Reliability of Fatigue Damaged Structure Using FORM, SORM and Fatigue Model

Ouk Sub Lee and Dong Hyeok Kim

***Abstract*—The methodologies to calculate failure probability and to estimate the reliability of fatigue loaded structures are developed. The applicability of the methodologies is evaluated with the help of the fatigue crack growth models suggested by Paris and Walker. The probability theories such as the FORM (first order reliability method), the SORM (second order reliability method) and the MCS (Monte Carlo simulation) are utilized. It is found that the failure probability decreases with the increase of the design fatigue life and applied minimum stress, and the decrease of initial edge crack size, applied maximum stress and slope of Paris equation. Furthermore, according to the sensitivity analysis of random variables, it is found that the slope of Pairs equation affects the failure probability dominantly among other random variables in the Paris and the Walker models.**

***Index Terms*—Fatigue, Reliability, Failure Probability, Sensitivity, FORM, SORM, Monte Carlo simulation**

I. Introduction

In the fatigue design, the use of S-N curves is well established, since the repeated loads may lead to failure of material even when the load level is lower than the ultimate limit states. The structures must be repaired, if the crack is discovered. These curves predict fatigue failure under constant amplitude cyclic loading, but cannot incorporate information related to crack detection and/or measurement [1,2].

However, the use of fracture mechanics techniques can be successfully applied to this problem. The fracture mechanics needs the information about the defects, or cracks to be used in the analysis. Since the size and location of defects are quite random, the deterministic analysis may provide incomplete results about the structure safety. Also the randomness of loads, geometry and material properties influence significantly the reliability of a structure. Therefore, the fracture mechanics with a probabilistic method provide a useful tool to solve these problems [3-7].

In this paper, fatigue models suggested by Paris and Walker are used to formulate the LSF (limit state function) for assessing the failure of fatigue loaded structures. And the failure probability is estimated by using the FORM (first order reliability method) and the SORM (second order reliability method). The reliability is assessed by using this failure probability, and the application of these methods to the reliability estimation is given for a case study. Furthermore, the sensitivity of each random variable, which is quantifying the effect on the failure probability, is estimated. And the results obtained by using the FORM and the SORM are compared with those estimated by using the MCS (Monte Carlo simulation) and systematically investigated to assess the accuracy of the reliability.

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II. Fatigue Models

The fatigue crack growth rate, da/dN, versus the applied stress intensity factor range, ΔK, can be obtained from fatigue crack propagation experiments. The corresponding applied stress intensity factor range, ΔK, is calculated when the crack length, a, and the applied stress range, ΔS, are measured in the experiments as below [1,3,4].

$$\begin{aligned}\Delta K_I = \Delta K = K_{\max} - K_{\min} &= S_{\max}\sqrt{\pi a}\alpha - S_{\min}\sqrt{\pi a}\alpha \\ &= (S_{\max} - S_{\min})\sqrt{\pi a}\alpha = \Delta S\sqrt{\pi a}\alpha\end{aligned} \tag{1}$$

Where α is the geometry factor. Since the stress intensity factor is undefined in the compression, $K_{\min}$ is taken as zero if $S_{\min}$ is compressive. The correlation for constant amplitude loading is usually a log-log plot of the fatigue crack growth rate, da/dN, in $m/cycle$, versus the opening mode stress intensity factor range, ΔK_I (or ΔK), in $MPa\sqrt{m}$.

The typical log-log plot of fatigue crack growth rate versus stress intensity factor range as shown schematically in Fig. 1 has a sigmoid shape that can be divided into three major regions. Region I is the near threshold region and indicates a threshold value, ΔK_{th}, and there is no observable crack growth below this value. This threshold occurs at crack growth rates on the order of $1\times10^{-10}\, m/cycle$ or less. Region II shows essentially a linear relationship between log da/dN and log ΔK, which corresponds to the formula suggested by Paris [1,3,4].

$$\frac{da}{dN} = C(\Delta K)^n \tag{2}$$

Where n, C are material constants. n is the slope of the line and C is the coefficient found by extending the straight line to

$\Delta K = 1MPa\sqrt{m}$. Region II fatigue crack growth corresponds to stable macroscopic crack growth that is typically controlled by the environment. Microstructure and mean stress have less influence on fatigue crack growth behavior in region II than in region I. In region III, the fatigue crack growth rates are very high as it approaches instability, and little fatigue crack growth life is involved. This region is controlled primarily by fracture toughness K_C or K_{IC}, which depends on the microstructure, mean stress, and environment.

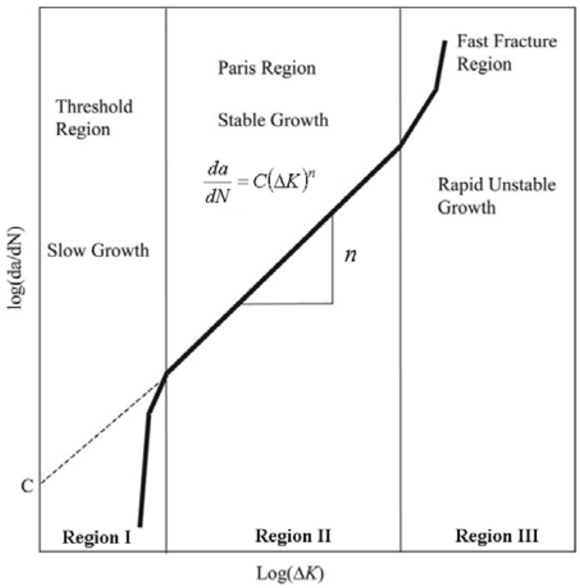


Fig. 1 Schematic behavior of fatigue crack growth rate versus stress intensity factor range

Conventional S-N or ε -N fatigue behavior is usually referenced to the fully reversed stress or strain conditions ($R = -1$). However, fatigue crack growth data are usually referenced to the pulsating tension condition with $R = 0$ or approximately zero.

The general influence of mean stress on fatigue crack growth behavior can be estimated by using the stress ratio, $R = K_{\min} / K_{\max} = S_{\min} / S_{\max}$, which is used as the principal parameter and has the positive value, $R \geq 0$. It should be recognized that the effect of the R ratio on the fatigue crack growth behavior is strongly material dependent.

A common empirical relationship used to describe mean stress effects with $R \geq 0$ is the Walker equation as below [1,3-5].

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1-R)^{n(1-\lambda)}} = C''(\Delta K)^n \tag{3}$$

Where C and n are the coefficient and slope of Paris equation for $R = 0$, respectively, and λ is a material constant. Paris equation and Walker equation are basically similar, with different coefficients of the equations, C and C'', as below.

$$C'' = \frac{C}{(1-R)^{n(1-\lambda)}} \tag{4}$$

Because the effect of R on fatigue crack growth is known as material dependent, it is necessary to determine the material constant, λ. Value of λ for various metals ranges from 0.3 to nearly 1, with a typical value of around 0.5.

The fatigue failure life, N_f, can be obtained by integrating the fatigue crack growth rate formula at the domain from initial crack, a_i, to final crack, a_f. And the final crack can calculate using the fracture toughness as below.

$$a_f = \frac{1}{\pi}\left(\frac{K_c}{S_{\max}\alpha}\right)^2 \tag{5}$$

III. PROBABILITY THEORY

A. *FORM (first order reliability method)*

The failure probability is calculated by using the FORM, which is one of the methods utilizing the reliability index. The FORM method is based on the first-order Taylor series approximation of a limit state function (LSF), which is defined as below [6-10].

$$Z = RE - LO \tag{6}$$

Where, RE is the resistance normal variable, and LO is the load normal variable. Assuming that RE and LO are statistically independent, normally distributed random variables, the variable Z is also normally distributed. The failure occurs when $RE < LO$, i.e., $Z < 0$. The failure probability is given as below.

$$PF = P[Z<0] = \int_{-\infty}^{0} \frac{1}{\sigma_Z\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{Z-\mu_Z}{\sigma_Z}\right)^2\right\} dZ$$
$$= \int_{-\infty}^{-\beta} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{U^2}{2}\right\} dU = \Phi(-\beta) \tag{7}$$

Where μ_Z and σ_Z are the mean and standard deviation of the variable Z, respectively, and Φ is the cumulative distribution function for a standard normal variable, and β is the safety index or reliability index and the coefficient of variation (C.O.V) denoted as below.

$$\beta = \frac{\mu_Z}{\sigma_Z} = \frac{\mu_R - \mu_L}{\sqrt{\sigma_R^2 + \sigma_L^2}}, \; C.O.V = \frac{\sigma_X}{\mu_X} \tag{8}$$

(8) can be used when the system has a linear LSF. Actually, most real systems and cases do not have linear LSF but rather a nonlinear LSF. So, for a system that has a nonlinear LSF, (8) cannot be used to calculate the reliability index. Rackwitz and Fiessler proposed a method to estimate the reliability index that uses the procedure shown in Fig. 2 for a system having a nonlinear LSF. In this paper, we iterate the loop, as shown in Fig. 2, to determine a reliable reliability index until the reliability index converges to a desired value ($\Delta\beta \leq 0.001$) [9, 10].

The LSF must be defined to formulate the FORM and evaluate the reliability. In this paper, the LSF can be defined by using the fatigue models as below [6,7].

$$Z = N_D - N_f \tag{9}$$

Where, N_D is the design fatigue life and N_f is the fatigue life estimated from the fatigue crack growth models such as Paris and Walker models using (2) or (3).

The sensitivity index, which is used to evaluate the effect of random variables on the failure probability, is denoted as below [9, 10].

$$SI = \frac{\left(\frac{\partial Z}{\partial X}\right)}{\left(\sqrt{\sum\left(\frac{\partial Z}{\partial X}\right)^2}\right)} \qquad (10)$$

Where $\partial Z / \partial X$ is the partial derivative of a random variable X .

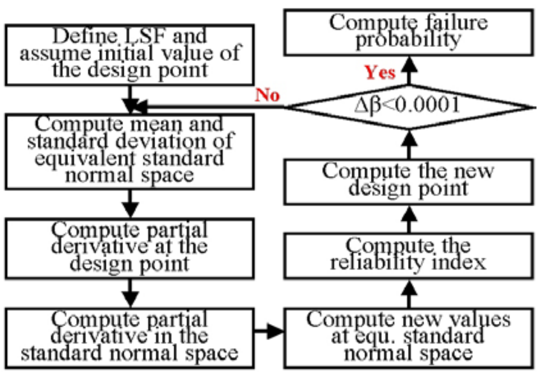


Fig. 2 Computation process of the reliability index

B. SORM (second order reliability method)

The computations required for reliability analysis of systems with linear LSF are relatively simple. However, the LSF could be nonlinear either due to a nonlinear relationship the random variables in the LSF or due to some variables being non-normal.

The FORM approach will give the same reliability index for both linear and nonlinear limit state cases, if the minimum distance point is same. But it is apparent that the failure probability of the nonlinear limit state would be less than that of the linear limit state, due to the difference in the failure domains. The curvature of the limit state around the minimum distance point determines the accuracy of the first order approximation in the FORM. The SORM improves the FORM result by including additional information about the curvature of the limit state.

The SORM approach was first explored by Fiessler using various quadratic approximations. A simple closed form solution for probability computation using a second order approximation and adopting the theory of asymptotic approximation was given by Breitung [6,7,9].

$$PF_{SORM} = \Phi(-\beta)\prod_{i=1}^{n-1}(1-\beta\kappa_i)^{-1/2} \qquad (10)$$

Where κ_i denotes the principal curvatures of the LSF at the minimum distance point and β is the reliability index calculated by using the FORM. The principal curvatures are computed by using steps shown in Fig. 3.

C. MCS (Monte Carlo Simulation)

Unlike many engineering analytical results, the ones obtained by probabilistic methods are difficult to verify experimentally. However, the adequacy of the results out of the FORM and the SORM may be required to be verified somehow. We use the MCS technique to do this job performed by the steps shown in Fig. 4 [6,7,9,10].

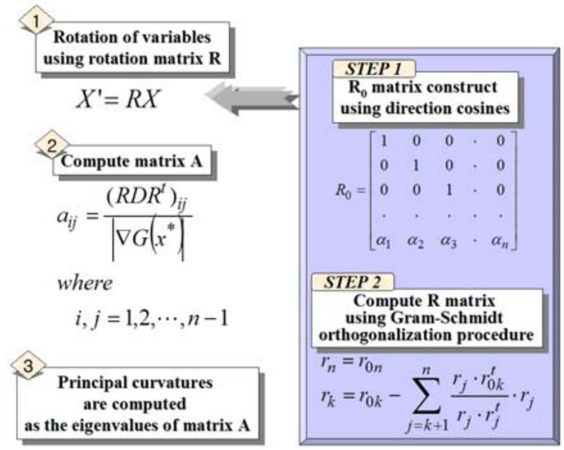


Fig. 3 Process of computing the principal curvatures

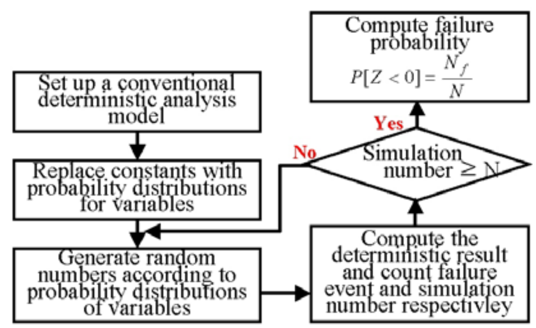


Fig. 4 Computation process of the failure probability by the Monte Carlo simulation

IV. A CASE STUDY

In this paper, we formulate the LSF using the fatigue models, and the failure probability is estimated by using the FORM and the SORM for fatigue experiment data with a single edge crack shown in Fig. 5. The specimen is a very wide SAE 1020 cold-rolled thin plate subjected to constant amplitude uniaxial cyclic loads. The random variables and their values to apply at fatigue models are listed in Table 1 [1-5].

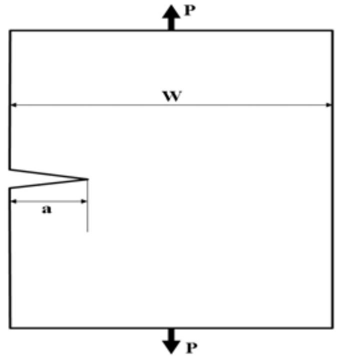


Fig. 5 The geometry of single edge crack specimen

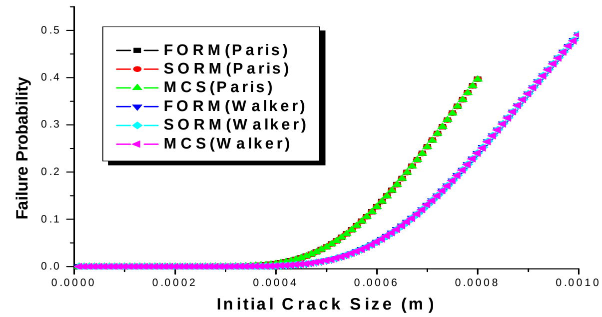
Table 1. Random variables and its statistical values used in a case study.

Valuable	Mean	C.O.V
$S_{\max}$	$200\,MPa$ (Paris)	0.002
	$300\,MPa$ (Walker)	0.002
$S_{\min}$	$-50\,MPa$ (Paris)	0.002
	$100\,MPa$ (Walker)	0.002
S_y	$630\,MPa$	-
S_u	$670\,MPa$	-
E	$207\,GPa$	-
K_c	$104\,MPa\sqrt{m}$	-
a_i	$0.001\,m$	0.01
α	1.12	-
C	6.9×10^{-12}	0.02
n	3.0	0.02
N_D	$129{,}000\,cycle$ (Paris)	0.003
	$65000\,cycle$ (Walker)	0.003
λ	0.5	-

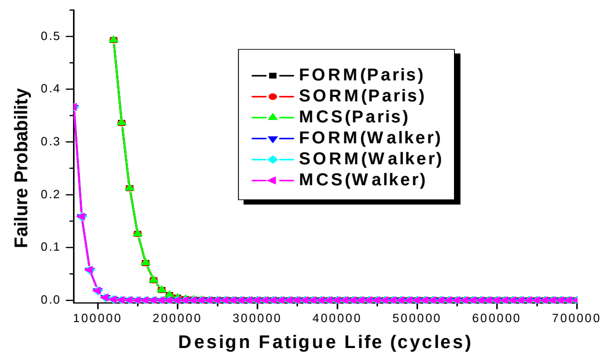
V. RESULTS AND DISCUSSION

In this paper, the LSF is formulated by using the fatigue crack growth models suggested by Paris and Walker. And the failure probability is estimated by using the values of random variables listed in Table 1 and probability theories such as the FORM, the SORM and the MCS.

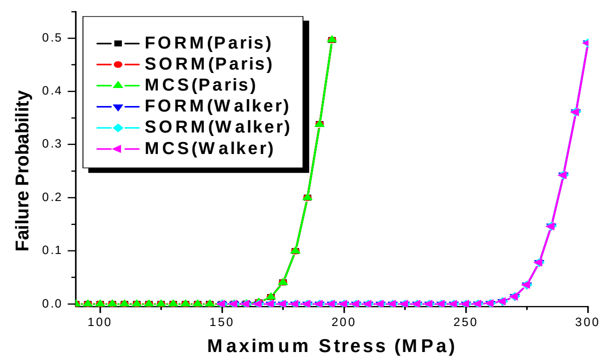
The relationship between failure probability and variation of random variables is shown in Fig. 6 corresponding to the fatigue models and the probability theories. It is found from Fig. 6 that the failure probability decreases with the increase of the design fatigue life and applied minimum stress, and the decrease of initial edge crack size, applied maximum stress and slope of Paris equation. The specific statistical values are used in a deterministic case study to compare the results out of the Pairs model to the Walker model. It is found in Fig. 6 that the failure probabilities based on the Paris and Walker models are turned out to be very similar for variation of the initial edge crack size and the slope of Paris equation. However, the Paris and Walker models show the different failure probabilities with variation of the design fatigue life, because they have different fatigue lives corresponding to the maximum and minimum stresses.



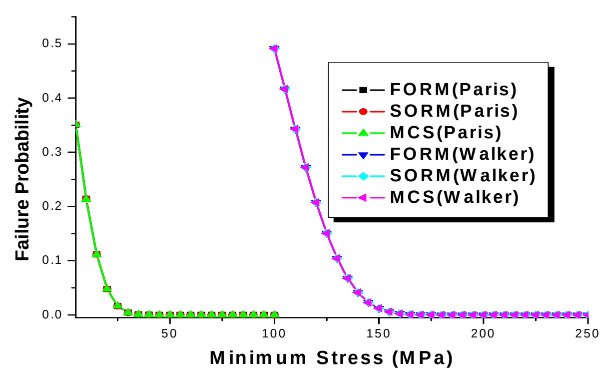
(a) Initial edge crack size



(b) Design fatigue life



(c) Applied maximum stress



(d) Applied minimum stress

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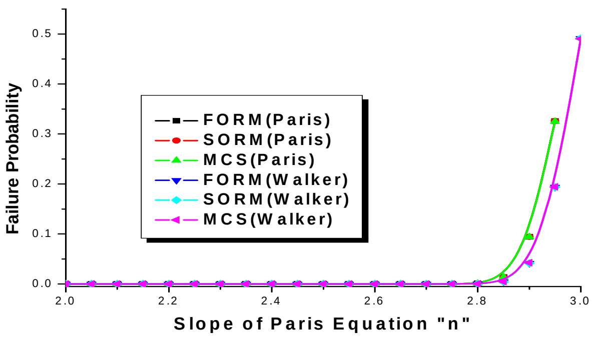
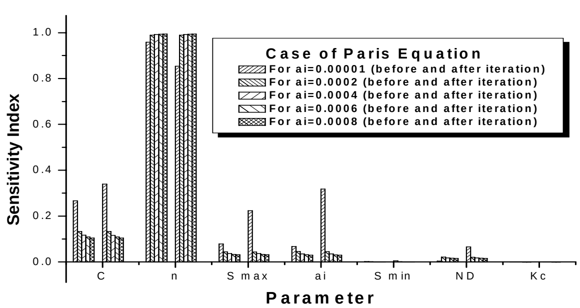


(e) Slope of Paris equation

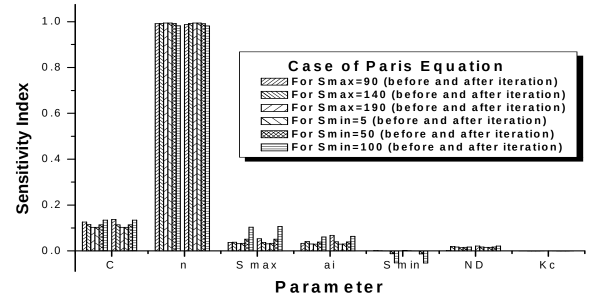
Fig. 6 Relationship between failure probability and various random variables according to the FORM, the SORM and the MCS

It is found from Fig. 6 that the FORM, the SORM and the MCs show similar failure probability for Paris and Walker models. Table 2 quantitatively shows the mean percentile differences among the results of the FORM, the SORM and the MCS for Paris and Walker models, respectively. It is recognized for the Paris and the Walker models from Table 2 that the FORM and the SORM show the similar failure probability for varying random variables. On the other hand, it is found that the difference of failure probability between the FORM and the MCS are similar with those between the SORM and the MCS.

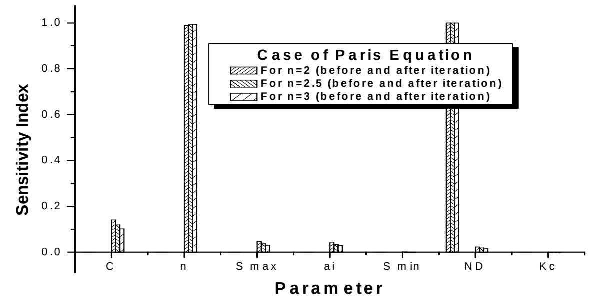
It is also recognized from Table 2 that the Walker model shows slightly larger differences of failure probability among the FORM, the SORM and the MCS than the Paris model for the variation of the initial edge crack size, the applied minimum stress and the slope of Paris equation. On the other hand, the Paris model shows slightly larger differences of failure probability among the FORM, the SORM and the MCS than the Walker model for the variation of the applied maximum stress and the design fatigue life.



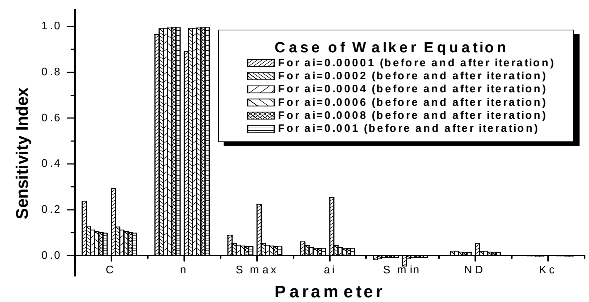
(a) change of initial edge crack



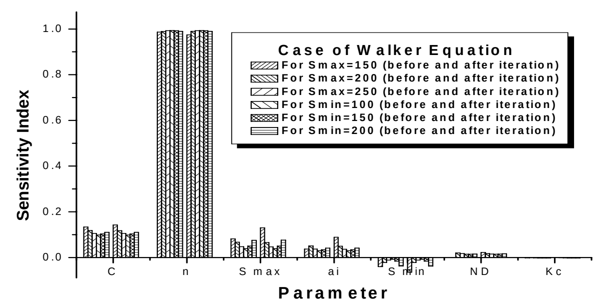
(b) change of applied maximum and minimum stresses



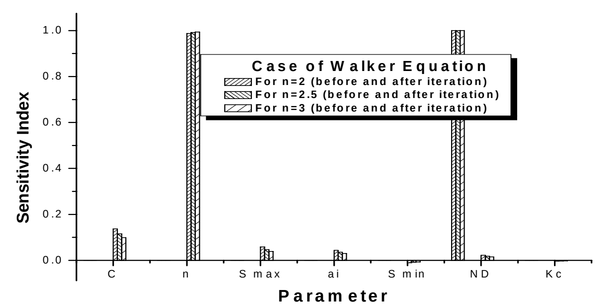
(c) change of slope of Paris equation



(d) change of initial edge crack



(e) change of applied maximum and minimum stresses



(f) change of slope of Paris equation

Fig. 7 Sensitivity of parameters according to the variation of random variables about Paris and Walker models

Although the differences among results for the variation of design fatigue life and the stress ratio are large, the differences are not distinguished clearly in Fig. 6, because the absolute values of the failure probability estimated by the FORM, the SORM and the MCS are very small.

The some typical diagrams for the effects of each random variable on the failure probability are shown in Fig. 7 as the sensitivity index. It is recognized that the slope of Pairs equation, n, affects dominantly on the failure probability with the variation of the initial edge crack size, the applied maximum and minimum stresses in the Paris and Walker models. However, it is found that the effects of the slope of Paris equation, n, on the failure probability become larger with increases of the initial edge crack size, on the other hand, the effects of other random variables such as the coefficient of Paris equation, C, the applied maximum stress, $S_{\max}$, the initial edge crack size, a_i, the applied minimum stress, $S_{\min}$, the design fatigue life, N_D and the fracture toughness, K_C, on the failure probability become smaller with increases of the initial edge crack size. Therefore, it is essential that the material constant such as the slope of Pairs equation, n, and the coefficient of Paris equation, C, must be estimated very carefully using the appropriate method from fatigue experiment.

Table 2. Comparison of the mean percentile differences among results obtained by using the FORM, the SORM and the MCS

		FORM vs. MCS [%]	SORM vs. MCS [%]	FORM vs. SORM [%]
Paris Model	Initial Edge Crack	2.3182	2.3183	5.1E-04
	Maximum Stress	4.4140	4.4141	2.6E-04
	Minimum Stress	3.6939	3.6939	2.2E-05
	Design Fatigue Life	6.6352	6.6353	1.3E-04
	Slope of Paris Eq.	0.5834	0.5834	1.5E-04
Walker Model	Initial Edge Crack	4.8437	4.8437	3.7E-05
	Maximum Stress	1.0207	1.0207	1.4E-05
	Minimum Stress	4.8429	4.8429	6.9E-06
	Design Fatigue Life	1.1532	1.1532	6E-06
	Slope of Paris Eq.	1.4407	1.4407	9E-06

And it is found from Fig. 7 that the effects of random variable on the failure probability don't vary before and after iteration with the change of initial edge crack size, the applied maximum and minimum stresses in the Paris and Walker models. However, the design fatigue life affects dominantly on the failure probability before iteration with the variation of the slope of Paris equation, and the slope of Paris equation affects dominantly on the failure probability after iteration in the Paris and Walker models.

VI. Conclusion

In this paper, the fatigue crack growth models suggested by Pairs and Walker are used to formulate the limit state function (LSF) and the FORM (first order reliability method) and the SORM (second order reliability method) are used to estimate the failure probability. And the MCS (Monte Carlo simulation) is used to evaluate the applicability of the FORM and the SORM by comparing the failure probability. Moreover, the effects of various random variables on the failure probability are systematically studied using the sensitivity index and the following results are obtained:

1. It is found that the failure probability decreases with the increase of the design fatigue life and applied minimum stress, and the decrease of initial edge crack size, applied maximum stress and slope of Paris equation.

2. It is recognized that the FORM and the SORM show the similar failure probability in the Paris and the Walker models..

3. It is recognized that the slope of Pairs equation, n, affects dominantly on the failure probability with the variation of random variables in the Paris and Walker models.

Acknowledgment

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