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On Maximal Non-Diagonal CNF-Base Hypergraphs

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***Abstract*—The class of maximal non-diagonal CNF-base hypergraphs is introduced and investigated. It resides below the hierarchy of diagonal base hypergraphs, and is extreme in the respect that its members are only one edge away from diagonality. Here we prove a general existence criterion for maximal non-diagonality, provide connections to minimal diagonality, and discuss the relationship to maximal satisfiable CNF formulas.**

***Index Terms*—hypergraph, CNF-satisfiability, orbit**

I. Introduction

THE genuine and one of the most important NP-complete problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas [6]. More precisely, SAT is the natural NP-complete problem and thus lies at the heart of computational complexity theory. Moreover, numerous computational problems can be encoded as equivalent instances of CNF-SAT via reduction [7]. From a theoretical point of view on the one hand subclasses are to be detected for which SAT can be decided efficiently. There are known several of them such as quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested, co-nested formulas, and exact linear formulas etc. [2], [4], [5], [8], [9], [10], [11], [17], [18]. On the other hand it might be purposeful to reveal the structural aspects of CNF-SAT from diverse perspectives in order to attack the complexity issues among others. In [15] a hierarchy of diagonal (CNF-)base hypergraphs has been considered, such that $\hat{\mathfrak{H}}_i$ is the class of all instances with exactly i members in the orbit space of the diagonal fibre-transversals with respect to the action of the complementation group on clauses. In the present paper the class of maximal non-diagonal base hypergraphs is introduced and studied to some extent. Such instances are extreme among all the members below the first level of the mentioned hierarchy: By definition they are only one hyperedge away from diagonality. The connection to minimal diagonal base hypergraphs is considered which are diagonal but none of its subhypergraphs has this property. Further, it is shown that not every maximal non-diagonal base hypergraph is derived from a minimal diagonal one, as might be expected. Also a general equivalent criterion is proven for maximal non-diagonality based on the concept of a minimal transversal meeting all minimal diagonal subhypergraphs of a given diagonal base hypergraph. The relationship to the concept of maximal satisfiable formulas as defined in [13] is exhibited. In that context new parameters for base hypergraphs are discussed. Considering this paper as a first proposal towards the research on maximal non-diagonality we finally refer in a concluding section to several directions for future work on this topic.

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II. Notation and Preliminaries

A Boolean or propositional variable, for short variable, x taking values from $\{0,1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$ also called the *complemented* variable. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of literals over mutually distinct variables and it is represented as a set $c = \{l_1, \ldots, l_k\}$, or simply, as a literal *sequence*: $c = l_1 \cdots l_k$. A conjunctive normal form formula, for short formula, C is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. Let CNF be the collection of all formulas. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Given $C \in$ CNF, SAT means to decide whether there is a truth assignment $t : V(C) \rightarrow \{0,1\}$ such that there is no $c \in C$ all literals of which are set to 0. Such a t is a model of C; let $\mathcal{M}(C)$ be the space of all models of C. Let SAT $\subseteq$ CNF denote the collection of all clause sets for which there is a model, and UNSAT := CNF $\setminus$ SAT. Given a set V of propositional variables, an assignment t can be regarded as the clause $\{x^{t(x)} : x \in V\}$ of length $|V|$, where $x^0 := \bar{x}$, $x^1 := x$. Similarly, for $b \subseteq V$, we identify the restriction $t|b =: t(b)$ with the clause $\{x^{t(x)} : x \in b\}$. The collection of all clauses of length $|V|$ is denoted as W_V which therefore also can be regarded as the set of all mappings $V \rightarrow \{0,1\}$. For a clause c we denote by c^γ the clause in which all its literals are complemented. In case of an assignment $t \in W_V$, we have the correspondence of t^γ to the assignment $1 - t : V \rightarrow \{0,1\}$ complementing all truth values. Similarly, let $C^\gamma = \{c^\gamma : c \in C\}$ denote the complemented clause set version of C. As introduced in [12] a clause set C determines its base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ where $B(C) = \{V(c) : c \in C\}$. Let $C_b = \{c \in C : V(c) = b\}$ denote the fibre of C over b, thus $C = \bigcup_{b \in B(C)} C_b$. Also a given hypergraph $\mathcal{H} = (V, B)$ yields a CNF-base hypergraph when regarding its vertices as Boolean variables such that for each $x \in V$ there is a (hyper)edge $b \in B$ containing x. Thus every $t \in W_V$ yields a clause set over B, namely $t(B) := \{t(b) : b \in B\}$. Let $\mathfrak{H}$ be the collection of all (CNF-)base hypergraphs, and let $\mathfrak{H}^c$ be the subclass of all connected instances. A base hypergraph is linear if distinct hyperedges pairwise intersect in at most one vertex; if 1 is the size of all these intersections it even is exact linear. A base hypergraph is loopless if none of its hyperedges consists of a unique vertex. A hypergraph is Sperner if no hyperedge is a proper subset of another one [3]. A formula without unit clauses is (exact)

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linear if its base hypergraph is (exact) linear [17]. As usual $K_{\mathcal{H}} := \bigcup_{b\in B} W_b$ is the set of all clauses over $\mathcal{H}$. A $\mathcal{H}$-based formula is $C \subseteq K_{\mathcal{H}}$ such that $C_b := C \cap W_b \neq \emptyset$, for each $b \in B$. Given a $\mathcal{H}$-based formula $C \subseteq K_{\mathcal{H}}$ with the additional property that $\bar{C}_b := W_b \setminus C_b \neq \emptyset$ holds, for each $b \in B$, then its $\mathcal{H}$-based complement formula is $\bar{C} := \bigcup_{b\in B} \bar{C}_b = K_{\mathcal{H}} \setminus C$ with fibres $\bar{C}_b$. In that case it is $\mathcal{H}(\bar{C}) = \mathcal{H}(C)$. A fibre-transversal of $K_{\mathcal{H}}$ is a $\mathcal{H}$-based formula $F \subset K_{\mathcal{H}}$ such that $|F \cap W_b| = 1$, for each $b \in B$. Hence F is a formula containing exactly one clause of each fibre W_b of $K_{\mathcal{H}}$; let that clause be refered to as $F(b)$. An important type of fibre-transversals F are the compatible ones, i.e., $\bigcup_{b\in B} F(b) \in W_V$, collected in $\mathcal{F}_{\text{comp}}(\mathcal{H}) \subseteq \text{SAT}$. A fibre-transversal F is diagonal if $F \cap F' \neq \emptyset$, for all $F' \in \mathcal{F}_{\text{comp}}(\mathcal{H})$. Let $\mathcal{F}_{\text{diag}}(\mathcal{H})$ be the set of all diagonal fibre-transversals of $K_{\mathcal{H}}$. Observe that exactly the members of $\mathcal{F}_{\text{diag}}(\mathcal{H})$ provide unsatisfiable fibre-transversals at all. A base hypergraph $\mathcal{H}$ is diagonal if $\mathcal{F}_{\text{diag}}(\mathcal{H}) \neq \emptyset$, and it is minimal diagonal if no subhypergraph of $\mathcal{H}$ is diagonal. Let $\mathfrak{H}_{\text{diag}}$ be the class of all diagonal base hypergraphs, and $\mathfrak{H}_{\text{mdiag}}$ denote the subcollection of all its minimal diagonal instances. The number of orbits in $\mathcal{F}_{\text{diag}}(\mathcal{H})$ with respect to the action of the group of variable complementation induced on the space of all CNF formulas is denoted as $\delta(\mathcal{H})$ [14]; for short the term orbit is used in the sequel. Clearly, it is $\delta = 0$ for all non-diagonal instances, and specifically, a base hypergraph with $\delta = 1$ is called simple. We use $[n] = \{1, \ldots, n\}$, where n is a positive integer. As usual, 'iff' means 'if and only if'. Next we collect several useful properties of minimal unsatisfiable formulas. To that end the following result proven in [12] is needed, which characterizes the satisfiability of a formula C in terms of the compatible fibre-transversals in its based complement formula $\bar{C}$. A compatible fibre-transversal of a $\mathcal{H}$-based formula $C \subset K_{\mathcal{H}}$ simply is a compatible fibre-transversal $K_{\mathcal{H}}$ that is contained in C.

Theorem 1: [12] For $\mathcal{H} = (V, B)$, let $C \subset K_{\mathcal{H}}$ be a $\mathcal{H}$-based formula such that $\bar{C}$ is $\mathcal{H}$-based, too. Then C is satisfiable if and only if $\bar{C}$ admits a compatible fibre-transversal F. Moreover, the union of all clauses in F^{γ} is a model of C.

Recall that $C \in \text{UNSAT}$ is minimal unsatisfiable if $C \setminus \{c\}$ is satisfiable, for every $c \in C$ [1]. We denote the class of exactly those instances by $\mathcal{I} \subset \text{UNSAT}$.

Lemma 1: Let $C \in \mathcal{I}$ with $\mathcal{H}(C) =: (V, B) = \mathcal{H}(\bar{C})$ then:

(a) For every $t \in W_V$ it is $t(B) \cap C \neq \emptyset$.
(b) There is a $t \in W_V$, s.t. $|t(B) \cap C| = 1$.
(c) For every $b \in B$ there is a $t \in W_V$, s.t. $t(B) \cap C = \{t(b)\}$.
(d) In general there are $t \in W_V$ s.t. $|t(B) \cap C| > 1$, and also $c \in C$ with $|\mathcal{M}(C \setminus \{c\})| > 1$.
(e) Let $t \in W_V$. Then $|t(B) \cap C| = 1$, and specifically there is $b \in B$ s.t. $t(B) \cap C = \{t(b)\}$ iff $t^{\gamma} \in \mathcal{M}(C')$ where $C' := C \setminus \{t(b)\}$.

PROOF. As any $t \in W_V$ can be identified with a compatible fibre-transversal, (a) is a direct consequence of Thm. 1. Let $C' := C \setminus \{c\}$, for any fixed $c \in C$. According to the previous result there is a compatible fibre-transversal F'^{γ} of $\bar{C}'$ such that $\bigcup_{b\in B} F'(b) \in \mathcal{M}(C')$ and $F'^{\gamma}(V(c)) = c \in \bar{C}'$ because $C \in \text{UNSAT}$. Since F'^{γ} cannot contain another clause of C it follows (b). Let $b \in B$ and $c \in C_b$ be arbitrary, as above it follows $t'^{\gamma}(b) = c$, for any $t' \in \mathcal{M}(C')$, with $C' := C \setminus \{c\}$, which is the unique clause having this property, so (c). Next consider $C = \{xy_1, xy_2, \bar{x}y_3, \bar{x}y_4, \bar{y}_1\bar{y}_2, \bar{y}_3\bar{y}_4\}$ whose membership to $\mathcal{I}$ can be verified easily. Over $B := B(C)$ define $t_0(B) := \{xy_1, xy_2, x\bar{y}_3, x\bar{y}_4, y_1y_2, \bar{y}_3\bar{y}_4\}$ then $|t_0(B) \cap C| = 3$. Now let $c = \bar{y}_3\bar{y}_4$ and $t_1(B) := \{xy_1, x\bar{y}_2, xy_3, xy_4, y_1\bar{y}_2, y_3y_4\}$, $t_2(B) := \{x\bar{y}_1, xy_2, xy_3, xy_4, \bar{y}_1y_2, y_3y_4\}$ then, regarded as truth assignments, obviously $t_i \in \mathcal{M}(C \setminus \{c\})$, $i = 1, 2$, thus we obtain (d). Finally (e) directly follows from (b), (c) and Thm. 1, finishing the argumentation. □

III. Maximal Non-Diagonality Derived from Minimal Diagonality

What is the structure of the class of all non-diagonal base hypergraphs? Those members clearly reside below the hierachy of diagonal base hypergraphs. In Proposition 4 [16], a criterion is stated under which conditions a non-diagonal base hypergraph $\mathcal{H}$ having $\delta(\mathcal{H}) = 0$ becomes a diagonal instance $\mathcal{H}'$ with $\delta(\mathcal{H}') = i$, for $i > 0$ arbitrary, by adding exactly one hyperedge to $\mathcal{H}$. On this basis and to gain more insight into this class the following notion for some of its most extreme members is introduced:

Definition 1: A non-diagonal CNF-base hypergraph $\mathcal{H} = (V, B)$ is called *maximal non-diagonal* if there is a diagonal base hypergraph $\mathcal{H}' = (V, B')$, with $B \subseteq B'$, such that for every $b \in B' \setminus B$ it is $\delta(\mathcal{H} \cup \{b\}) > 0$. Let $\mathfrak{H}_{\text{maxnd}}$ denote the class of all *maximal non-diagonal* base hypergraphs.

The non-trivial existence of a maximal non-diagonal base hypergraph can be established on the basis of any minimal diagonal instance.

Lemma 2: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_{\text{mdiag}}$ then $\mathcal{H}_b := \mathcal{H} \setminus \{b\} \in \mathfrak{H}_{\text{maxnd}}$, for every $b \in B$.□

Thm. 6 in [15] provides the equivalence that a diagonal base hypergraph $\mathcal{H}$ is minimal diagonal iff $\mathcal{F}_{\text{diag}}(\mathcal{H}) \subseteq \mathcal{I}$. Moreover, for the subclass of simple, connected base hypergraphs collected in $\mathfrak{H}_1^c$ one has according to Cor. 3 in [15] that $\mathfrak{H}_1^c \subseteq \mathfrak{H}_{\text{mdiag}}$. So, one obtains:

Corollary 1: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_1^c$ then $\mathcal{H} \setminus \{b\} \in \mathfrak{H}_{\text{maxnd}}$, for every $b \in B$. □

Besides $\mathfrak{H}_1$ there are higher levels according to the hierarchy of diagnal base hypergraphs as introduced in [15]. On this basis $\mathfrak{H}_{\text{maxnd}}$ formally decomposes into the following subclasses.

Definition 2: Let $i \geq 1$ be an integer then $\mathcal{H} = (V, B)$ with $\delta(\mathcal{H}) = 0$ is called a *maximal i-non-diagonal* base hypergraph if there is $\mathcal{H}' = (V, B')$ with $B \subseteq B'$, $\delta(\mathcal{H}') = i$, and for every $b \in B' \setminus B$ it is $\delta(\mathcal{H} \cup \{b\}) \in [i]$. Let $\mathfrak{H}^i_{\text{maxnd}}$ denote the class of all *maximal i-non-diagonal* base hypergraphs. Specifically, we set $\hat{\mathfrak{H}}^i_{\text{maxnd}} \subseteq \mathfrak{H}^i_{\text{maxnd}}$ for all maximal i-non-diagonal members $\mathcal{H} = (V, B)$ such that for all $b \in B' \setminus B$ it is $\delta(\mathcal{H} \cup \{b\}) = i$.

Note that $\mathcal{H}'$ in the previous definition belongs to the class $\hat{\mathfrak{H}}_i$. So far it is unknown whether these classes are non-trivial for an arbitrary integer $i > 0$. However, as shown in [15] there are arbitrary large i such that $\hat{\mathfrak{H}}_i \neq \emptyset$. Similarly, the non-trivial existence of the classes $\hat{\mathfrak{H}}^i_{\text{maxnd}}$ defined above needs to be established. According to Prop. 4 [16] a maximal non-diagonal base hypergraph $\mathcal{H} = (V, B) \subset \mathcal{H}'$ where $\delta(\mathcal{H}') = i$ must have the property that for each $b \in B' \setminus B$

there are exactly i distinct orbits in $\mathcal{F}(\mathcal{H})$ and for each orbit there is a member F containing $c(F) \in W_b$ with $t(b) = c(F)$, for all $t \in \mathcal{M}(F)$. Then $\delta(\mathcal{H} \cup \{b\}) = i$ is ensured, hence $\mathcal{H} \in \hat{\mathfrak{H}}^i_{\mathrm{maxnd}}$.

Let $\mathcal{H}_i \in \mathfrak{H}_{\mathrm{mdiag}}$, $i \in [2]$, with $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ then obviously $\mathcal{H}_1 \cup \mathcal{H}_2 \notin \mathfrak{H}_{\mathrm{mdiag}}$. On this basis one is able to derive a maximal non-diagonal base hypergraph which is no subhypergraph of a minimal diagonal instance.

Proposition 1: For any fixed integer $r > 1$, let $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}_{\mathrm{mdiag}}$, $i \in [r]$, be mutually disjoint, i.e., $V_i \cap V_j = \emptyset, i \neq j$, and $\mathcal{H}' = (V, B') := \bigcup_{i\in[r]} \mathcal{H}_i$. Setting $B := \bigcup_{i\in[r]} B_i \setminus \{b_i\}$, for any selection $b_i \in B_i$, $i \in [r]$, it is $\mathcal{H} := (V, B) \in \mathfrak{H}_{\mathrm{maxnd}}$. □

Observe that the instances above are disconnected by construction. The next result also provides members belonging to the connected class.

Theorem 2: For any fixed integer $r > 1$, and mutually disjoint $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}_{\mathrm{mdiag}}$, $i \in [r]$, let $\mathcal{H}_0 = (V_0, B_0) := \bigcup_{i\in[r]} \mathcal{H}_i$, choose $b_0 \subset V_0$ such that $\forall i \in [r] : |b_0 \cap V_i| = 1$ and let $y \notin V_0$. Then $\mathcal{H}' = (V, B') := \mathcal{H}_0 \cup \{b\} \in \mathfrak{H}^c$ is diagonal, where $b := b_0 \cup \{y\}$. Further for any selection $b_i \in B_i$, for all $i \in [r]$, and $B := \{b\} \cup \bigcup_{i\in[r]} B_i \setminus \{b_i\}$ it is $\mathcal{H} := (V, B) \in \mathfrak{H}_{\mathrm{maxnd}}$. Moreover if $b_i \cap b_0 = \emptyset$, for all $i \in [r]$, then $\mathcal{H} \in \mathfrak{H}^c$.

PROOF. Since $\delta(\mathcal{H}_i) > 0$, $i \in [r]$, according to La. 1 (ii) in [15] it follows that $\delta(\mathcal{H}_0) > 0$. Clearly $b_0 \notin B_i, i \in [r]$. Moreover each $\mathcal{H}_i$ is minimal diagonal, thus also connected. Adding the new edge $b = b_0 \cup \{y\}$ enlarged by the new variable y to the union $\mathcal{H}_0$ of $\mathcal{H}_i$, $i \in [r]$, provides the connected base hypergraph $\mathcal{H}' = \mathcal{H}_0 \cup \{b_0 \cup \{y\}\}$. Clearly $\mathcal{H}_0 \subset \mathcal{H}'$ and because of the monotony of the mapping δ, as stated in Prop. 6 (1) in [16], one has $\delta(\mathcal{H}') > 0$. Since $|B_i| > 1$, the selection of the b_i specifically can be performed such that $b_i \cap b_0 = \emptyset$, for all $i \in [r]$, maintaining the connectedness of $\mathcal{H}$. Since the new variable y ensures that any clause over b can be satisfied independently and because of the fact that the instances $\mathcal{H}_i$, $i \in [r]$, all are minimal diagonal, the rest of the theorem follows directly from Prop. 1. □

Observe that the new variable y added to b_0 above, in general, cannot be omitted. Consider e.g. $\mathcal{H}_i := (V_i, B_i)$ with $V_i := \{u_i, v_i\}$, $B_i := \{u_i, v_i, u_i v_i\}$, $i \in [2]$, $b_0 := v_1 u_2$, and the selection $b_1 := u_1 \in B_1$, $b_2 := v_2 \in B_2$ where the hyperedges for simplicity are denoted as sequences of there (variable-)vertices. Clearly, $\mathcal{H}_i$, $i \in [2]$, both are simple and minimal diagonal base hypergraphs. Then $\mathcal{H} := (V, \tilde{B}) \notin \mathfrak{H}_{\mathrm{maxnd}}$, it even remains a diagonal base hypergraph in case of $\tilde{B} := \{b_0\} \cup \bigcup_{i\in[r]} B_i \setminus \{b_i\}$ instead of $B := \{b\} \cup \bigcup_{i\in[r]} B_i \setminus \{b_i\}$ where $b := b_0 \cup \{y\}$ as required in Thm. 2. Indeed, $\tilde{B}$ specifically contains the edges $v_1 u_2$, u_2, and v_1 yielding a simple, hence diagonal subhypergraph.

Theorem 3: There are Sperner, and even linear, maximal non-diagonal base hypergraphs. There also exist exact linear and maximal non-diagonal base hypergraphs.

PROOF. Consider the formula $C \in \mathcal{I}$ used in the proof of Lemma 1. $\mathcal{H}(C)$ obviously is loopless, linear, hence Sperner. Moreover it is connected and simple as shown in the proof of Thm. 8 [15], hence it is minimal diagonal. Therefore by Lemma 2, $\mathcal{H}(C) \setminus \{b\} \in \mathfrak{H}_{\mathrm{maxnd}}$ is a linear base hypergraph, for every $b \in B(C)$, implying the first claim. Regarding the second statement, let $V = \{u, x, y\}$ and $B = \{x, xy, yu, ux\}$. We claim that $\mathcal{H} = (V, B) \in \mathfrak{H}^c_1$ where the connectedness is obvious. Observe that $\mathcal{H} \setminus \{x\}$ is exact linear, hence it is non-diagonal [17]. Thus a minimal fibre-transversal of $\mathcal{H}$ can occur only via the literal over x and indeed obviously, e.g. $\{x, \bar{x}y, \bar{y}u, \bar{u}\bar{x}\} \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ which easily can be verified to belong to $\mathcal{I}$. Now, any unsatisfiable formula in a distinct orbit can occur only if a bifurcation [15] is performed at the clause only containing the literal over x. But since the x-clause is forced to have a unique value in a model there can exist only one orbit of unsatisfiable fibre-transversals. Thus $\mathcal{H} \in \mathfrak{H}^c_1$ implying that $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$. Therefore by Lemma 2 it is $\mathcal{H} \setminus \{x\} \in \mathfrak{H}_{\mathrm{maxnd}}$. □

IV. THE GENERAL CASE

In this section maximal non-diagonal base hypergraphs are constructed which do not necessarily rely on minimal diagonal ones. For generalizing the previous discussion the following concept is crucial.

Definition 3: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$. Any subhypergraph $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ with $\tilde{\mathcal{H}} \in \mathfrak{H}_{\mathrm{mdiag}}$ is called a *diagonal germ*. Let $\mathfrak{G}(\mathcal{H})$ be the collection of all diagonal germs of $\mathcal{H}$. Any $G \subset B$ is called a *transversal of diagonal germs (TDG) of* $\mathcal{H}$ if $G \cap B(\tilde{\mathcal{H}}) \neq \emptyset$ for every $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$. A TDG G of $\mathcal{H}$ is *minimal* if it does not contain a proper TDG of $\mathcal{H}$.

As a concrete example for the previous terminology, consider the non-minimal diagonal base hypergraph $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$ with $V = \{u, v, x, y\}$, $B = \{x, y, xy, xu, uv, u, v\}$. Then one has $\mathfrak{G}(\mathcal{H}) = \{\mathcal{H}_l = (V_l, B_l) : l \in [6]\}$, where

$$\begin{aligned} &B_1 := \{x, y, xy\}, \quad B_2 := \{u, v, uv\}, \\ &B_3 := \{x, u, xu\}, \quad B_4 := \{x, xu, uv, v\}, \\ &B_5 := \{y, xy, xu, u\}, \quad B_6 := \{y, xy, xu, uv, v\} \end{aligned}$$

which all belong to $\mathfrak{H}_{\mathrm{mdiag}}$ as can be verified easily. The collection of all minimal TDG of $\mathcal{H}$ is provided by $\{G_i : i \in [20]\}$, where

$$\begin{aligned} &G_1 = \{x, y, u\}, \quad G_2 = \{x, y, v\}, \\ &G_3 = \{u, v, x\}, \quad G_4 = \{u, v, y\}, \\ &G_5 = \{x, v, xy\}, \quad G_6 = \{u, x, xu\}, \\ &G_7 = \{x, y, uv\}, \quad G_8 = \{x, u, xy\}, \\ &G_9 = \{u, v, xy\}, \quad G_{10} = \{u, x, uv\}, \\ &G_{11} = \{u, y, uv\}, \quad G_{12} = \{u, y, xu\}, \\ &G_{13} = \{v, x, xu\}, \quad G_{14} = \{v, y, xu\}, \\ &G_{15} = \{x, uv, xu\}, \quad G_{16} = \{u, uv, xy\}, \\ &G_{17} = \{x, xy, uv\}, \quad G_{18} = \{y, uv, xu\}, \\ &G_{19} = \{u, xy, xu\}, \quad G_{20} = \{v, xy, xu\} \end{aligned}$$

The following facts provide the connections to (minimal) diagonality.

Proposition 2: Let $\mathcal{H} \in \mathfrak{H}$ then:

(a) $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$ iff $\mathfrak{G}(\mathcal{H}) \neq \emptyset$,
(b) $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ iff $\mathfrak{G}(\mathcal{H}) = \{\mathcal{H}\}$.

PROOF. Let $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$ then it contains a minimal diagonal subhypergraph hence $\mathfrak{G}(\mathcal{H}) \neq \emptyset$. Reversely, let $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$ then $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ is diagonal so $\mathcal{H}$ is diagonal, so (a) is true. By definition $\{\mathcal{H}\} \subseteq \mathfrak{G}(\mathcal{H})$. Let $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H})$ then $\tilde{\mathcal{H}} \subseteq \mathcal{H}$ and $\tilde{\mathcal{H}} \in \mathfrak{H}_{\mathrm{mdiag}}$ hence $\tilde{\mathcal{H}} = \mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$, so (b) is true. □

If $\mathcal{H} = (V, B)$ is a diagonal base hypergraph, and G is one of its minimal TDG, we set $\mathcal{H} \setminus G := (V, B \setminus G)$ in accordance with the next fact.

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Lemma 3: Let $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$. If a minimal TDG of $\mathcal{H}$ is removed from B then the resulting hypergraph has the same vertex set V.

PROOF. Let $\mathcal{H}' = (V', B')$ be the resulting base hypergraph and assume there is $x \in V \setminus V'$. Then there is a $\tilde{b} \in G$ containing x, and there is $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B}) \in \mathfrak{G}(\mathcal{H})$ containing $\tilde{b}$. Hence $\tilde{\mathcal{H}} \setminus \{\tilde{b}\} \subseteq \mathcal{H}'$ is not diagonal and its vertex set does not contain x. So, extending any fibre-transversal in $\mathcal{F}(\tilde{\mathcal{H}} \setminus \{\tilde{b}\})$ over $\tilde{b}$ by an arbitrary clause yields a fibre-transversal in $\mathcal{F}(\tilde{\mathcal{H}})$ whose additional clause can be satisfied independently of the other clauses via the literal over x. Therefore $\mathcal{F}(\tilde{\mathcal{H}})$ does not contain any diagonal fibre-transversal yielding a contradiction. □

Towards a characterization of maximal non-diagonality the next result relying on the previous one turns out to be useful.

Lemma 4: Let $\mathcal{H}' = (V, B') \in \mathfrak{H}_{\mathrm{diag}}$ and $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{maxnd}}$ where $B \subseteq B'$ then $B' \setminus B$ is a minimal TDG of $\mathcal{H}'$.

PROOF. Assume that $G := B' \setminus B$ is no TDG of $\mathcal{H}'$. Then there is a minimal diagonal $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B}) \subseteq \mathcal{H}'$ such that $\tilde{B} \cap G = \emptyset$. Thus $\tilde{\mathcal{H}} \in \mathfrak{G}(\mathcal{H}' \setminus G) \neq \emptyset$. However since $\mathcal{H}' \setminus G = \mathcal{H}$ relying on La. 3, by Prop. 2 (a) it follows that $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$, yielding a contradiction, hence G is a TDG of $\mathcal{H}'$. Next assume there is a proper sub-TDG $\tilde{G} \subset G$ of $\mathcal{H}'$. Thus there is $b \in G \setminus \tilde{G}$ and we claim that $\mathcal{H} \cup \{b\}$ remains non-diagonal yielding a contradiction to $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$. To verify the claim assume by Prop. 2 (a) that $\mathcal{H}_g = (V_g, B_g) \in \mathfrak{G}(\mathcal{H} \cup \{b\})$. As by definition and La. 3 $\mathcal{H} = \mathcal{H}' \setminus G$, it follows $\mathcal{H} \cup \{b\} = \mathcal{H}' \setminus (G \setminus \{b\}) \subseteq \mathcal{H}' \setminus \tilde{G}$. Hence it is $\mathcal{H}_g \in \mathfrak{G}(\mathcal{H}' \setminus \tilde{G})$ providing a contradiction because by definition $\tilde{G} \cap B_g \neq \emptyset$. □

Theorem 4: Let $\mathcal{H}' = (V, B') \in \mathfrak{H}_{\mathrm{diag}}$. Then $\mathcal{H} = (V, B) \subseteq \mathcal{H}'$ is maximal non-diagonal iff $B' \setminus B$ is a minimal TDG of $\mathcal{H}'$.

PROOF. The necessity directly is implied by Lemma 4. For the sufficiency let $G := B' \setminus B$ be a minimal TDG of $\mathcal{H}'$ and $\mathcal{H} := \mathcal{H}' \setminus G$. According to La. 3 it is $V(\mathcal{H}) = V$. First assume that $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$, and let $\mathcal{H}_g \in \mathfrak{G}(\mathcal{H})$. Since $\mathcal{H}' \setminus G \subseteq \mathcal{H}'$ it follows that $\mathcal{H}_g \in \mathfrak{G}(\mathcal{H}')$ yielding a contradiction because by definition $G \cap B(\mathcal{H}_g) \neq \emptyset$, so $\mathcal{H}$ is non-diagonal. Next, assume that there is any $b \in G$ such that $\mathcal{H} \cup \{b\}$ remains non-diagonal, i.e., $\mathfrak{G}(\mathcal{H} \cup \{b\}) = \emptyset$. Let $\mathcal{H}_g \in \mathfrak{G}(\mathcal{H}')$ be arbitrary. Then it is $B(\mathcal{H}_g) \cap (G \setminus \{b\}) \neq \emptyset$, otherwise $\mathcal{H}_g \subseteq \mathcal{H} \cup \{b\}$, so $\mathcal{H}_g \in \mathfrak{G}(\mathcal{H} \cup \{b\})$ which is impossible. Therefore it follows that $G \setminus \{b\}$ is a TDG of $\mathcal{H}'$ which obviously is a sub-TDG of G providing a contradiction to its minimality and settling that $\mathcal{H} \in \mathfrak{H}_{\mathrm{maxnd}}$. □

As a direct consequence one obtains.

Corollary 2: For every diagonal base hypergraph there is a maximal non-diagonal subhypergraph. More precisely, let $\mathcal{H}' = (V, B') \in \mathfrak{H}_{\mathrm{diag}}$ then for every minimal TDG G of $\mathcal{H}'$ it is $\mathcal{H}(G) := \mathcal{H}' \setminus G \in \mathfrak{H}_{\mathrm{maxnd}}$. □

V. A CONNECTION TO MAXIMAL SATISFIABLE FORMULAS

Recall that for $C' \in$ UNSAT a subformula $C \in$ SAT is C'-maximal satisfiable if by definition $C \cup \{c\} \in$ UNSAT for every $c \in C' \setminus C$ [13]. As a non-trivial example consider the total clause set $K_{\mathcal{H}} \in$ UNSAT over any (even non-diagonal) base hypergraph $\mathcal{H} = (V, B)$. However, for every $t \in W_V$ with $K_{\mathcal{H}} \setminus t(B)$, a $K_{\mathcal{H}}$-maximal formula is provided [13]. As $\mathcal{H}(K_{\mathcal{H}} \setminus t(B)) = \mathcal{H}$ there can arise maximal non-diagonal base hypergraphs from a C'-maximal satisfiable formula C only in case $\mathcal{H}(C') \neq \mathcal{H}(C)$. As another example, let $I \in \mathcal{I}$. For any $c \in I$ it is $I \setminus \{c\}$ an I-maximal satisfiable formula [13]. So for $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$, every $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ provides an F-maximal satisfiable formula $F \setminus \{c\}$, for any $c \in F$. Whether there are deeper connections between both concepts shall be the topic next. A useful concept here for the instances in UNSAT is the parameter $\mu(C) := \min\{|t(B(C)) \cap C| : t \in W_{V(C)}\} > 0$, together with $W_{V(C)}^{\mu(C)} := \{t \in W_{V(C)} : |t(B() \cap C| = \mu(C)\}$ [13]. Obviously $\mu(C) = 0$ means that there is a compatible fibre-transversal in the complement formula which by Thm. 1 implies $C \in$ SAT contradicting the assumption. A first result here is:

Theorem 5: Let $C' \in$ UNSAT with $|C'_b| = 1$ for all $b \in B(C')$. If there is $t \in W_{V(C')}^{\mu(C')}$ such that $\mathcal{H} := \mathcal{H}(C' \setminus t(B(C')))$ is non-diagonal then $\mathcal{H} \subset \mathcal{H}(C')$ already is maximal non-diagonal.

PROOF. The fibre-condition on C' ensures that it is a fibre-transversal of its base hypergraph, hence $\mathcal{H}' := \mathcal{H}(C') =: (V', B') \in \mathfrak{H}_{\mathrm{diag}}$. According to the proof of Thm. 7 in [13] it is $C := C' \setminus t(B')$ a C'-maximal satisfiable formula. Assume there is $x \in V' \setminus V(C)$ then there also is $c \in C' \setminus C$ containing a literal over x which can be satisfied independent of C. Thus $C \cup \{c\} \in$ SAT providing a contradiction. Hence it is $V(C) = V' = V(\mathcal{H})$. By assumption $\mathcal{H} =: (V', B)$ is diagonal. Let $b \in B' \setminus B$ be arbitrary then there is $c \in C'_b \cap W_b$ which is not in C. But as C is C'-maximal satisfiable, $C \cup \{c\} \in$ UNSAT which according to the condition on C' is a fibre-transversal of $\mathcal{H} \cup \{b\}$. Thus $\delta(\mathcal{H} \cup \{b\}) > 0$, for every $b \in B' \setminus B$. □

Restricting μ to diagonal fibre-transversals induces the following parameters on diagonal base hypergraphs.

Definition 4: Let $\mathcal{H} = (V, B)$ be a diagonal base hypergraph. Let $\lambda(\mathcal{H}) := \min\{|t(B) \cap F| : t \in W_V, F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$ be the *lower intersection index* of $\mathcal{H}$. Similarly the *upper intersection index* of $\mathcal{H}$ is defined by $\nu(\mathcal{H}) := \max\{|t(B) \cap F| : t \in W_V, F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$. Moreover let $\mathcal{W}^{\tau(\mathcal{H})} := \{(t, F) \in W_V \times \mathcal{F}_{\mathrm{diag}}(\mathcal{H}) : |t(B) \cap F| = \tau(\mathcal{H})\}$, for $\tau \in \{\lambda, \nu\}$.

Considering both λ, ν as integer-valued mappings on $\mathfrak{H}_{\mathrm{diag}}$ one has.

Lemma 5: There is no upper bound for the values of ν on $\mathfrak{H}^{c}_{\mathrm{diag}}$, and the lower bound of λ on $\mathfrak{H}_{\mathrm{diag}}$ is 1. Moreover, restricted to $\mathfrak{H}_{\mathrm{mdiag}}$, λ equals the constant 1.

PROOF. Regarding the first claim again consider the simple hypergraph $\mathcal{H}_0 := (V_0, B_0)$ with $V_0 := \{u, v\}$, $B_0 := \{u, v, uv\}$. For the diagonal fibre-transversal $F_0 := \{u, v, \bar{u}\bar{v}\}$ and the truth assignment $t_0(B_0) := \{u, v, uv\}$ one has $|t_0(B_0) \cap F_0| = 2 = \lceil |B_0|/2 \rceil = |B_0| - 1$ which also coincides with $\nu(\mathcal{H}_0)$ as can be verified easily. Now let $r > 0$ be an integer, and let $\mathcal{H}_i = (V_i, B_i) \in \mathfrak{H}$, $i \in [r]$, be arbitrary, mutually disjoint base hypergraphs such that also $V_i \cap V_0 = \emptyset$, for all $i \in [r]$. Set $V := \bigcup_{i \in [r]} V_i$, $B := \bigcup_{i \in [r]} B_i$ and let $t \in W_V$, hence $t(B)$ yields a compatible fibre-transversal over B. Then $\tilde{t}'(B') := t_0(B_0) \cup t(B)$ provides a truth assignment $t' \in W_{V'}$ where $V' := V_0 \cup V$, $B' := B_0 \cup B$, and $F' := F_0 \cup t(B) \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}')$ where $\mathcal{H}' := (V', B')$. Moreover it is $|t'(B') \cap F'| = |B'| - 1$. Finally, introduce a new variable u and select exactly one variable from $u_i \in V_i$, for all $i \in [r]$. Defining a new edge $\tilde{b} := \{u, u_1, \ldots u_r\}$ and setting $\tilde{B} := B' \cup \{\tilde{b}\}$ obviously yields a connected diagonal base hypergraph $\tilde{\mathcal{H}} = (\tilde{V}, \tilde{B})$. Finally, setting $\tilde{t}(\tilde{B}) := t'(B') \cup \{\tilde{b}\}$ and $\tilde{F} := F' \cup \{\tilde{b}\}$ establishes the first claim. The lower bound statement for λ directly follows from Thm. 1. Next observe that $\lambda(\mathcal{H}) = \min\{\mu(F) : F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$. Lemma 1 (e) directly implies that $\mu(C) = 1$ for every $C \in \mathcal{I}$. Since $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \subseteq \mathcal{I}$, for every minimal diagonal base hypergraph it follows that the restriction $\lambda|\mathfrak{H}_{\mathrm{mdiag}} = 1$. □

Proposition 3: For any diagonal base hypergraph $\mathcal{H} = (V, B)$ it is $\mathcal{H}(F \setminus t(B))$ non-diagonal, for every $(t, F) \in \mathcal{W}^{\nu(\mathcal{H})}$.

PROOF. Let $\mathcal{H}_0 = (V_0, B_0) := \mathcal{H}(F \setminus t(B))$ and suppose there is $F_0 \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H}_0)$. Then by definition there is $b \in B_0 \subseteq B$ such that $F_0(b) = t(b)$ as $t(B) \setminus \{t(b) : b \in B \setminus B_0\}$ is a compatible fibre-transversal over B_0. Clearly $t(b) \neq F(b)$. Hence extending F_0 to $\tilde{F}_0$ over B by setting $\tilde{F}_0(b') := F(b')$ for all $b' \in B \setminus B_0$ yields a diagonal fibre-transversal in $\mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ with $|t(B) \cap \tilde{F}_0| > \nu(\mathcal{H})$ which is a contradiction providing the claim. □

Theorem 6: For all $\mathcal{H} \in \mathfrak{H}_{\mathrm{diag}}$ one has:

(a) $\lambda(\mathcal{H}) = 1$,
(b) $\nu(\mathcal{H}) > \lambda(\mathcal{H})$, if $\mathcal{H} \notin \mathfrak{H}_{\mathrm{mdiag}}$.
(c) $\nu(\mathcal{H}) = \lambda(\mathcal{H})$ implies $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ such there is no pair $b, b' \in B$ with $b \cap b' = \emptyset$.

PROOF. Consider an arbitrary diagonal base hypergraph $\mathcal{H} = (V, B)$, and let $\mathcal{H}_g = (V_g, B_g) \in \mathfrak{G}(\mathcal{H})$ be fixed due Prop. 2. If $\mathcal{H} = \mathcal{H}_g$ we are done because of La. 5. Otherwise set $\mathcal{H}_0 := \mathcal{H} \setminus \mathcal{H}_g = (V(B \setminus B_g), B \setminus B_g)$. Choose an arbitrary $(t_g, F_g) \in \mathcal{W}^{\lambda(\mathcal{H}_g)}$, and fix an arbitrary $F_0 \in \mathcal{F}_{\mathrm{comp}}(\mathcal{H}_0)$. Defining $t \in W_V$ via $t(B) := t_g(B_g) \cup F_0^{\gamma}$, and $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ via $F := F_g \cup F_0$, as $F_g \in$ UNSAT, we claim that $|t(B) \cap F| = 1$ proving that $\lambda(\mathcal{H}) = 1$. To verify the claim, we first have $\lambda(\mathcal{H}_g) = 1$ according to La. 5. Thus $|t_g(B_g) \cap F_g| = 1$, and clearly, $F_0^{\gamma} \cap F_0 = \emptyset$. Since $B(\mathcal{H}) \cap B(\mathcal{H}_0) = \emptyset$ it also follows that $t_g(B_g) \cap F_0 = \emptyset$ and $F_0^{\gamma} \cap F_g = \emptyset$ finishing the verification of (a). For (b) suppose that $\mathcal{H} \notin \mathfrak{H}_{\mathrm{mdiag}}$ then there are $\mathcal{H}_g = (V_g, B_g) \in \mathfrak{G}(\mathcal{H})$ and $\mathcal{H}_0 := \mathcal{H} \setminus \mathcal{H}_g = (V(B \setminus B_g), B \setminus B_g)$. Fix any $F_0 \in \mathcal{F}_{\mathrm{comp}}(\mathcal{H}_0)$, define $t \in W_V$ via $t(B) := t_g(B_g) \cup F_0$, and $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ via $F := F_g \cup F_0$. As $|F_0| \geq 1$ it follows $|t(B) \cap F| = |t_g(B_g) \cap F_g| + |F_0| \geq 2$ according to La. 5, hence $\nu(\mathcal{H}) > 1 = \lambda(\mathcal{H})$.

Regarding (c) let $\mathcal{H} \in \mathfrak{H}_{\mathrm{mdiag}}$ then $|B| > 1$. Assume there are $b_0, b_0' \in B$ with $b_0 \cap b_0' = \emptyset$ and let $F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})$ be arbitrary. Define $t \in W_V$ via $t(b_0) = F(b_0)$, $t(b_0') = F(b_0')$ and for all $b \in B \setminus \{b_0, b_0'\}$ by setting $t(b)$ such that $V(t(b_0) \cap t(b)) = b_0 \cap b$ and $V(t(b_0') \cap t(b)) = b_0' \cap b$. Hence it follows $|F \cap t| = 2$. □

VI. Conclusion and Open Problems

The class of maximal non-diagonal base hypergraphs has been introduced. Whereas some structural properties could be revealed, numerous questions remain open so far. So, observe that the hypergraph $\mathcal{H}$ in the proof of the second statement of Thm. 3 contains a loop which also is the crucial part of this argumentation. Therefore one should try to construct a loopless diagonal linear base hypergraph yielding an exact linear, and maximal non-diagonal instance. Here the Lemma 18 in [17] might be helpful. Next, there arise several computational problems along with their complexities: First of all, given any base hypergraph $\mathcal{H}$. What is the complexity for deciding whether it is diagonal? This problem is closely related to the problem: Given $\mathcal{H}$ and any of its fibre-transversals F, decide whether F is diagonal. Observe that testing whether F is compatible can be performed in linear time in the size of the formula relying on appropriate data structures. Suppose it could be efficiently decided whether a non-compatible F is satisfiable. Then also the decision whether a linear formula is satisfiable was easy, at it appears to be a fibre-transversal of its (linear) base hypergraph. On the other hand, the latter problem is well known to be NP-complete [17]. Thus the decision whether a fibre-transversal is diagonal in general at least is NP-complete, too. We conclude that a test of a base hypergraph for diagonality should not rely on testing fibre-transversals. And it arises the question whether there is another approach. Also the complexity for deciding the minimal diagonality of an instance $\mathcal{H}$ is unknown. Here it might be helpful to clarify whether the criterion $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \subset \mathcal{I}$ for minimal diagonality of $\mathcal{H}$ could be relaxed to $\mathcal{F}_{\mathrm{diag}}(\mathcal{H}) \cap \mathcal{I} = \emptyset$. The latter would be equivalent to $\mathfrak{H}_1^c = \mathfrak{H}_{\mathrm{mdiag}}$. Next suppose it could be tested fast whether the fibre-transversal of a linear base hypergraph is minimal unsatisfiable. Then clearly the same was true for an arbitrary linear formula, implying that also SAT for linear formulas was decidable easily. So the question how minimal diagonality or at least simplicity could be decided efficiently remains open. Due to Cor. 2 every diagonal base hypergraph admits a maximal non-diagonal subhypergraph. Thus, given any diagonal $\mathcal{H}'$ and $\mathcal{H} \subset \mathcal{H}'$, the decision whether $\mathcal{H}$ is maximal non-diagonal could be performed by testing whether $B(\mathcal{H}') \setminus B(\mathcal{H})$ is a minimal TDG of $\mathcal{H}'$. Hence the complexity for deciding whether a given set of hyperedges forms a minimal TDG has to be investigated.

Further it is unclear whether the values of ν restricted to the subspace of minimal diagonal base hypergraphs admits an upper bound depending on the size of the corresponding B. Here we conjecture that this upper bound is given by $\lceil |B|/2 \rceil$. Note that $\nu(\mathcal{H}_0)$ exactly equals this bound, as provided in the proof of La. 5. Given $\mathcal{H} = (V, B) \in \mathfrak{H}_{\mathrm{diag}}$, one might ask whether there is a deeper structural relationship between those instances $\mathcal{H}_i := \mathcal{H} \setminus t_i(B)$, for different pairs $(t_i, F_i) \in \mathcal{W}^{\nu(\mathcal{H})}$, $i \in [2]$. Regarding the relationship between λ, μ, the assumption $C \in$ UNSAT in general does not mean that $\mathcal{H}(C) \in \mathfrak{H}_{\mathrm{diag}}$. Moreover, μ does not equal 1 on all of UNSAT: As e.g., the total clause set $K_{\mathcal{H}}$ even of a non-diagonal base hypergraph $\mathcal{H} = (V, B)$ clearly fulfills $\mu(K_{\mathcal{H}}) = |B|$. Additionally defining a function $\sigma :$ UNSAT $\rightarrow \mathbb{N}$ via $\sigma(C) := \max\{|t \cap C| : t \in W_{V(C)}\}$ yields the relationship $\nu(\mathcal{H}) = \max\{\sigma(F) : F \in \mathcal{F}_{\mathrm{diag}}(\mathcal{H})\}$ to the upper intersection index of a diagonal base hypergraph. Observe that also $\sigma(K_{\mathcal{H}}) = |B|$, so the question whether there is an instance, for which both mappings, μ, σ become equal, must be answered positive. However, according to Thm. 6 it remains open whether there are minimal diagonal base hypergraphs such that $\nu(\mathcal{H}) = \lambda(\mathcal{H}) = 1$.

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