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The Management of an Investment Portfolio in Financial Markets within the Framework of an Approach Alternative to Self-financing Strategy

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***Abstract*— The robust feed-back control schemes to provide the sustainable growth of investor capital are introduced. These schemes are based on the current dynamics of the asset prices. It is assumed that the price of asset follows rather general stochastic differential equation. In contrast to the generally used self-financing strategy the control is realized within the framework of an open system. The latter implies the possibility to invest cash into the portfolio in the process of trading.**

***Index Terms*— Portfolio management, assets trading, stochastic control methods.**

I. Introduction

The attempts to apply classical methods of optimization based on the theory of optimal and adaptive control to realize the management of an investment portfolio very often tumble over serious problems. For instance the application of control theory as the stochastic version of dynamic programming approach implies the detailed information about the structure of factors in stochastic differential equations describing the dynamics of constituting portfolio assets. The latter information in contemporary financial markets seems hardly to be available. The methods of adaptive control theory are also not very often applicable because of the strongly nonstationary behavior of parameters of these or those modeling equations describing the dynamics of portfolio value.

Because of the aforesaid it is not surprising that the problem to create special control methods adapted to the investment portfolio management has long drawn the attention of researchers. Usually such methods imply the creation of control providing in a particular sense the positive dynamics of profit along with the minimization of quantitative and qualitative information about the structure of modeling equations. Moreover one of the most common models for assets pricing is the model of geometrical Brownian motion. Nevertheless when following this way to create the control of investment portfolio there arise a number of difficulties which may be formulated as follows. The heart of the matter is that the designing of control up till now has been based as a rule on the principles of self-financing strategy (see for instance [1]-[4]). The latter means that the purchase or sale of any assets automatically implies sale or purchase of a volume in the equivalent money terms of other assets constituting portfolio.

It is essential to note that realization of any circuit of management based on self-financing strategy implies (at least in terms of the literature available to the authors of the present work) the required number of assets in the portfolio significantly depends not only on the prices of struck bargains but also on the volatilities of corresponding assets.

The latter fact causes some inquires that seem to be an impediment in implementation of corresponding control systems. The point is that for majority of liquid assets the values of their volatilities have strongly non-stationary and pronounced palpitating character. It makes the tracking of their values with arbitrary precision in real time hardly possible. It is also important to keep in mind the property of delay inherent in each control system based on continuous model of pricing and the necessity to realize discrete procedure for their implementation. In this connection it is clear that the occurrence of essential mistakes is possible while defining the amount of assets included in a portfolio. How significantly such errors can affect the ultimate goal of management to provide the profitableness of portfolio remains not clear.

The aforesaid makes reasonable to pose the problem of creating the management of portfolio with a feed-back control based only on the prices of struck bargains to provide in some sense portfolio profitableness on a certain time interval and within the framework of the pricing model corresponding to geometrical Brownian motion.

The main goal of the present study is to solve the problem under consideration within the framework of a management alternative to self-financing strategy. It implies the possibility to invest additional cash from outside during the whole period of portfolio management. Moreover the release of cash as a result of trading allows its reinvestment to acquire new required assets.

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II. Formulation of the Problem and the Main Result

Originally consider elementary structure of the investment portfolio including only one type of assets. Assume that the price of asset x_t is a stochastic process on a time interval $[0,T]$ and follows stochastic differential equation

$$dx_t = c_t x_t dt + \sigma_t x_t dW_t , \qquad (1)$$

where $\sigma_t = \sigma(t)$ is a factor of volatility which is considered as a nonrandom function of time, W_t is a standard Wiener process, $c_t = c(t,\omega)$ is a measurable random function.

Portfolio value is set by a parity

$$f_t = a_t x_t + m_t , \qquad (2)$$

where $a_t = a(t,\omega)$ is a measurable random function defining the number of assets, $m_t = m(t,\omega)$ is a measurable random function responding to some money equivalent which economical sense is given below.

Further, to avoid misunderstanding the realization of any random process in contrast to the process itself will be denoted by the corresponding letter with wave, as for example $\tilde{x}_t$ and x_t.

Consider the control defined for each moment t by the relationship

$$df_t = a_t dx_t + l(t, x_t)dt , \qquad (3)$$

where dx_t is defined by the right hand side of equation (1) while the existence of stochastic differential df_t is supposed.

The second term in dependence (3) is interpreted as cash flow on the time interval dt invested and processed by the control system, while $l(t,x_t) \geq 0$. Consequently $l(t,x_t)$ is regarded as a regulator of the amount of cash processed by the control system and acts as control function.

Appling to the left and right hand sides of relationship (2) the procedure of calculating the stochastic differential, which implies the existence of stochastic differentials da_t and dm_t, one arrives to the relationship

$$df_t = a_t dx_t + x_t da_t + dx_t da_t + dm_t .$$

The latter one by making use of parity (3) may be rewritten as follows

$$dm_t = -x_{t+dt} da_t + l(t, x_t)dt ,$$

where x_{t+dt} is defined as $x_{t+dt} := x_t + dx_t$, or in the integral form

$$m_t = -\int_0^t x_{\tau+d\tau} da_\tau + \int_0^t l(\tau, x_\tau)d\tau . \qquad (4)$$

Sufficient conditions to provide the existence of stochastic integral in relationship (4) as the limit of corresponding sums will be clarified below.

The first term in (4) taken with minus is the value of assets as the result of effected trading.

Define profit $\tilde{p}_t$ for the observable value of price $\tilde{x}_t$ as the difference between the current price of assets and the value of assets as a result of effected trading

$$\tilde{p}_t = \tilde{a}_t \tilde{x}_t - \int_0^t \tilde{x}_{\tau+d\tau} d\tilde{a}_t . \qquad (5)$$

Keeping in mind formulas (2), (4) the latter dependence is equivalent to the relationship

$$\tilde{p}_t = \tilde{f}_t - \int_0^t l(\tau, \tilde{x}_\tau)d\tau , \qquad (6)$$

where $\tilde{f}_t$ is a portfolio value for the observable price.

For the initial instant the portfolio is considered to be empty containing neither assets nor cash.

Consider the notions of the lower and upper bounds of sensitivity which are considered as the respective borders of the price band symmetric with respect to the price of the first bargain struck by the control system. Further, it is supposed the price of the asset is inside the pointed out band during the whole period of control $[0,T]$. For the utility and brevity of calculations the price of asset will be made dimensionless and scaled with respect to the lower bound of sensitivity, thus, defining the aforementioned price band as an interval $(1,\beta)$ where $\beta > 1$ is fixed.

We say that control provides profitableness of an investment portfolio on the time interval $[0,T]$ if $\tilde{p}_T > 0$.

Pose the problem of the existence and realization of portfolio control to provide its profitableness on a given time interval $[0,T]$.

Theorem. Let the following conditions hold on the time interval $[0,T]$, where $T > 0$:

1. The price of asset x_t follows stochastic differential equation (1), moreover volatility σ_t is considered as a nonrandom function of time and consequently one can put down $\sigma_t = \tilde{\sigma}_t$.

2. Integrated volatility is subjected to the following condition of growth: $\int_\tau^T \sigma_s^2 ds \geq \gamma(T-\tau)$ for arbitrary $\tau \in [0,T]$, where γ is strictly positive number.

3. The observable realization of price $\tilde{x}_t$ does not pierce the borders of the price band $(1,\beta)$, where $\beta > 1$ is an arbitrary finite number.

Then if fixed T and β correspond to sufficiently large γ there exists control providing the profitableness of an investment portfolio on the time interval $[0,T]$. Moreover, within the framework of such control the amount of assets in the portfolio for each instant depends on the prices of struck bargains but does not depend explicitly on the volatility values.

Proof. Seek unknown f_t as the function of two variables $f_t = f(t, x_t)$, where x_t follows equation (1). Applying to $f(t,x_t)$ Ito's formula and comparing it with ratio (3) one arrives to the parities

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma_t^2 x_t^2\frac{\partial^2 f}{\partial x_t^2}=l(t,x_t)\ , \tag{7}$$

$$a_t=\frac{\partial f}{\partial x_t} \quad .(8)$$

The control $l(t,x_t)$ is set according to the relationship

$$l(t,x_t)=r(t)\varphi(x_t), \tag{9}$$

where $\varphi(x)$ is the eigenfunction corresponding to the first eigenvalue λ_1 of the following Sturm-Liouville problem

$$\frac{d^2\varphi}{dx^2}+\frac{\lambda_1^2}{x^2}\varphi=0, \tag{10}$$

$$\varphi(1)=\varphi'(\beta)=0. \tag{11}$$

The structure of function $r(t)$ will be clarified bellow.

As for the initial instant of control $t=0$ the portfolio is empty then

$$f(0,x_t)=0\,. \tag{12}$$

Besides the following boundary conditions are introduced

$$\frac{\partial f}{\partial x_t}\to 0 \text{ as } x_t\to\beta\,, \tag{13}$$

$$f(t,x_t)\to 0 \text{ as } x_t\to 1\,. \tag{14}$$

Owing to ratio (8) the fulfillment of boundary condition (13) implies the system of control takes long position, i.e. $a_t\geq 0$, and tends to get rid of assets when the price converges to the upper bound of sensitivity.

To clarify boundary condition (14) make use of relationship (4). The management efficiency implies the value of assets to exceed the cash flow spent for their acquisition, i.e. the inequality $m_t<0$ is to be valid. When asset price converges to the lower bound of sensitivity it is reasonable that the whole amount of profit generated in the process of trading be fixed and invested in purchasing assets, namely $a_t\to-\frac{m_t}{x_t}$ as $x_t\to 1$, what precisely matches, owing to relationship (2), the fulfillment of boundary condition (14).

Taking into account relationship (9) seek solution to the initial-boundary value problem (7), (12), (13), (14) in the form

$$f(t,x_t)=K(t)\varphi(x_t)\,,$$

where $K(t)$ is the unknown function.

As the result of trivial transformations ultimately one arrives to the relationship

$$f(t,x_t)=\int_0^t e^{\frac{1}{2}\lambda_1^2\int_\tau^t\sigma_s^2ds}r(\tau)d\tau\cdot\varphi(x_t)\,, \tag{15}$$

while the value of λ_1 and structure of function $\varphi(x_t)$ are determined by the parities [5]

$$\lambda_1^2=b^2+\frac{1}{4}\,,\ \varphi(x_t)=\sqrt{x_t}\,\sin(b\ln x_t)\,, \tag{16}$$

where b is the minimal strictly positive root to the equation

$$tg(b\ln\beta)=-2b\,. \tag{17}$$

By introducing the new variable $z=b\ln\beta$ equation (17) may be rewritten as follows

$$tg(z)=-\frac{2z}{\ln\beta}\,. \tag{18}$$

The graphical solution to the derived transcendental equation is presented at Fig. 1.

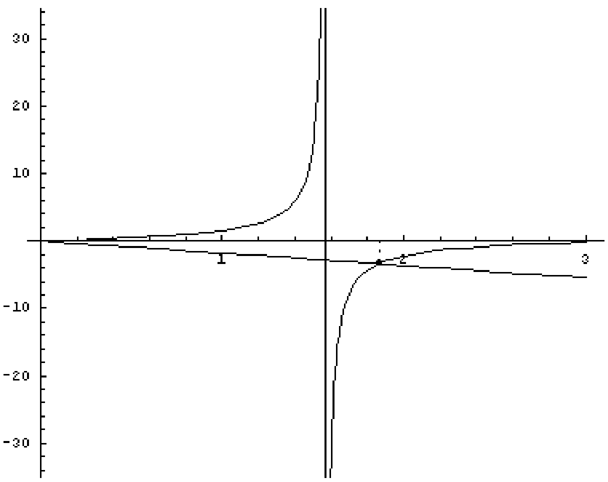

Fig. 1. Graphical solution to the transcendental equation.

Note that relationships (16), (17) describe the whole set of eigenvalues and eigenfunctions of Sturm-Liouville problem (10), (11). However the choice of the first eigenvalue provides, as one can easily note, the corresponding eigenfunction to be separated from zero inside the specified price band $(1,\beta)$.

Relationships (8), (15) define the amount of assets in the portfolio according to the formula

$$\tilde{a}_t=\left(\frac{\partial f}{\partial x_t}\right)_{x_t=\tilde{x}_t}=\int_0^t e^{\frac{1}{2}\lambda_1^2\int_\tau^t\sigma_s^2ds}r(\tau)d\tau\cdot\varphi'(x_t)\Big|_{x_t=\tilde{x}_t}\,. \tag{19}$$

Partition the time interval $[0,T]$ in n parts as follows $0=T_0<T_1<\ldots<T_n=T$.

Define function $r(\tau)$ as the limit of pointwisely converging sequence of functions determined by the relationship

$$r_n(\tau)=\frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)}e^{-\frac{1}{2}\lambda_1^2\int_\tau^{T_i}\sigma_s^2ds}\,, \tag{20}$$

where $\tau\in(T_{i-1},T_i]$, $u_n(\tau)$ are given functions, while the sequence $u_n(\tau)$ as $n\to+\infty$ is supposed to converge pointwisely to the function $u(\tau)$ for a uniform partition.

Substituting in (19) instead of $r(\tau)$ sequence (20) one arrives to the relationship

$$\tilde{a}^n_{T_j} = \sum_{i=1}^{j} \int_{T_{i-1}}^{T_i} \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \cdot \varphi'(x_t)\Big|_{x_t = \tilde{x}_{T_j}} ,$$

or

$$\tilde{a}^n_{T_j} = \int_0^{T_j} \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \cdot \varphi'(x_t)\Big|_{x_t = \tilde{x}_{T_j}} .$$

Ultimately, realizing limit transition as $n \to +\infty$ and within the framework of uniform partition one arrives to the formula describing the continuous distribution of the amount of assets in time under the observable realization of asset price $\tilde{x}_t$:

$$\tilde{a}_t = \int_0^t \frac{u(\tau)}{\sqrt{\tilde{x}_\tau}\,\sin(b\ln \tilde{x}_\tau)} d\tau \cdot \frac{\partial}{\partial x_t}\left(\sqrt{x_t}\,\sin(b\ln x_t)\right)\Bigg|_{x_t = \tilde{x}_t} . \quad (21)$$

By making use of the same arguments and taking into account relationships (6), (9), (15), (20) write down the value of profit at the moment of time T for the observable realization of asset price $\tilde{x}_t$:

$$\tilde{p}_T = \int_0^T \frac{u(t)}{\sqrt{\tilde{x}_t}\,\sin(b\ln \tilde{x}_t)} dt \cdot \sqrt{\tilde{x}_T}\,\sin(b\ln \tilde{x}_T) - \int_0^T e^{-\frac{1}{2}\lambda_1^2 \int_t^T \sigma_s^2 ds} u(t)dt. \quad (22)$$

Note that the transition to the control function $u(t)$ makes it possible to get rid of the explicit dependence on $\tilde{x}_t$ in the second term of formula (22) corresponding to the cash flow processed by the control system by the moment T .

When the control function $u(t)$ is represented by any a priori given piecewise constant nonnegative function which is not identically equal to zero, the first term in parity (22) is strictly positive while the second term may be taken arbitrary small because of the second condition of the Theorem. Thus, the constructed portfolio management really provides the profitableness of the investment portfolio on the time interval $[0,T]$ that makes the proof of the Theorem completed. ■

Remark 1. It is worth noting that relationship (21) explicitly does not depend both on c_t and on volatility σ_t from equation (1). Thus, to construct the required management defining the amount of assets in portfolio there is no necessity to identify the pointed out factors to provide the profitableness of portfolio. On the other hand from formula (22) one can see that the increasing of integrated volatility leads to the essential growth of profit in time. ■

Remark 2. One can easily check that nonnegative values of the function $u(t)$ provide the system of control to take the long position, i.e. $a_t \geq 0$ for arbitrary t . ■

Remark 3. Note that the constructed management provides under certain conditions the profitableness of portfolio but the optimality of such management is not guaranteed. In other words the existence of some other management providing higher profitableness is possible. ■

Remark 4. Note that the stochastic integral on the right hand side of formula (5) is regarded as the limit of the following sums sequence

$$\tilde{S}_n = \sum_{i=1}^{n} \tilde{x}_{t_i}\left(\tilde{a}_{t_i} - \tilde{a}_{t_{i-1}}\right) \quad (23)$$

obtained in the process of time interval $[0,t]$ uniform partition and converging in the mean. The values $\tilde{a}_{t_i} = a\left(t_i, \tilde{x}_{t_i}\right)$ are defined by formula (21). Thus, the supposition of sums (23) convergence in mean imposes certain restrictions on the process x_t . By making use of Ito's formula the stochastic integral on the right hand side of relationship (5) may be presented as the sum of Riemann integral determined on the trajectories of random process x_t and Ito's integral:

$$\int_0^t x_{\tau+d\tau} da_\tau = \int_0^t \psi_1(\tau, x_\tau)d\tau + \int_0^t \psi_2(\tau, x_\tau)dW_\tau ,$$

where ψ_1, ψ_2 are smooth functions defined by the relationship $a_t = a(t, x_t)$ according to formula (21). Thus, to provide the mean convergence of sums in relationship (23) one may use standard sufficient conditions of the existence of corresponding integrals either in the form of restrictions on the process x_t itself or in the form of restrictions on the factors c_t and $\sigma_t = \sigma(t)$ of stochastic differential equation (1). In particular it suffices the process x_t to be integrable in Ito's sense while its trajectories are supposed to be continuous functions. ■

It is worth noting that the construction of control function $u(t)$ as well as the width of the price band want further detailing as they are to be matched to the duration of investment, the distribution of invested cash flow in time and the global dynamics of integrated volatility within the framework of condition 2 of the Theorem.

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