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Discrete Pre-Bundles and CNF-SAT

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Abstract— **In this paper a discrete pre-bundle concept is discussed, of which several properties are established. Existence results are proven for pre-bundles of CNF base hypergraphs. Also pre-bundles are constructed for clause sets and CNF formulas.**

Keywords: CNF, satisfiability, bundle, hypergraph

1 Introduction

The propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas settles the basis for the NP-completeness concept of the computational complexity theory [5]. Due to the high expressiveness of the CNF language numerous computational problems can be encoded as equivalent instances of CNF-SAT via reduction [8]. In applications most often the modelling CNF formulas are of a specific structure for which fast algorithms are required. Also from a theoretical point of view one is interested in classes for which SAT can be solved in polynomial time. There are known several classes, for which SAT can be tested efficiently, such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested, co-nested formulas, and exact linear formulas etc. [1, 3, 4, 6, 9, 10, 11, 12, 17, 18, 20]. In this paper we investigate a discrete (pre-)bundle approach in order to gain structural insight into CNF-SAT investigating the fibre-view on clause sets. First a pre-bundle hierarchy is established and several of its properties are investigated. Existence results regarding discrete pre-bundles of base hypergraphs of CNF formulas are discussed on the basis of the orbit spaces of their fibre-transversals with respect to the complementation group. Finally we consider pre-bundles and sections of total clause sets with a fibre-stable action of the complementation group.

2 Notation and Preliminaries

A Boolean or propositional variable, for short variable, x taking values from $\{0, 1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$ also called the *flipped* or *complemented* variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of different literals and it is represented as a set $c = \{l_1, \ldots, l_k\}$. A conjunctive normal form formula, for short formula, C is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. Let CNF be the collection of all formulas. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Let CNF_+ denote that part of CNF containing only clauses with no negated variables. Given $C \in$ CNF, SAT asks whether there is a truth assignment $t : V(C) \to \{0, 1\}$ such that there is no $c \in C$ all literals of which are set to 0. If such an assignment exists it is called a *model* of C. Let SAT $\subseteq$ CNF denote the collection of all clause sets for which there is a model, and UNSAT $:=$ CNF $\setminus$ SAT. For a (not necessarily finite) set M, let 2^M be its power set. The set of all positive integers is denoted by $\mathbb{N}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. Let $\mathbb{P}$ denote the set of all prime numbers and $\mathbb{M}_{\mathbb{P}}$ its subset of all Mersenne primes. For a given (partial) mapping f, let $\mathrm{dom}(f)$ denote its domain, and $\mathrm{im}(f)$ its image. Further denote the (proper) restriction of f to a subset $A \subset \mathrm{dom}(f)$ by $f|_A$. As usual a *total* map is defined on the whole pre-image set. For a group G acting on a space M, meaning the existence of a map $G \times M \to M$: $(g, m) \mapsto m^g$, let $\mathcal{O}(m) := \{m^g : g \in M\}$ denote the orbit of $m \in M$ (under G) (cf. e.g. [19]). Given a non-empty set $A \subseteq M$ and $g \in G$, we set $A^g := \{m^g : m \in A\}$, and by convention $\emptyset^g := \emptyset$, for all $g \in G$. For $m \in M$, respectively $A \subseteq M$, let $G(m) := \{g \in G : m^g = m\}$, $G(A) := \{g \in G : A^g = A\}$ denote the isotropy group of m, respectively A. If $G(m) = m$, respectively $G(A) = A$, m respectively A is a fixed point [19]. Finally, as usual 'iff' is an abbreviation for 'if and only if'.

3 A Hierarchy of Discrete Pre-Bundles

In this section we provide the basics of a discrete pre-bundle concept. (Continuous) fibre bundles are an important concept of topology and geometry (e.g. [7]) and the concept restricted to discrete structures might be fruitful as well. Let I be any non-empty discrete (index) set. If I is infinite it is bijective to $\mathbb{N}$. In case I is finite it is bijective to $[n]$ where $n := |I|$. Note that I may have a discrete structure so that in general (even in the finite case) I needs not to be isomorphic to $\mathbb{N}$ (respectively $[n]$). Consider a disjoint union $K := \bigcup_{k\in I} K_k$, called the *total space*, of certain non-empty, mutually disjoint and (not necessarily finite) discrete spaces K_k, for every $k \in I$, called the *fibres*. Let $\pi : K \to I$ be a total map such that for every $\kappa \in K$ one requires $\pi(\kappa) := k$ iff $\kappa \in K_k$ ensuring the identity $K_k = \pi^{-1}(k)$. Hence

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π is surjective and is called the *(discrete bundle) projection* onto the *discrete base* I. The triple (K, I, π) is called a *discrete pre-bundle over* I. A *(partial) section* of the discrete pre-bundle is a (partial) mapping $s : I \to K$ such that $\pi|_{\mathrm{im}(s)} \circ s = \mathrm{id}_{\mathrm{dom}(s)}$. A total section s, we shall also call a *(discrete) fibre-transversal*, because $\mathrm{im}(s)$ contains exactly one member from every fibre. Let $\mathcal{S}(I, K)$ denote the set of all total sections of (K, I, π). Let $\mathcal{K} := \{\hat{\kappa} \in 2^K : \exists k \in I \forall \kappa \in \hat{\kappa}, \pi(\kappa) = k\}$ and define the total map $\hat{\pi} : \mathcal{K} \to I$ induced by π such that for every $\hat{\kappa} \in \mathcal{K}$ one has $\hat{\pi}(\hat{\kappa}) := k$ iff $\hat{\kappa} \subseteq K_k$ and $\hat{\kappa} \neq \emptyset$. Set $(\mathcal{K})_k := \hat{\pi}^{-1}(k)$. A (partial) *(multi-)section* is a (partial) mapping $\hat{s} : I \to \mathcal{K}$ such that $\hat{\pi}|_{\mathrm{im}(\hat{s})} \circ \hat{s} = \mathrm{id}_{\mathrm{dom}(\hat{s})}$. Hence for every $k \in \mathrm{dom}(\hat{s})$ one has $\hat{s}(k) \subseteq K_k$. As is explained next, a multi-section is no distinct concept. We set $\mathcal{K}_0 := K$, $\pi_0 := \pi$, and $\mathcal{K}_1 := \mathcal{K}$, $\hat{\pi} := \pi_1$. Furthermore, for any integer $\nu \geq 2$, defining

$$\mathcal{K}_\nu := \left\{\hat{\kappa} \in 2^{\mathcal{K}_{\nu-1}} : \exists k \in I \forall \kappa \in \hat{\kappa}, \pi_{\nu-1}(\kappa) = k\right\}$$

and $\pi_\nu : \mathcal{K}_\nu \to I$ such that for every $\hat{\kappa} \in \mathcal{K}_\nu$ one sets $\pi_\nu(\hat{\kappa}) := k$ iff $\emptyset \neq \hat{\kappa} \subseteq (\mathcal{K}_{\nu-1})_k$, where $(\mathcal{K}_{\nu-1})_k := \pi_{\nu-1}^{-1}(k)$, we obtain a hierarchy of discrete pre-bundles as follows:

Lemma 1 *If $(\mathcal{K}_0, I, \pi_0)$ is a discrete pre-bundle over I then also $(\mathcal{K}_\nu, I, \pi_\nu)$ is a discrete pre-bundle over I, for every $\nu \in \mathbb{N}$.*

PROOF. The proof proceeds by induction on ν, where the base is clear. For fixed $\nu > 0$ assume that $(\mathcal{K}_{\nu-1}, I, \pi_{\nu-1})$ is a discrete pre-bundle over I. By definition of $\pi_{\nu-1}$, K_ν does not contain the emptyset. Now suppose there are $k_1, k_2 \in I$ such that $(\emptyset \neq)\hat{\kappa} \in \pi_\nu^{-1}(k_1) \cap \pi_\nu^{-1}(k_2)$ then $\hat{\kappa} \subseteq (\mathcal{K}_{\nu-1})_{k_1}$, $\hat{\kappa} \subseteq (\mathcal{K}_{\nu-1})_{k_2}$, by definition meaning that for all $\kappa \in \hat{\kappa}$ we have $\pi_{\nu-1}(\kappa) = k_1$ and also $\pi_{\nu-1}(\kappa) = k_2$. As $\hat{\kappa}$ contains at least one member one has $k_1 = k_2$ ensuring that π_ν is a well-defined (partial) map which of course is total by construction. Finally, let $k \in I$ then there is $\kappa \in \mathcal{K}_{\nu-1}$ with $k = \pi_{\nu-1}(\kappa)$ by its surjectivity therefore $\{\kappa\} \in \mathcal{K}_\nu$ by its definition and so π_ν also is surjective. □

In view of the preceding discussion a (partial) multi-section of $(\mathcal{K}_{\nu-1}, I, \pi_{\nu-1})$ appears as a (partial) section of $(\mathcal{K}_\nu, I, \pi_\nu)$, for every fixed $\nu \in \mathbb{N}$. Let G be a group acting on the total space K of the pre-bundle (K, I, π) such that each fibre remains invariant, hence one requires $\kappa^g \in K_k$ iff $\kappa \in K_k$ for every $k \in I$ and $g \in G$. Let us call this action *fibre-stable*. Using the notation as above one obtains:

Proposition 1 *If G acts fibre-stable on $(\mathcal{K}_0, I, \pi_0)$ then a fibre-stable G-action on $(\mathcal{K}_\nu, I, \pi_\nu)$, $\nu \in \mathbb{N}$, is induced.*

PROOF. Proceeding by induction on ν, fix $\nu > 0$ and assume that G acts fibre-stable on the discrete pre-bundle $(\mathcal{K}_{\nu-1}, I, \pi_{\nu-1})$. For any $\hat{\kappa} \in \mathcal{K}_\nu$ and $g \in G$ we set by induction $\hat{\kappa}^g := \{\kappa^g : \kappa \in \hat{\kappa}\}$. As K_ν does not contain the empty set, on basis of Lemma 1 one then has $\hat{\kappa}^g \in (\mathcal{K}_\nu)_k$ iff $k = \pi_\nu(\hat{\kappa}^g)$ iff $k = \pi_{\nu-1}(\kappa^g)$, for all $\kappa^g \in \hat{\kappa}^g$ iff, by the induction hypothesis, $k = \pi_{\nu-1}(\kappa)$, for all $\kappa \in \hat{\kappa}$ iff $k = \pi_\nu(\hat{\kappa})$ iff $\hat{\kappa} \in (\mathcal{K}_\nu)_k$, from which the claim follows. □

Given a fibre-stable action of G, let $\varphi_\nu := \{\kappa \in \mathcal{K}_\nu : \forall g \in G, \kappa^g = \kappa\}$ denote the set of fixed points in $\mathcal{K}_\nu$, $\nu \in \mathbb{N}_0$.

Proposition 2 *Let G act fibre-stable on $(\mathcal{K}_0, I, \pi_0)$, and let $\kappa \in \mathcal{K}_{\nu-1}$, for fixed $\nu \in \mathbb{N}$, then there is a unique $k \in I$ such that the G-orbit of κ satisfies $\mathcal{O}(\kappa) \in (\mathcal{K}_\nu)_k$. Moreover $2^{\varphi_{\nu-1}} \subseteq \varphi_\nu$.*

PROOF. Clearly by the pre-bundle property due to Lemma 1 there is a unique $k \in I : \pi_{\nu-1}(\kappa) = k$, for all $\nu > 0$. Thus $\kappa \in (\mathcal{K}_{\nu-1})_k$, and by the fibre-stable action of G one has $\{\kappa^g : g \in G\} = \mathcal{O}(\kappa) \subseteq (\mathcal{K}_{\nu-1})_k$, from which the first claim follows. Moreover if $\varphi_{\nu-1} = \emptyset$ then the second claim obviously holds true, as by convention $\emptyset^g = \emptyset$, $g \in G$. Otherwise let $\emptyset \neq \hat{\kappa} \in 2^{\varphi_{\nu-1}}$ then $\hat{\kappa}^g = \{\kappa^g : \kappa \in \hat{\kappa}\} = \hat{\kappa}$, for all $g \in G$. Thus $\hat{\kappa} \in \varphi_\nu$. □

Regarding the isotropy group of sections one has:

Theorem 1 *Let $\nu \in \mathbb{N}_0$ and $s \in \mathcal{S}(I, \mathcal{K}_\nu)$ arbitrary. Then the isotropy group of* $\mathrm{im}(s)$ *is given by:* $G(\mathrm{im}(s)) = \bigcap_{k \in I} G(s(k))$.

PROOF. Let $g \in G(\mathrm{im}(s))$, then $\mathrm{im}(s) = \mathrm{im}(s)^g = \{s(k)^g : k \in I\}$. Now $s(k) \in (\mathcal{K}_\nu)_k$ is equivalent with $s(k)^g \in (\mathcal{K}_\nu)_k$, $k \in I$, because G acts fibre-stable. Since the spaces $(\mathcal{K}_\nu)_k$, $k \in I$, are mutually disjoint one obtains $s(k) = s(k)^g$ for all $k \in I$. Therefore $g \in \bigcap_{k \in I} G(s(k))$ which clearly is a subgroup of G. The reverse inclusion is obvious and the assertion is proven. □

Next we briefly describe how the pre-bundle notion can be extended to a discrete fibre-bundle structure. Let (K, I, π) be a discrete pre-bundle and let $U \neq \emptyset$ be such that there are bijections $\phi_k : U \to K_k$, for every $k \in I$. Clearly, if every K_k is equipped with a specific structure one would require that U carries the same structure and that these bijections are isomorphisms in the categorical sense. Now the tuple $(K, I, \pi, U, \mathrm{Aut}(U))$ is called a discrete (fibre) bundle, where $\mathrm{Aut}(U)$ is the group of all automorphisms of U. Let $\{I_\mu\}$ be an arbitrary family of *discrete local neighborhoods* where $\emptyset \neq I_\mu \subset I$, $|I_\mu| < \infty$, and the μ are taken from any suitable index set J. For every $\mu \in J$ there is a unique positive integer n_μ with $|I_\mu| = n_\mu$. Further assume $\bigcup_{\mu \in J} I_\mu = I$ and that there are bijections $\phi_\mu : I_\mu \times U \to \pi^{-1}(I_\mu)$ such that $\phi_\mu(k, u) \in K_k$, for every $(k, u) \in I_\mu \times U$. Every pair (I_μ, ϕ_μ), $\mu \in J$, is called a *local trivialisation* of the discrete bundle. For fixed $k \in I_\mu$ let

$$\phi_{\mu,k} := \phi_\mu(k, \cdot) : U \to K_k$$

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we then have for any $k \in I_{\mu_1} \cap I_{\mu_2} \neq \emptyset$,

$$\phi_{\mu_2,k}^{-1} \circ \phi_{\mu_1,k} : U \to U$$

which obviously is a member of $\mathrm{Aut}(U)$. The collection of all pairs $\{(I_\mu, \phi_\mu)_{\mu \in J}\}$ plays the role of a *discrete bundle atlas*. If specifically U is finite with $N := |U|$, then also $|K_k| = N$, for all $k \in I$, and $\mathrm{Aut}(U) = S_N$ becomes the finite symmetric group of degree N and of order $N!$. Conversely, i.e., no pre-bundle is given, suppose that there is a (structure-preserving) bijection $\phi_k : U \to K_k$ mapping a space U to every member of a collection of mutually disjoint spaces K_k, $k \in I$, where I is a discrete (possibly infinite) index set. Then the following mapping is induced for every finite $I_n = \{k_1, \ldots, k_n\} \subset I$:

$$\phi_n : I_n \times U \ni (k_j, u) \mapsto \phi_{k_j}(u) \in \bigcup_{j=1}^{n} K_{k_j}$$

Now we have $J = \mathbb{N}$ for the index set. The latter mappings are (structure-preserving) bijections, because each ϕ_{k_j}, for fixed $k_j \in I_n$ is such a bijection on the whole of K_{k_j}. Further one has

$$\pi^{-1}(I_n) := \bigcup_{j=1}^{n} K_{k_j}$$

as a disjoint union, and for fixed $\kappa \in \pi^{-1}(I_n)$, there is a unique $j \in [n]$ such that $\kappa \in K_j$. Now $\phi_{n,k} = \phi_k, \forall k \in I_n$, and every positive integer n. It follows that the map

$$\pi : \bigcup_{k \in I} K_k \to I$$

thereby defined via $\pi(\kappa) = k$ iff $\kappa \in K_k$ is total and surjective. Note that the fibre-bundle constructed in this manor is trivial in the sense that K is bijective to $I \times U$. Here one typically would have $I \in \{\mathbb{N}, \mathbb{Z}\}$, but also other possibly structured discrete index sets might be useful.

4 Discrete Pre-Bundles of Base Hypergraphs

Next we consider the question of the existence of base hypergraphs associated to arbitrary given parameter values. Recall that the hyperedge set $B(C)$ of the base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ of a formula $\emptyset \neq C \in \mathrm{CNF}$ is $B(C) := \{V(c) : c \in C\} \in \mathrm{CNF}_+$. Also a given hypergraph $\mathcal{H} = (V, B)$ serves as a base hypergraph if its vertex set V is a finite non-empty set of Boolean variables such that for every $x \in V$ there is a $b \in B$ containing x. Thus ensuring $B \neq \emptyset$, which is assumed throughout. Recall that a loop is a hyperedge containing exactly one vertex [2]. Let $\mathfrak{H}$ denote the space of all (finite) base hypergraphs of non-empty formulas. By $W_b := \{c : V(c) = b\}$ denote the collection of all clauses over a fixed $b \in B$. As usual $C_b = C \cap W_b$ is the fibre over b of a formula $C \in \mathrm{CNF}$ [13]. A hypergraph $\mathcal{H} = (V, B)$ is *linear* if $|b \cap b'| \leq 1$, for all distinct $b, b' \in B$. Let $\mathfrak{H}_{\mathrm{lin}}$ denote the subclass of all connected, loopless and linear base hypergraphs. The set of all clauses over $\mathcal{H}$ is the total clause set $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$. A *fibre-transversal* F of $K_{\mathcal{H}}$ contains exactly one member of every W_b, $b \in B$, namely $F(b)$ and $\mathcal{F}(K_{\mathcal{H}})$ is the set of all fibre-transversals [13]. Note that F can be viewed as a map or as a formula which formally results as the image of the map F. It shall become clear which view of F is meant in either context. A *compatible* fibre-transversal satisfies $\bigcup_{b \in B} F(b) \in W_V$, let those be collected in $\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$. A *diagonal* fibre-transversal has a non-empty intersection with every member of $\mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$, these are collected in $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$. A base hypergraph $\mathcal{H}$ is called diagonal [14] iff $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}}) \neq \emptyset$. As defined in [15] let $\beta : \mathfrak{H} \to \mathbb{N}_0$ where $\beta(\mathcal{H}) = \sum_{b \in B} |b| - |V|$, for every $\mathcal{H} \in \mathfrak{H}$, assuming that $V \neq \emptyset \neq B$. Let $\beta_{\mathrm{lin}} := \beta|_{\mathfrak{H}_{\mathrm{lin}}}$.

Proposition 3 *$(\mathfrak{H}, \mathbb{N}_0, \beta)$ is a discrete pre-bundle. Moreover $(\mathfrak{H}_{\mathrm{lin}}, \mathbb{N}_0, \beta_{\mathrm{lin}})$ is a discrete pre-bundle.*

PROOF. It suffices to verify the second claim which implies the first. So let $i = 0$, then $\mathcal{H}_0 := (\{x_1, x_2\}, \{\{x_1, x : 2\}\})$ satisfies $\beta(\mathcal{H}_0) = 0$ and $\mathcal{H}_0 \in \mathfrak{H}_{\mathrm{lin}}$. For $i \in \mathbb{N}$ take $V_i = \{x_j : j \in [i+2]\}$, and $B_i = \{b_l : l \in [i+1]\}$ such that $|b| = 2$, for every $b \in B_i$. Setting $b_l = \{x_l, x_{l+1}\}$, for $l \in [i+1]$, obviously yields $V(B_i) = V_i$ and $\sum_{l \in [i+1]} |b_l| = 2i+2$ hence $\beta(\mathcal{H}_i) = i$ where $\mathcal{H}_i := (V_i, B_i)$ and $\mathcal{H}_i \in \mathfrak{H}_{\mathrm{lin}}$. Thus β is a projection onto $\mathfrak{H}_{\mathrm{lin}}$. $\square$

We identify $\mathfrak{H} =: \mathfrak{H}_0$ respectively $\mathfrak{H}_{\mathrm{lin}} =: \mathfrak{H}_{\mathrm{lin}0}$ with $\mathcal{K}_0$, and also $\beta =: \beta_0$ respectively $\beta_{\mathrm{lin}} =: \beta_{\mathrm{lin}0}$ with π_0. Further let $\mathfrak{H}_\nu$, respectively $\mathfrak{H}_{\mathrm{lin}\nu}$, be identified by $\mathcal{K}_\nu$, and β_ν respectively β_{lin_ν} by π_ν, for every $\nu > 0$. On basis of Lemma 1 and the previous result one concludes:

Corollary 1 *$(\mathfrak{H}_\nu, \mathbb{N}_0, \beta_\nu)$, $(\mathfrak{H}_{\mathrm{lin}\nu}, \mathbb{N}_0, \beta_{\mathrm{lin}_\nu})$ are discrete pre-bundles for every integer $\nu \geq 0$.*

Recall that for a set V of variables, clause c^X results from c via complementing all variables in $X \cap V(c)$, for $X \subseteq V(C)$. As considered in [15] let $G_V := (2^V, \oplus)$ with neutral element $\emptyset$ denote the (finite) group inducing this flipping action on CNF by observing that $\{c\} \in \mathrm{CNF}$. By $\mathcal{O}(C) := \{C^X : X \in G_V\}$ denote the $(G_V$-)orbit of C in CNF. Given $\mathcal{H} \in \mathfrak{H}$, as defined in [15], $\omega(\mathcal{H})$ denotes the number of all such orbits in $\mathcal{F}(K_{\mathcal{H}})$, and $\delta(\mathcal{H})$ is the number of those orbits in $\mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$. In [16] a further map on $\mathfrak{H}$ is defined, namely $\rho : \mathfrak{H} \to \mathbb{N}_0$, where for any $\mathcal{H} = (V, B) \in \mathfrak{H}$, $\rho(\mathcal{H})$ denotes the number of orbits with respect to the G_V-action of all fibre-transversals in $\mathcal{F}(K_{\mathcal{H}})$ which are neither compatible nor diagonal.

Lemma 2 *For $\mathcal{H} \in \mathfrak{H}$ one has $\rho(\mathcal{H}) = 0$ iff $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{F}_{\mathrm{comp}}(K_{\mathcal{H}})$ iff $\omega(\mathcal{H}) = 1$. In this case also $\delta(\mathcal{H}) = 0$.*

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PROOF. Let $\mathcal{H} = (V, B) \in \mathfrak{H}$ then $\omega(\mathcal{H}) = 1+\rho(\mathcal{H})+\delta(\mathcal{H})$ [16]. If $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ then $\rho(\mathcal{H}) = 0 = \delta(\mathcal{H})$ implying $\omega(\mathcal{H}) = 1$. If $\omega(\mathcal{H}) = 1 = 2^{\beta(\mathcal{H})}$ [15] then $\beta(\mathcal{H}) = 0$ implying $\mathcal{F}(K_{\mathcal{H}}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. If $\rho(\mathcal{H}) = 0$ then $\omega(\mathcal{H}) = 1 + \delta(\mathcal{H})$. Suppose $\delta(\mathcal{H}) \geq 1$ then there is a $F \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ containing a minimal unsatisfiable subformula $F' \subseteq F$. As $\delta(\mathcal{H}(F')) \geq 1$, $B(F')$ cannot consist of loops only. So, select any $b \in B(F')$ with $|b| \geq 2$. According to [16], Lemma 6, $F'_c := (F' \setminus \{F'(b)\}) \cup \{c\}$ is satisfiable, for any $c \in W_b$ with $c \neq F'(b)$. Hence choose c such that F'_c is non-compatible. This is always possible: let t be a model of $F' \setminus \{F'(b)\}$ then every literal in $F'(b)$ is set by t to 0, and must occur outside $F'(b)$ as a complemented literal. As $|b| \geq 2$, one can choose c as required. Now let t' be a model of F'_c and extend F'_c over the remaining part of B compatible with t' yielding $F_1 \in \mathcal{F}(K_{\mathcal{H}}) \cap \text{SAT}$. Thus we obtain $\rho(\mathcal{H}) > 0$ yielding a contradiction implying $\delta(\mathcal{H}) = 0$ thus $\omega(\mathcal{H}) = 1$. □

Theorem 2 *Let $\mathcal{H} \in \mathfrak{H}$. (1) If $\delta(\mathcal{H}) = 1$ then $\mathcal{H}$ is connected. (2) There exist disconnected $\mathcal{H}$ such that $\rho(\mathcal{H}) < \delta(\mathcal{H})$ and $\delta(\mathcal{H}) \equiv 1 \mod 2$.*

PROOF. For $\mathcal{H}$ ($\mathcal{H}_i$) set $\alpha(\mathcal{H}) := \alpha$ ($\alpha(\mathcal{H}_i) := \alpha_i$), $\alpha \in \{\rho, \delta, \omega\}$. Assume that $\mathcal{H}$ is composed of two disjoint components $\mathcal{H}_i$, $i = 1, 2$ such that at least $\mathcal{H}_1$ has $\delta_1 \geq 1$. In general due to [16], Lemma 1, one has for such a disjoint union $\delta(\mathcal{H}) = \delta_1\omega_2 + \delta_2\omega_1 - \delta_1\delta_2$ and $\rho(\mathcal{H}) = \rho_1 + \rho_2 + \rho_1\rho_2$. Therefore here, as $\omega_1 > \delta_1$, $\delta > \delta_1\omega_2 \geq \delta_1$. It also follows that if $\delta_i = 0$, $i = 1, 2$ then $\delta = 0$. Hence for a disconnected $\mathcal{H}$ either $\delta = 0$ or $\delta > 1$ thus (1). Regarding (2) consider $\mathcal{H}' = (V', B')$ with $V' = \{x_1, x_2\}$, $B' = \{b_1, b_2, b_3\}$ and $b_1 = \{x_1\}$, $b_2 = \{x_2\}$, $b_3 = \{x_1, x_2\}$, for which $\omega' = 4$, $\delta' = 1$ (as is easy to see) and $\rho' = 2$. Let $\mathcal{H}_1$ be two disjoint copies of $\mathcal{H}'$. Again using the combination formulas as above one obtains $\omega_1 = \omega'^2 = 16$, $\delta_1 = 7$ and $\rho_1 = 8$. Adding another disjoint copy of $\mathcal{H}'$ to $\mathcal{H}_1$ yields a disconnected $\mathcal{H}$ with $\omega = 64$, and odd $\delta = 37 > \rho = 26$. □

Definition 1 *Let $\mathcal{H}_0 = (V_0, B_0)$, non-empty $b \notin B_0$ with $\mathcal{H} := (V, B)$, where $V := V_0 \cup b$, $B := B_0 \cup \{b\}$. Then the* fluctuation f_b *is the number of all G_{V_0}-orbits in $\mathcal{F}(K_{\mathcal{H}_0}) \cap$* SAT *which become G_V-orbits in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$.*

By Lemma 1 in [16], $f_b = 0$ if $b \cap V_0 = \emptyset$. Setting $\alpha(\mathcal{H}) =: \alpha, \alpha \in \{\rho, \delta\}$ then for the remaining case at least one obtains:

Theorem 3 *Let $j := |b \cap V_0| \in \mathbb{N}$. (1) $\delta = 2^j\delta_0 + f_b$. (2) If $b \setminus V_0 \neq \emptyset$ then $f_b = 0$. (3) If for every $x \in b$ there is $\{x\} \in B_0$ then $f_b = \rho_0 + 1$. (4) The condition in (3), and $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$ contributes exactly 1 to f_b, are equivalent.*

PROOF. (1) If $\delta_0 = 0$ then clearly $\delta = f_b$ by definition. Assume that $\delta_0 > 0$. Take an arbitrary fixed fibre-transversal F_k, refered to as the orbit base, of every G_{V_0}-orbit in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$. These F_k, $k \in [\delta_0]$, are mutually distinct as their orbits are mutually disjoint. Let $b =: b' \cup \tilde{b}$ as disjoint union where $b' := V_0 \cap b$. Defining $F_k(c) := F_k \cup \{c\} \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ where $c := d \cup \tilde{b}$, for every $d \in W_{b'}$, yields the collection of 2^j mutually disjoint G_V-orbit bases in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ because all variables of b' occur as constant fixed literals in F_k. The negation of any member of $\tilde{b}$, if non-empty, results in an orbit member, hence yields no additional orbit base. Thus in the same manner running through all members of $\{F_k : k \in [\delta_0]\}$ yields exactly $2^j\delta_0$ further distinct G_V-orbits in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ as their defining parts F_k are mutually distinct. In summary we obtain $\delta = 2^j\delta_0 + f_b$. (2) If $|b \setminus V_0| > 0$ then any $F \in \mathcal{F}(K_{\mathcal{H}_0}) \cap \text{SAT} =: \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ yields $F \cup \{c\} \in \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}})$, as every $c \in W_b$ can be satisfied independently of all clauses in F. Thus $f_b = 0$. (3) First it is shown that the condition is sufficient for that every orbit in $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ contributes at least 1 to f_b meaning $f_b \geq |\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})|$. So let $b = \{x_i : i \in [r]\}$, for $r \in \mathbb{N}$ appropriately, let $b_i := \{x_i\} \in B_0$, $i \in [r]$, and let $F_0 \in \mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ be an arbitrarily fixed orbit base. There are fixed clauses $c_i \in W_{b_i}$ such that $\{c_i : i \in r\} \subseteq F_0$. Let $c \in W_b$ be selected such that it contains the complements of the literal in c_i, for every $i \in [r]$ then $F_0 \cup \{c\} \in \text{UNSAT}$. As all orbit bases in $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ are mutually distinct one has $f_b \geq |\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})|$. Finally observe that a given orbit in $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ can contribute at most 1 to f_b. Indeed suppose there is an orbit base F_0 and distinct clauses $c_i \in W_b$ such that $F_i := F_0 \cup \{c_i\} \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$, $i = 1, 2$. Then any model of F_0 simultaneously sets all literals in c_1 and c_2 to 0 which is impossible. So $f_b \leq |\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})|$ and in summary $f_b = |\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})| = \rho_0 + 1$. (4) Since (3) means that any G_{V_0}-orbit in $\mathcal{F}_{\text{SAT}}(K_{\mathcal{H}_0})$ increases f_b of exactly 1 it specifically is sufficient for $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$. Regarding the necessity, let $F_0 \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$ and $c \in W_b$ be arbitrary, and let t_0 be the model of F_0 setting every literal in F_0 to 1. Assume there is $x \in b$ but $\{x\} \notin B_0$ then every edge in B_0 contains a variable distinct to x. So, modifying t_0 such that the literal over x in c is set to 1 yields $F_0 \cup \{c\} \in \text{SAT}$ and the proof is finished by contraposition. □

Corollary 2 *Let b and $\mathcal{H}_0, \mathcal{H} \in \mathfrak{H}$ as in Definition 1, then the fluctuation always ranges in $0 \leq f_b \leq \rho_0 + 1$. Moreover there are instances for which the boundary values are valid.*

Theorem 4 *There are connected $\mathcal{H} \in \mathfrak{H}$ such that $\delta(\mathcal{H}) > \rho(\mathcal{H})$.*

PROOF. Consider the disconnected base hypergraph $\mathcal{H} = (V, B)$ with $\omega = 64$, $\delta = 37$, and $\rho = 26$ as constructed in the proof of Theorem 2, (2). Recall that $\mathcal{H}$ is the union of three disjoint copies of $\mathcal{H}' = (V', B')$ with $V' = \{x_1, x_2\}$, $B' = \{\{x_1\}, \{x_2\}, \{x_1, x_2\}\}$. So one may assume that

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$V = \{x_i : i \in [6]\}$, and $B = \{\{x_i\} : i \in [6]\} \cup \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$. Setting $\tilde{B} := B \cup \{b\}$ with $b := V$ yields a connected base hypergraph $\tilde{\mathcal{H}} = (V, \tilde{B})$, for which the condition (3) of Theorem 3 is valid. Thus $f_b = \rho + 1 = 27$, $\tilde{\omega} = 2^{12}$, and $\tilde{\delta} = 2^6 \cdot 37 + 27 = 2395 > \tilde{\rho} = 2^{12} - 2395 - 1 = 1700$. □

Lemma 3 *Let $i \in \mathbb{N}_0$. (1) There is $\mathcal{H} \in \mathfrak{H}_{\text{lin}}$ with $\omega(\mathcal{H}) = 2^i$. (2) If there is a $\mathcal{H}_0 \in \mathfrak{H}$ with $\rho_0 = i$ then there is $\mathcal{H}_1 \in \mathfrak{H}$ with $\rho_1 = 2i + 1$.*

PROOF. (1) directly follows from Proposition 3 due to $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$, $\mathcal{H} \in \mathfrak{H}$. For (2) consider $\mathcal{H}_1 = (V_1, B_1)$ with $V_1 = \{x_1, x_2\}$ and $B_1 = \{b_1, b_2\}$ with $b_1 = \{x_1\}$, $b_2 = \{x_1, x_2\}$ yielding $\omega_1 = 2$, $\delta_1 = 0$ and $\rho_1 = 1$. Next let $\mathcal{H}_2 = (V_2, B_2) \in \mathfrak{H}$ be arbitrary but disjoint to $\mathcal{H}_1$. Then for $\mathcal{H} := \mathcal{H}_2 \cup \mathcal{H}_1$ one obtains $\rho(\mathcal{H}) = 1 + 2\rho_2$. □

According to [15] a base hypergraph $\mathcal{H}$ is simple if $\delta(\mathcal{H}) = 1$. Set $\mathfrak{H}_{\text{simp}}$ for the class of all simple base hypergraphs. Recall that due to Theorem 2, (1) all members of $\mathfrak{H}_{\text{simp}}$ are connected. Further there is no upper bound on ρ in $\mathfrak{H}_{\text{simp}}$. To state it more precisely, set $\mathbb{M}_{-1} := \{m - 1 : m \in \mathbb{M}\}$ where $\mathbb{M}$ denotes the set of all Mersenne numbers excluding $0, 1$. Further let $\rho_{\text{simp}} := \rho|_{\mathfrak{H}_{\text{simp}}}$, then one obtains:

Theorem 5 $(\mathfrak{H}_{\text{simp}}, \mathbb{M}_{-1}, \rho_{\text{simp}})$ *is a discrete pre-bundle.*

PROOF. First it is to verify that ρ_{simp} cannot take values outside $\mathbb{M}_{-1} = \{2(2^{k-1} - 1) : k \in \mathbb{N} \setminus \{1\}\}$. Clearly $\omega(\mathcal{H}) = 1 + 1 + \rho(\mathcal{H})$ for any simple base hypergraph. In general we have $\rho(\mathcal{H}) = \delta(\mathcal{H}) = 0$ according to Lemma 2 only in case $\omega(\mathcal{H}) = 1$. Thus $\omega(\mathcal{H}) = 2$ is excluded for simple base hypergraphs. Hence $\rho(\mathcal{H}) = 2^k - 2$, for $k \geq 2$ is the only possible range of values, meaning that ρ_{simp} cannot take values outside $\mathbb{M}_{-1}$. It remains to verify that ρ_{simp} is a surjection. Take $\mathcal{H}' \in \mathfrak{H}_{\text{simp}}$ as defined in the proof of Theorem 2 with $\rho(\mathcal{H}') = 2$ being the smallest member in $\mathbb{M}_{-1}$, for $k = 2$. For $k \geq 3$ take $\mathcal{H}_k = (V_k, B_k)$ such that $V_k := \{x_i : i \in [k]\}$ and $B_k := \{\{x_i\} : i \in [k]\}$. Then $\beta_k = 0$ therefore $\omega_k = 1$ and due to Lemma 2 it follows that $\rho_k = \delta_k = 0$. Now for $\mathcal{H}'_k := (V_k, B'_k)$ with $B'_k := B_k \cup \{b\}$ and $b := V_k \notin B_k$, one has $\beta(\mathcal{H}'_k) = 2k - k = k$ thus $\omega(\mathcal{H}'_k) = 2^k$. Moreover for $\mathcal{H}'_k$ the condition in (3) of Theorem 3 is valid. Therefore $f_b = \rho_k + 1 = 1$ and by (1) of Theorem 3, $\delta(\mathcal{H}'_k) = 2^k \delta_k + 1 = 1$, hence $\rho(\mathcal{H}'_k) = 2^k - 2 \in \mathbb{M}_{-1}$. □

Theorem 6 *If for every prime number $p \geq 5$ such that $p \in \mathbb{P} \setminus \mathbb{M}_{\mathbb{P}}$ there can be constructed $\mathcal{H} \in \mathfrak{H}$ with $\rho(\mathcal{H}) = p - 1$ then $(\mathfrak{H}, \mathbb{N}_0, \rho)$ is a discrete pre-bundle.*

PROOF. To verify the surjectivity of ρ by induction on $i \in \mathbb{N}_0$, note that for $\mathcal{H}_0 \in \mathfrak{H}$ with mutually disjoint hyperedges, one has $\mathcal{F}(K_{\mathcal{H}_0}) = \mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$. Hence $\rho_0 = 0$ by Lemma 2. For $i = 1$ we refer to $\mathcal{H}_1$ in the proof of Lemma 3 with $\rho_1 = 1$. For $i = 2$, consider $\mathcal{H}'$ as defined in the proof of Theorem 2 having $\rho' = 2$. For the induction step, let $i + 1 \geq 3$ be fixed and assume that the claim is verified for all integers $\leq i$. If $i + 1$ is odd then there is a unique integer $i > k \geq 1$ with $(i + 1) = 2k + 1$. By the induction hypothesis there is a $\mathcal{H}_k$ such that $\rho_k = k$. On behalf of Lemma 3, (2) then there also is a $\mathcal{H} \in \mathfrak{H}$ with $\rho(\mathcal{H}) = 2k + 1 = i + 1$. If $i + 1$ is even then $i + 2$ is odd. If $i + 2 \in \mathbb{M}_{\mathbb{P}}$ there is a (simple) $\mathcal{H}$ with $\rho(\mathcal{H}) = i + 1$ according to Theorem 5. If $i + 1 \in \mathbb{P}$ is a non-Mersenne prime we are done by the assumption. Else let $q_i \leq (i+2)/3$, $i \in [r]$, for appropriate $r \in \mathbb{N}$, be all the (not necessarily distinct) prime factors of $i + 2$. Note that $q_i \leq i$, $i \in [r]$ as $i \geq 2$. Hence by the induction hypothesis there are instances $\mathcal{H}_i$ such that $\rho(\mathcal{H}_i) = q_i - 1$, $i \in [r]$. And we can assume that all these instances are chosen mutually disjoint. According to [16], Lemma 1 (iii) for their union $\mathcal{H}$ one has

$$\begin{aligned} \rho(\mathcal{H}) &= -1 + \Pi_{i=1}^{r}(1 + \rho(\mathcal{H}_i)) \\ &= -1 + \Pi_{i=1}^{r} q_i \\ &= -1 + (i + 2) \end{aligned}$$

and the assertion follows.□

5 Clause Bundles and Sections

Let $\mathcal{H} = (V, B)$ be a base hypergraph and identify V with the mapping $V : K_{\mathcal{H}} \to B$ which assigns to a clause its set of variables, then one obtains:

Proposition 4 $(K_{\mathcal{H}}, B, V)$ *is a (finite) discrete pre-bundle on which the flipping group G_V acts fibre-stable.*

PROOF. The first assertion is clear and for any $b \in B$, $V^{-1}(b) = W_b = (K_{\mathcal{H}})_b$. Let $c \in W_b$ and $g \in G_V$ then $c^g = c^{g \cap V(c)} \in W_b$ hence G_V acts fibre-stable. □

Observe that the base B has a discrete structure. Note that any fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ is a total section of $(K_{\mathcal{H}}, B, V)$. Again identifying $K_{\mathcal{H}} =: K_{\mathcal{H}0}$ with $\mathcal{K}_0$, $V =: V_0$ with π_0 and then defining $K_{\mathcal{H}\nu}$, V_ν corresponding to $\mathcal{K}_\nu$, π_ν for every integer $\nu > 0$, on basis of Lemma 1, and Proposition 1 one obtains due to the previous result:

Corollary 3 $(K_{\mathcal{H}\nu}, B, V_\nu)$ *is a discrete pre-bundle, on which G_V acts fibre-stable, for every $\nu \in \mathbb{N}_0$.*

Given $K_{\mathcal{H}} = (V, B)$, any total section s of $(K_{\mathcal{H}1}, B, V_1)$ yields a collection $\text{im}(s) = \{C_b : b \in B\}$ of fibre-formulas over B. For this setting by adapting Theorem 1 here we directly have $G_V(\text{im}(s)) = \bigcap_{b \in B} G_V(s(b))$, $s(b) = C_b$, $b \in B$. Using the fibre-decomposition [13] one has $C =$

$\bigcup_{b\in B(C)} C_b$. As these C_b are mutually disjoint objects in the total space $K_{\mathcal{H}1}$, one can identify C with the section $s \in \mathcal{S}(B, K_{\mathcal{H}1})$ such that $s(b) = C_b$, $b \in B$.

6 Conclusions and Open Problems

From Theorem 6 and Theorem 5 in Section 4 one directly concludes via accordingly adapting the settings prior to Lemma 1:

Corollary 4 *(1) If for every prime number $p \geq 5$ such that $p \in \mathbb{P}\backslash\mathbb{M}_{\mathbb{P}}$ there can be constructed a base hypergraph $\mathcal{H}$ with $\rho(\mathcal{H}) = p - 1$ then $(\mathfrak{H}_\nu, \mathbb{N}_0, \rho_\nu)$ is a discrete pre-bundle, for every integer $\nu \geq 0$. (2) $(\mathfrak{H}_{\mathrm{simp}_\nu}, \mathbb{M}_{-1}, \rho_{\mathrm{simp}_\nu})$ is a discrete pre-bundle, for every integer $\nu \geq 0$.*

There remain several directions for future work. Theorem 6, respectively Corollary 4 (1), admit a strong assumption which should be established explicitely. Analogously to Theorem 3 it remains to provide similar results for the higher components in the hierarchy of diagonal base hypergraphs [16]. In all these cases one also should investigate whether the subclass of loopless, connected or even linear base hypergraphs already admit pre-bundles over $\mathbb{N}_0$. Further it remains open whether also δ induces a pre-bundle with integer base, i.e., whether for every nonnegative integer i there is a diagonal base hypergraph with $\delta = i$. Finally the properties of the fluctuation parameter should be investigated in more detail.

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