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Stochastic Urban Rapid Transit Network Design

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Abstract— **The rapid transit network design problem considers at upper level the list of potential transit corridors and stations to design the network as a discrete space of alternatives. At lower level the alternatives are evaluated based on the route and mode user decisions. The optimization objective is to maximize the transit demand considering the user's behaviour and the network design constraints. The stochastic extension considers the demand as a random variable, the formulation considering "a priori" and "a posteriori" models are considered. The model is proved experimenting the formulations using Branch and Bound in test networks.**

Index Terms— **rapid transit network design, stochastic optimization.**

I. Introduction

The rapid transit network design (RTND) problem considers the location decisions and the user decisions: at upper level, a list of potential rapid transit corridors and stations are assessed on the basis of its own constraints and, at lower level, the user traffic behaviour is considered. The way of selecting and comparing these network alternatives is performed by considering that the demand chooses path and mode depending on the network supplied.

The main efforts in this line of research have been aimed at determining the alignments and the location of stations. Reference [10] expands on the previous models by incorporating the station location problem, the alternative of several lines and defining the model using the maximum coverage of the public demand as an objective function and the budget constraints as side constraints. Reference [11] is an extension of the above paper, where the train lines are not initially given and the lines do not have fixed origins and destinations.

Uncertainty is modelled under the assumption that the demand is a random vector. Under the approach of scenario analysis, Benders decomposition is an appealing algorithm that replaces the very large problems posed in scenario optimization with large sequences of relatively small problems.

Different algorithms have been proposed for generating cuts. The first algorithm of this class is the so-called "L-shaped" decomposition, which works on a finite set of outcomes. This concept was generalized by the stochastic decomposition algorithm, which generates cuts from a potentially infinite sample space. This algorithm converges almost surely to the optimal solution, but the rate of convergence on practical applications remains an open question [1], [3].

Stochastic linearization techniques [8] are easy to compute and store, but they lack of stability. We may consider that linear approximations are attractive for transportation applications because they retain the structure of the original problem. Then if the first stage is a network problem, adding a linear adjustment term retains this property. The instability of pure linearization techniques may be solved by employing a nonlinear stabilization term [5].

Another way of tackling the problem is to approximate the recourse function without considering the convergence to the exact function. Then we consider an adaptive functional estimator. For no differentiable problems, the result of its use is an algorithm that should be considered as nearly optimal with a much faster rate of convergence.

The choice of the best algorithm for two-stage resource allocation problems is an open question [2]. They appear frequently but in transportation applications it is usual to deal with multistage problems, as is our case.

Other formulations to include the uncertainties are being considered: the chance constraints [4], and the use of risk functions [12].

The paper is organized as follows: In Section 2 the deterministic RTND model is discussed. In Section 3 we define the stochastic approach to the RTND problem. Finally the conclusions and the references are considered.

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II. Deterministic Rapid Transit Network Design

The RTND model is defined in [10] and [11]. Its detailed formulation may be found in the references. A short RTND formulation description is here considered.

The data required for the model are the following:

1. The set of potential locations (**N**) and the set of edges (**A**) linking them. Therefore, we have a potential

network from which the optimum rapid transit network is selected.

2. As well as locating the stations, they must be connected with a finite number of transit lines that do not have predetermined origins and destinations. Although the number of lines is a variable, the maximum number of them is known.
3. The distances between every pair of nodes, as they will be used to define the public cost.
4. The demand corresponding to an origin/destination (O-D) pair of nodes (**w**) or, equivalently, to a commodity. The set of O-D pairs is **W.**
5. Each edge **(i,j)** has a capacity $\mathbf{q_{ij}}$ given.
6. The costs of constructing each station or section of line. The set of lines is **L.** The construction costs have lower and upper budget bounds.
7. The generalized cost of satisfying the demand through the private network. Note that the cost through the public network depends on the topology of the constructed network and therefore on the edges that are selected.

The decisions that will be considered are described by:

- $y_i^l = 1$, if line $l \in L$ is located using the node **i**, 0 otherwise.
- $x_{ij}^l = 1$, if line $l \in L$ is defined using the edge **(i,j)**, 0 otherwise.
- $h_l = 1$, if the line **l** has at least a link located, 0 otherwise.
- $f_{ij}^w = 1$, if the demand of the pair **w** uses edge **(i,j)** in the public network, 0 otherwise.
- $p_w = 1$, if the demand **w** uses the public mode, 0 otherwise.

The objective function is defined as follows:

$$z = -\eta\, z_{dpub} + \left(1-\eta/2\right) z_{rc} + \left(1-\eta/2\right) z_{lc}$$

, where $\boldsymbol{\eta}$ is typically about 0.9, because the maximization of the public trip covering, z_{dpub}, is the main objective. The routing and location costs, z_{rc} and z_{lc} respectively, are also minimized.

The disaggregate level of the constraints is mentioned between parentheses:

- Budget Cost Constraint (**BCC**), that bounds the construction cost.
- In the Routing Demand Constraints (**RDC(i,w)**) the multicommodity flow conservation at each node **i** is assumed for each demand **w**.
- The Line Location Constraints (**LLC(l)**) are needed so that line **l** can be constructed.
- The Mode Demand Splitting Constraints (**MDSC(w)**) produce an all or nothing mode assignment for each demand **w**: if private cost is inferior to public cost then the demand is assigned to private mode; otherwise, it is assigned to public mode.
- The Location-Allocation Constraints (**LAC((i,j),w)**) guarantee that a demand **w** is routed on an edge **(i,j)** only if this edge belongs to the public network and the edge has enough capacity.

The **RTND** may then be expressed, in terms of the above constraints, as follows:

$$\begin{aligned} &\underset{x,y,h,f,p\in\{0,1\}}{Min.}\ z \\ &subject\ to: \quad CCC,\ RDC(i,w),\ LLC(l), \\ &\qquad\qquad MDSC(w),\ LAC((i,j),w) \end{aligned}$$

III. STOCHASTIC RAPID TRANSIT NETWORK DESIGN

Uncertainties that appear in constraints RDC will be taken into account using Stochastic Optimization. Two usual approaches are: chance constrained and recourse models.

The chance constrained models assume that the constraints that involve random variables will hold with a given threshold probability.

The stochastic RDC (SRDC) in RTND (SRDC) is formulated by:

$$\xi = Bf_{ij}^w = d_i^w(\vartheta), \forall i \in N, \forall w \in W$$

, where $d_i^w(\vartheta)$ is the random demand.

The SRDC under chance constrained model are formulated as:

$$P\left(\xi \ge d_i^w\right) \ge 1-\alpha_w, \forall w \in W$$

If we assume that the random variables appearing in different constraints are independent. We can write the above constraints as:

$$\prod_{w\in W} P\left(\xi \ge d_i^w\right) \ge 1-\alpha$$

The stochastic SRTND is defined by the RTND constraints but substituting RDC by SRDC.

In SRTND the decisions must be taken before the values of the random variables are known (a priori decision) and an adaptative action is allowed when these uncertainties disappear (a posteriori decision). The adaptative step is defined by a second level optimization in the context of a bilevel programming. Resource decomposition is adequate to split the bilevel to two optimization problems of one level. The second or lower level represents the future resources and it is formulated depending on the “posteriori” policy to represent.

The resource decomposition considers at first, or master level, the “difficult” or “priori” variables, $x_1 = (x, y, h)$ and at second, or submodel level, the “easy” or “posteriori” variables, $x_2 = (f, p)$. In SRTND context the first variables are the design variables and the second the user routing variables. Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be the feasible sets that depend only on variables x_1 and x_2, respectively. Being E the mathematical expectation, SRTND may be formulated as:

$$\underset{x_1\in X_1}{Min}\ z_{lc}(x_1) + \mathrm{E}(\varphi(x_1,\vartheta))$$

$$\varphi(x_1,\vartheta) = \underset{x_2\in X_2}{Min}\ (-z_{dpub}(x_2) + z_{rc}(x_2))$$

$$\xi = Bf_{ij}^w = d_i^w(\vartheta), \forall i \in N, \forall w \in W$$

$$\sum_{w\in W} f_{ij}^w \le \sum_{l\in L} q_{ij}\, x_{ij}^l, \forall ij \in A$$

We deal with a discrete time stochastic process defined on a probability space {Ω,ℑ, P} where the variable is {ξ}. If the demands are considered, the variable will be discrete. If a percentage is taking into consideration, ξ will by a real valued vector.

We will substitute the terms depending on x_2 in the objective function by their expectation. We should derive its distribution function or approximate it by another one ψ. In the latter case, it is necessary to establish the convergence of the expectation computed to the real one.

Using this fact we should be able to fix an upper bound to the approximating error

$$\text{arg.min} \left(E(\varphi(x_1,\vartheta))\text{-}E(\varphi(x_1,\vartheta)/\psi)\right) = \varepsilon(\psi)$$

This is a recommended approach for solving stochastic programs, [7] and [8]. When ψ is an estimate of the unknown density we should look for the bound of $E(\varepsilon(\psi))$, say $B(\varepsilon(\psi))$ and establish under which condition its limit is zero. Once established an algorithm we must study again its convergence and evaluate its behaviour using Monte Carlo experiments.

The look for a deterministic equivalent of a stochastic programming usually determines a dynamic programming formulation; see the results of [13] and [14], hence Bender's decomposition or some other method should be considered for prediction. Benders decomposition is an appealing algorithm, which performs, satisfactorily in large problems appearing in scenario optimization. Different algorithms can be used for generating the cuts. A very popular one is the "L-shaped" decomposition algorithm. It deals with a not too large finite set of outcomes. A linear program must be solved for each outcome. A generalization of it is the so called stochastic decomposition algorithm; see [15] and [16]. It generates cuts from a sample space which is considered infinite. These types of algorithms converge almost surely to the optimal solution. Which is the real rate of convergence of them is not known. Hence in there are not recommendations for practical applications.

A large number of academic analyses of stochastic programming use the niceties of scenario methods but, in applications, they are too costly. It seems that algorithms based on Benders decomposition, stochastic liberalization with nonlinear stabilization strategies, or nonlinear functional approximations are the solutions for solving large problems as ours. Some challenging experiences are reported by [6].They have been considered when we developed our algorithms.

Some authors have quoted that Benders decomposition is probably limited in transportation applications to resource allocation problems, because of the usual use of approximations based on linear programs. We should give a response on the speed of convergence of Benders decomposition. Some statistics on it should be made. Notice that in reality we should obtain integer solutions an evaluation of errors generated by rounding should be made therefore the accuracy of the solutions should be studied too.

A stochastic, decomposition based on Bender has been suggested, see [9], the decomposition, requires smoothing the coefficients. Despite some technical considerations, Benders decomposition is a very promising technique for developing the computer algorithms. We should test its performance wit expect to other alternative approximation strategies and stochastic variants of Benders decomposition algorithms. The literature suggests that they should be almost optimal and that they should works at a faster rate of convergence than has been achieved using the current Benders decomposition.

The previous formulations will be considered in detail and some first computational experiments comparing them will be presented in the meeting.

IV. Conclusion

In this paper, a first version of the stochastic approach of the Rapid Transit Network Design has been defined. Some constraints alternatives extensions and the resource approaches have been discussed. Some computational experiments are included. Further research considers a number of ways where the models can be strengthened, and new efficiently methods may be studied.

References

[1] J. Birge & F. Louveaux, I*ntroduction to Stochastic Programming*. New York: Springer-Verlag, 1997.

[2] T. A. Carvalho & W. B. Powell, "A multiplier adjustment method for dynamic resource allocation problems", in *Transportation Science*, vol. 34, 2000, pp. 150–164.

[3] Culioli, J.-C. & Cohen, G. (1990), 'Decomposition/coordination algorithms in stochastic optimization', SIAM Journal of Control and Optimization 28, pp.1372–1403.

[4] D. Dentcheva, A. Prékopa, and A. Ruzczynski, "On stochastic integer programming under probabilistic constraints", in *Rutcor Research Report RR*, 1998, pp. 29-98.

[5] Y. Ermoliev, "Stochastic quasigradient methods", in *Numerical Techniques for Stochastic Optimization*, Y. Ermoliev & R. Wets, Ed. Berlin: Springer-Verlag, 1988.

[6] Godfrey, G. and Powell, W. B. (2002), 'An adaptive, dynamic programming algorithm for stochastic resource allocation problems I: Single period travel times', Transportation Science 36, 21–39.

[7] P. Kall, Bounds and approximization to Stochastic Linear Programming, in *Stochastic Programming methods and technical applications*, K. Marti, Ed. Berlin: Akademie Verlag, 1998, pp. 1-21.

[8] P. Kall, and J. Mayer, "On the role of bounds in Stochastic Linear Programming", in *Optimization*, vol. 47, 2000, pp. 287-301.

[9] Kall, P. and Mayer J (2005): Stochastic Linear Programming: models, theory and applications. Springer, N. York.G. Laporte, A. Marín, J. A. Mesa, and F. A. Ortega, "An Integrated Methodology for the Rapid Transit Network Design Problem", Lectures Notes Computer Sc, to be published.

[10] A. Marín, "Urban Rapid Transit Network Design", TOP, Spring*er Verlag*, to be published.

[11] A. Marín and J. Salmerón, "A risk function for the stochastic modeling of electric capacity expansion", in *Naval Research Logistic*, vol. 48, 2001, pp. 662-683.

[12] W. B. Powell & R. Cheung, "A network recourse decomposition method for dynamic networks with random arc capacities", in *Networks*, vol. 24, 1994, pp. 369–384.

[13] W. B. Powell & R. Cheung, "Stochastic programs over trees with random arc capacities", in *Networks*, vol. 24, 1994, pp. 161–175.

[14] W. B. Powell, A. Ruszczynski, & H. Topaloglu, “Learning algorithms for separable approximations of stochastic optimization problems”, Technical report, Princeton University, Department of Operations Research and Financial Engineering, 2002.

[15] H. Topaloglu, & W. B. Powell, “Dynamic programming approximations for stochastic, time-staged integer multicommodity flow problems”, Technical report, Princeton University, Department of Operations Research and Financial Engineering, 2002.