_{ij} > 0\) and \(\delta_{ij} > 0\) are constants and the mean value \(a_{kij}\) and \(b_{ij}\) vary depending on the scenarios \(s_k\) and \(t_l\). Moreover, we assume that a scenario \(s_k\) occurs with a probability \(p_{k}\), where \(\sum_{k=1}^{K} p_{k} = 1\), and a scenario \(t_l\) occurs with a probability \(p_{l}\), where \(\sum_{l=1}^{L} p_{l} = 1\).

Then, a multiobjective bimatrix game with fuzzy random payoffs can be formulated as follows, where \(T\) means transportation.

**P1**

\[
\begin{align*}
\text{maximize} & \quad (x^T \tilde{A}_1 y, \ldots, x^T \tilde{A}_K y) \\
\text{maximize} & \quad (x^T \tilde{B}_1 y, \ldots, x^T \tilde{B}_L y)
\end{align*}
\]

where

\[x \in X \quad \text{and} \quad y \in Y
\]

\[
X \equiv \{x \in \mathbb{R}^m \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, \ldots, m\}
\]

\[
Y \equiv \{y \in \mathbb{R}^n \mid \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1, \ldots, n\}
\]

are mixed strategies for Player 1 and Player 2. It should be noted here that, the expected payoffs for the scenarios \(s_k \in \{1, \ldots, S_k\}\) and \(t_l \in \{1, \ldots, T_l\}\) can be expressed as fuzzy numbers whose membership functions can be defined as follows [2].

**P2**

\[
\begin{align*}
\text{maximize} & \quad (\Pi_{x^T \tilde{A}_1 y}(\tilde{G}_1), \ldots, \Pi_{x^T \tilde{A}_K y}(\tilde{G}_1)) \\
\text{maximize} & \quad (\Pi_{x^T \tilde{B}_1 y}(\tilde{G}_2), \ldots, \Pi_{x^T \tilde{B}_L y}(\tilde{G}_2))
\end{align*}
\]

By applying an expectation model [15] to each objective function in P2, P2 can be transformed into a usual multiobjective bimatrix game as follows.

**P3**

\[
\begin{align*}
\text{maximize} & \quad (E[\Pi_{x^T \tilde{A}_1 y}(\tilde{G}_1)], \ldots, E[\Pi_{x^T \tilde{A}_K y}(\tilde{G}_1)]) \\
\text{maximize} & \quad (E[\Pi_{x^T \tilde{B}_1 y}(\tilde{G}_2)], \ldots, E[\Pi_{x^T \tilde{B}_L y}(\tilde{G}_2)])
\end{align*}
\]

From Assumption 1, the following relations always hold.

\[
0 \leq \Pi_{x^T \tilde{A}_k y}(\tilde{G}_k) < 1, \quad k = 1, \ldots, K,
\]

\[
\forall x \in X, \forall y \in Y
\]

\[
0 \leq \Pi_{x^T \tilde{B}_l y}(\tilde{G}_k) < 1, \quad l = 1, \ldots, T_l,
\]

\[
\forall x \in X, \forall y \in Y
\]

To define an equilibrium solution concept to P3, We assume that both players adopt the fuzzy decision [14], [22] to integrate multiple objectives in P3. Then, P3 can be reduced to the following bimatrix game.

**P4**

\[
\begin{align*}
\text{maximize} & \quad \min_{k=1, \ldots, K} E[\Pi_{x^T \tilde{A}_k y}(\tilde{G}_k)] \\
\text{maximize} & \quad \min_{l=1, \ldots, L} E[\Pi_{x^T \tilde{B}_l y}(\tilde{G}_2)]
\end{align*}
\]
Now, we can introduce an equilibrium solution concept to P4.

**Definition 1:** \((x^*, y^*) \in X \times Y\) is an equilibrium solution to P4, if the following inequalities hold.

\[
\begin{align*}
\min_{k=1, \ldots, K} E[\Pi^{x \rightarrow \tilde{A}_k} y(G_{1k})] \\
& \geq \min_{k=1, \ldots, K} E[\Pi^{x \rightarrow \tilde{A}_k} y(G_{1k})], \forall x \in X \\
& \geq \min_{l=1, \ldots, L} E[\Pi^{x \rightarrow \tilde{B}_l} y(G_{2l})], \forall y \in Y
\end{align*}
\]

From the definition of the membership functions (5), (6) and Assumption 1, \(E[\Pi^{x \rightarrow \tilde{A}_k} y(G_{1k})]\) and \(E[\Pi^{x \rightarrow \tilde{B}_l} y(G_{2l})]\) can be expressed as the following forms.

\[
\begin{align*}
E[\Pi^{x \rightarrow \tilde{A}_k} y(G_{1k})] &= \sum_{s_k=1}^{S_k} p_{1s_k} \cdot \Pi^{x \rightarrow \tilde{A}_k s_k} y(G_{1k}) \\
& = \sum_{s_k=1}^{S_k} p_{1s_k} \cdot \left( \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{k_{s_k}ij} + \beta_{k_{s_k}ij})x_{iyj} - E_{k10} \right) \\
& \geq \sum_{s_k=1}^{S_k} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \sum_{s_k=1}^{S_k} p_{1s_k} (a_{k_{s_k}ij} + \beta_{k_{s_k}ij})x_{iyj} - E_{k10} \right) \\
& = \frac{\sum_{s_k=1}^{S_k} \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{k_{s_k}ij} + \beta_{k_{s_k}ij})x_{iyj} - E_{k10}}{E_{k11} - E_{k10} + \sum_{s_k=1}^{S_k} \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{k_{s_k}ij}x_{iyj}} \\
& \geq \Pi^{x \rightarrow \tilde{A}_k (p_{1s_k})} y(G_{1k}) \\
& = \Pi^{x \rightarrow \tilde{B}_l} y(G_{2l}) \\
& = \sum_{t_{l}=1}^{T_l} p_{2t_{l}} \cdot \Pi^{x \rightarrow \tilde{B}_l t_{l}} y(G_{2l}) \\
& = \sum_{t_{l}=1}^{T_l} p_{2t_{l}} \cdot \left( \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{l_{t_{l}ij}} + \delta_{l_{t_{l}ij}})x_{iyj} - E_{l20} \right) \\
& \geq \sum_{t_{l}=1}^{T_l} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \sum_{t_{l}=1}^{T_l} p_{2t_{l}} (b_{l_{t_{l}ij}} + \delta_{l_{t_{l}ij}})x_{iyj} - E_{l20} \right) \\
& = \frac{\sum_{t_{l}=1}^{T_l} \sum_{i=1}^{m} \sum_{j=1}^{n} (b_{l_{t_{l}ij}} + \delta_{l_{t_{l}ij}})x_{iyj} - E_{l20}}{E_{l21} - E_{l20} + \sum_{t_{l}=1}^{T_l} \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{l_{t_{l}ij}}x_{iyj}} \\
& = \Pi^{x \rightarrow \tilde{B}_l (p_{2t_{l}})} y(G_{2l})
\end{align*}
\]

Consider the \(v^*_l\)-level set for the fuzzy numbers \(x \rightarrow \tilde{A}_k (p_{1s_k}) y^*\) and \(v^*_l\)-level set for the fuzzy numbers \(x^T \tilde{B}_l (p_{2t_{l}}) y^*\) as follows.

\[
\begin{align*}
A_{k, v^*_l} (p_{1s_k}) &= \left( \sum_{s_k=1}^{S_k} p_{1s_k} a_{k_{s_k}ij}, v^*_l \right) \\
A_{k, v^*_l} (p_{1s_k}) &= \left( \sum_{s_k=1}^{S_k} p_{1s_k} a_{k_{s_k}ij}, v^*_l \right) \\
B_{l, v^*_l} (p_{2t_{l}}) &= \left( \sum_{t_{l}=1}^{T_l} p_{2t_{l}} b_{l_{t_{l}ij}}, v^*_l \right) \\
B_{l, v^*_l} (p_{2t_{l}}) &= \left( \sum_{t_{l}=1}^{T_l} p_{2t_{l}} b_{l_{t_{l}ij}}, v^*_l \right)
\end{align*}
\]
Corresponding to (20) and (21), we consider the following bimatrix game, in which \((v^*_1, v^*_2)\) are given as parameters in advance.

**P6**\((v^*_1, v^*_2)\):

\[
\begin{align*}
\max_{x \in X} \quad & \min_{k=1, \ldots, K} \left\{ x^T A^R_{k,v_1} (p_{1k}) y - \mu_{G_{1k}}^{-1} (v^*_1) \right\} \\
\min_{y \in Y} \quad & \max_{l=1, \ldots, L} \left\{ x^T B^R_{l,v_2} (p_{2l}) y - \mu_{G_{2l}}^{-1} (v^*_2) \right\}
\end{align*}
\]

For \(P6(v^*_1, v^*_2)\), we introduce an equilibrium solution concept.

**Definition 2**: \((x^*, y^*)\) is an equilibrium solution to \(P6(v^*_1, v^*_2)\), if the following inequalities hold.

\[
\begin{align*}
\min_{k=1, \ldots, K} \left\{ x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (v^*_1) \right\} & \geq 0, \\
\max_{l=1, \ldots, L} \left\{ x^T B^R_{l,v_2} (p_{2l}) y - \mu_{G_{2l}}^{-1} (v^*_2) \right\} & \geq 0
\end{align*}
\]

Then, the following relationships between equilibrium solutions to \(P6(v^*_1, v^*_2)\) and equilibrium solutions to \(P5\) hold.

**Theorem 1**: If \((x^*, y^*, v^*_1, v^*_2)\) is an equilibrium solution to \(P5\), then \((x^*, y^*)\) is an equilibrium solution to \(P6(v^*_1, v^*_2)\).

**(Proof)**: Assume that \((x^*, y^*)\) is not an equilibrium solution to \(P6(v^*_1, v^*_2)\). Then, there exists some \(x \in X\) such that

\[
\begin{align*}
\min_{k=1, \ldots, K} \left\{ x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (v^*_1) \right\} & < 0, \\
\max_{l=1, \ldots, L} \left\{ x^T B^R_{l,v_2} (p_{2l}) y^* - \mu_{G_{2l}}^{-1} (v^*_2) \right\} & < 0
\end{align*}
\]

This contradicts the fact that \((x^*, y^*, v^*_1, v^*_2)\) is an equilibrium solution to \(P5\). Similarly, we can prove for the case that there exists \(y \in Y\) such that (25) is satisfied.

**Theorem 2**: If \((x^*, y^*)\) is an equilibrium solution to \(P6(v^*_1, v^*_2)\), where the following relations hold,

\[
\begin{align*}
\min_{k=1, \ldots, K} \left( x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (v^*_1) \right) & = 0, \\
\min_{l=1, \ldots, L} \left( x^T B^R_{l,v_2} (p_{2l}) y^* - \mu_{G_{2l}}^{-1} (v^*_2) \right) & = 0
\end{align*}
\]

then, \((x^*, y^*, v^*_1, v^*_2)\) is an equilibrium solution to \(P5\).

**(Proof)**: Assume that \((x^*, y^*, v^*_1, v^*_2)\) is not an equilibrium solution to \(P5\). Then, there exists some \(x \in X\) such that

\[
\begin{align*}
v^*_1 & = \min_{k=1, \ldots, K} E[I_{x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (v^*_1)}] \\
& < \min_{k=1, \ldots, K} E[I_{x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (\hat{v}^*_1)}]
\end{align*}
\]

or, there exists some \(y \in Y\) such that

\[
\begin{align*}
v^*_2 & = \min_{l=1, \ldots, L} E[I_{x^T B^R_{l,v_2} (p_{2l}) y^* - \mu_{G_{2l}}^{-1} (v^*_2)}] \\
& < \min_{l=1, \ldots, L} E[I_{x^T B^R_{l,v_2} (p_{2l}) y^* - \mu_{G_{2l}}^{-1} (\hat{v}^*_2)}]
\end{align*}
\]

From (15) and (16), this means that there exists some \(x \in X\) such that

\[
\begin{align*}
v^*_1 & = \min_{k=1, \ldots, K} \Pi_{x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (\hat{v}^*_1)} \\
& < \min_{k=1, \ldots, K} \Pi_{x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (\hat{v}^*_1)}
\end{align*}
\]

or, there exists some \(y \in Y\) such that

\[
\begin{align*}
v^*_2 & = \min_{l=1, \ldots, L} \Pi_{x^T B^R_{l,v_2} (p_{2l}) y^* - \mu_{G_{2l}}^{-1} (\hat{v}^*_2)} \\
& < \min_{l=1, \ldots, L} \Pi_{x^T B^R_{l,v_2} (p_{2l}) y^* - \mu_{G_{2l}}^{-1} (\hat{v}^*_2)}
\end{align*}
\]

Assume that there exists some \(x \in X\) such that (28) is satisfied. Then, the following relation holds.

\[
0 = \min_{k=1, \ldots, K} \left( x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (v^*_1) \right) < \min_{k=1, \ldots, K} \left( x^T A^R_{k,v_1} (p_{1k}) y^* - \mu_{G_{1k}}^{-1} (\hat{v}^*_1) \right)
\]

This contradicts the fact that \((x^*, y^*)\) is an equilibrium solution to \(P6(v^*_1, v^*_2)\). Similarly, we can prove for the case that there exists \(y \in Y\) such that (29) is satisfied.

From the above theorems, instead of solving \(P5\) directly, we can obtain an equilibrium solution to \(P5\) by solving \(P6(v^*_1, v^*_2)\), where \((v^*_1, v^*_2)\) satisfies the equality conditions (26) and (27). On the other hand, an equilibrium solution to \(P6(v^*_1, v^*_2)\) is obtained by solving the following nonlinear programming problem [12].

**P7**\((v^*_1, v^*_2)\):

\[
\begin{align*}
\max_{x \in X, \quad y \in Y, \quad \sigma_1, \sigma_2} & \quad \sigma_1 + \sigma_2 - p - q \\
\text{subject to} & \quad A^R_{k,v_1} (p_{1k}) y - \mu_{G_{1k}}^{-1} (v^*_1) - e_1 \leq p_{1k}, \quad k = 1, \ldots, K \\
& \quad x^T B^R_{l,v_2} (p_{2l}) y - \mu_{G_{2l}}^{-1} (v^*_2) - e_2 \leq q_{2l}, \quad l = 1, \ldots, L \\
& \quad x^T A^R_{k,v_1} (p_{1k}) y - \mu_{G_{1k}}^{-1} (\hat{v}^*_1) \geq \sigma_1, \quad k = 1, \ldots, K \\
& \quad x^T B^R_{l,v_2} (p_{2l}) y - \mu_{G_{2l}}^{-1} (\hat{v}^*_2) \geq \sigma_2, \quad l = 1, \ldots, L
\end{align*}
\]
The following theorem shows the relationship between an optimal solution to Problem P7($v^*_1, v^*_2$) and an equilibrium solution to Problem P5.

**Theorem 3:** Let $(x^*, y^*, p^*, q^*, \sigma^*_1, \sigma^*_2)$ be an optimal solution to Problem P7($v^*_1, v^*_2$). If $\sigma^*_1 = p^* = 0$, $\sigma^*_2 = q^* = 0$, then $(x^*, y^*)$ is an equilibrium solution for Problem P5.

**(Proof):** Since $(x^*, y^*)$, $p^* = q^* = \sigma^*_1 = \sigma^*_2 = 0$ is a feasible solution to Problem P7($v^*_1, v^*_2$), the following inequalities hold:

\[ A_{k, v_1^*}^R (p_{1k}) y^{*} - \mu^{-1}_{G_{1k}} (v^*_1) e_l \leq 0, \quad k = 1, \ldots, K \]  
\[ x^T B_{l, v_2^*}^R (p_{2l}) - \mu^{-1}_{G_{2l}} (v^*_2) e_2 \leq 0, \quad l = 1, \ldots, L \]  
\[ x^T A_{k, v_1^*}^R (p_{1k}) y^{*} - \mu^{-1}_{G_{1k}} (v^*_1) \geq 0, \quad k = 1, \ldots, K \]  
\[ x^T B_{l, v_2^*}^R (p_{2l}) y^{*} - \mu^{-1}_{G_{2l}} (v^*_2) \geq 0, \quad l = 1, \ldots, L \]

From (31c) and (31d), it holds that

\[ \min_{k=1, \ldots, K} \left( x^T A_{k, v_1^*}^R (p_{1k}) y^{*} - \mu^{-1}_{G_{1k}} (v^*_1) \right) = 0, \quad \min_{l=1, \ldots, L} \left( x^T B_{l, v_2^*}^R (p_{2l}) y^{*} - \mu^{-1}_{G_{2l}} (v^*_2) \right) = 0. \]

This means that the following equalities hold.

\[ v^*_1 = \min_{k=1, \ldots, K} \Pi x^T \bar{A}_{k}(p_{1k}) y^{*}, \quad G_{1k} \]
\[ v^*_2 = \min_{l=1, \ldots, L} \Pi x^T \bar{B}_{l}(p_{2l}) y^{*}, \quad G_{2l} \]

On the other hand, from (31a) and (31c), the following inequality holds.

\[ \min_{k=1, \ldots, K} \left( x^T A_{k, v_1^*}^R (p_{1k}) y^{*} - \mu^{-1}_{G_{1k}} (v^*_1) \right) \geq \min_{k=1, \ldots, K} \left( x^T A_{k, v_1^*}^R (p_{1k}) y^{*} - \mu^{-1}_{G_{1k}} (v^*_1) \right), \quad \forall x \in X \]

From (31b) and (31d), the following inequality holds.

\[ \min_{l=1, \ldots, L} \left( x^T B_{l, v_2^*}^R (p_{2l}) y^{*} - \mu^{-1}_{G_{2l}} (v^*_2) \right) \geq \min_{l=1, \ldots, L} \left( x^T B_{l, v_2^*}^R (p_{2l}) y^{*} - \mu^{-1}_{G_{2l}} (v^*_2) \right), \quad \forall y \in Y \]

The above inequalities (32a), (32b), (33a) and (34a) can be equivalently expressed as follows.

\[ v^*_1 = \min_{k=1, \ldots, K} \Pi x^T \bar{A}_{k}(p_{1k}) y^{*}, \quad G_{1k} \]
\[ \geq \min_{k=1, \ldots, K} \Pi x^T \bar{A}_{k}(p_{1k}) y^{*}, \quad G_{1k}, \quad \forall x \in X \]
\[ v^*_2 = \min_{l=1, \ldots, L} \Pi x^T \bar{B}_{l}(p_{2l}) y^{*}, \quad G_{2l} \]
\[ \geq \min_{l=1, \ldots, L} \Pi x^T \bar{B}_{l}(p_{2l}) y^{*}, \quad G_{2l}, \quad \forall y \in Y \]

From (15) and (16), it holds that

\[ \min_{k=1, \ldots, K} \text{E} [\Pi x^T \bar{A}_{k} y^{*}, \quad G_{1k}] \]
\[ \geq \min_{k=1, \ldots, K} \text{E} [\Pi x^T \bar{A}_{k} y^{*}, \quad G_{1k}], \quad \forall x \in X, \]
\[ \min_{l=1, \ldots, L} \text{E} [\Pi x^T \bar{B}_{l} y^{*}, \quad G_{2l}] \]
\[ \geq \min_{l=1, \ldots, L} \text{E} [\Pi x^T \bar{B}_{l} y^{*}, \quad G_{2l}], \quad \forall y \in Y. \]

This means that an optimal solution to Problem P7($v^*_1, v^*_2$) is an equilibrium solution to Problem P5.

Unfortunately, we cannot obtain an equilibrium solution to Problem P5 by solving Problem P7($v^*_1, v^*_2$), because the parameters ($v^*_1, v^*_2$) are unknown. However, since the first term $x^T A_{k, v_1^*}^R (p_{1k}) y$ in the left hand of the inequality constraint (30d) is strictly monotone decreasing with respect to $v^*_1$, and the second term $\mu^{-1}_{G_{1k}} (v^*_1)$ in the left hand of the inequality constraint (30d) is strictly monotone increasing with respect to $v^*_1$, there exists some value of $v^*_1$ such that $x^T A_{k, v_1^*}^R (p_{1k}) y = \mu^{-1}_{G_{1k}} (v^*_1)$. In a similar way, we can find $v^*_2$ such that $x^T B_{l, v_2^*}^R (p_{2l}) y = \mu^{-1}_{G_{2l}} (v^*_2)$.

From such a point of view, we can develop the algorithm to find the values of ($v^*_1, v^*_2$) such that $\sigma^*_1 = 0, \sigma^*_2 = 0$ by updating ($v^*_1, v^*_2$) sequentially, in which the conditions (26), (27) are satisfied. Using the bisection method with respect to ($v^*_1, v^*_2$), we can find the values of ($v^*_1, v^*_2$) such that $\sigma^*_1 = \sigma^*_2 = 0$.

**Algorithm 1**

1. Set the initial values of the parameter ($v^*_1, v^*_2$) as $(0.5, 0.5)$.
2. Solve Problem P7($v^*_1, v^*_2$), and obtain the optimal solution $(x^*, y^*, p^*, q^*, \sigma^*_1, \sigma^*_2)$.
3. If $\sigma^*_1 > 0$, then $v^*_1 \leftarrow v^*_1 + \Delta v_1$, else if $\sigma^*_1 < 0$, then $v^*_1 \leftarrow v^*_1 - \Delta v_1$. If $\sigma^*_2 > 0$, then $v^*_2 \leftarrow v^*_2 + \Delta v_2$, else if $\sigma^*_2 < 0$, then $v^*_2 \leftarrow v^*_2 - \Delta v_2$, where $\Delta v_1$ and $\Delta v_2$ are sufficiently small positive constants, and return to Step 2. If $|\sigma^*_1| \leq \epsilon$ and $|\sigma^*_2| \leq \epsilon$, then stop, where $\epsilon$ is a sufficiently small positive constant.

IV. A NUMERICAL EXAMPLE

To show the efficiency of the proposed algorithm, consider the following numerical example, in which each player has two kinds of fuzzy random payoff matrices $A_1, A_2, B_1, B_2$. Assume that under the occurrence of scenarios $s_k \in \{1, 2, 3\}, k = 1, 2$ and $t_l \in \{1, 2, 3\}, l = 1, 2$, realizations of fuzzy random payoff matrices are expressed as the following fuzzy payoff matrices $A_{1s_1}, A_{2s_2}, B_{1t_1}, B_{2t_2}$.
\[
B_{13} = \begin{bmatrix}
(150, 30, 30) & (35, 10, 10) \\
(60, 20, 20) & (120, 25, 25)
\end{bmatrix}
\]
\[
B_{21} = \begin{bmatrix}
(40, 20, 20) & (85, 25, 25) \\
(25, 10, 10) & (13, 5, 5)
\end{bmatrix}
\]
\[
B_{22} = \begin{bmatrix}
(65, 20, 20) & (70, 25, 25) \\
(35, 10, 10) & (15, 5, 5)
\end{bmatrix}
\]
\[
B_{23} = \begin{bmatrix}
(45, 20, 20) & (76, 25, 25) \\
(30, 10, 10) & (17, 5, 5)
\end{bmatrix}
\]

In the above matrices, each element is a triangular-type fuzzy number denoted as \((a_{kij}, b_{kij}, \alpha_{kij})\) and \((b_{kij}, \gamma_{kij}, \delta_{kij})\), respectively. The corresponding probabilities are set as \(p_{kij} = 1/3, k = 1, 2, s_k = 1, 2, 3\) and \(p_{kij} = 1/3, l = 1, 2, t_l = 1, 2, 3\), respectively. Assume that hypothetical players set their membership functions as follows.

\[
\mu_{G_{11}}(u) = \frac{u - 0}{230 - 0}, \quad \mu_{G_{12}}(u) = \frac{u - 0}{110 - 0}
\]
\[
\mu_{G_{21}}(v) = \frac{v - 0}{150 - 0}, \quad \mu_{G_{22}}(v) = \frac{v - 0}{90 - 0}
\]

The step sizes and the terminal condition at Step 3 of the proposed algorithm are set as \(\Delta t_1 = \Delta t_2 = 0.001\) and \(\epsilon = 0.1\). Then, applying Algorithm 1 proposed in the previous section, the equilibrium solution based on the fuzzy decision is obtained as follows.

\[
(x_1^*, x_2^*) = (0.322157, 0.677843)
\]
\[
(y_1^*, y_2^*) = (0.640382, 0.356918)
\]
\[
(E[\Pi x^T \hat{A}_1 y(G_{11})], E[\Pi x^T \hat{A}_2 y(G_{12})])
\]
\[
= (0.725596, 0.617737)
\]
\[
(E[\Pi x^T \hat{B}_1 y(G_{21})], E[\Pi x^T \hat{B}_2 y(G_{22})])
\]
\[
= (0.54552, 0.472824)
\]

V. Conclusion

In this paper, we have formulated multiobjective fuzzy random bimatrix games, and introduced an equilibrium solution concept based on the fuzzy decision, in which the expectation model and possibility measure are applied to deal with fuzzy random payoffs. By applying Algorithm 1, we can efficiently obtain such equilibrium solutions to multiobjective fuzzy random bimatrix games, in which the equilibrium conditions in the membership function space are transformed into the equilibrium conditions in the expected payoff space to circumvent the computational difficulties. In the proposed method, it is assumed that a realization of each element of fuzzy random payoffs is a triangular-type fuzzy number, and fuzzy goals for the objective functions of two player are expressed as linear membership functions. In the near future, we would like to deal with more generalized models of multiobjective fuzzy random bimatrix games, in which nonlinear membership functions are involved.

REFERENCES