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Efficient Interface Prediction- Correction Method For The Solution of Multi Dimensional Parabolic Problem Over Non Overlapping Subdomains

Daoud S Daoud & Nil Gurbuz *

Abstract—**In this article we present an efficient time lagging interface prediction method with implicit correction (TLIC) for the solution of multi dimensional parabolic problem defined over non overlapping subdomains. The time lagging method have been implemented with different integral multiples spatial spacing displacements from the interface line(s). We considered the additive splitting up method with respect to the spatial variables to solve the multi dimensional parabolic problem over each subdomains. The proposed method have been implemented to solve two dimensional model problem over multi non overlapping subdomains with respected to each spatial variable.**

Keywords: Non overlapping Domain Decomposition, Time Lagging, Multi Dimensional Parabolic Equation, Parallel Splitting

1 Introduction

Several time stepping methods have been introduced to approximate the two or three dimensional parabolic equations by the finite difference method. These time stepping methods treat the space variables individually. The alternating direction implicit method which was proposed by Douglas [4], J.Douglas and D. Peaceman [6] can be given as an example to this sort of method.

The concepts have been developed by the Russian mathematician Marchuk [11] and they present the fundamentals of the Locally One Dimensional(LOD) or Fractional Splitting (FS) methods. The classical FS methods reduce and simplify the computing of multi dimensional problem into fractional time steps according to each spatial variable.

So far, the studies have been motivated on improving the accuracy and modifying the algorithm for better performance. In 1991 W. Hundsdorfer [8] presented stabilization correction to obtain a more accurate and better approximate solution by explicit scheme. In 2001 J. Douglas and S. Kim [5] introduced modification on the alternation direction method and the FS methods by adding a correction term to achieve a second order accuracy in time. However non of these splitting methods provide parallelism for the solution with respect to each spatial variable and are classified as *Multiplicative Splitting Methods.*

In 1992, Lu et.al. [9] proposed a parallel fractional splitting method for multi dimensional parabolic problem. The algorithm poses an implicit first and second order splitting up into a series of independent one dimensional problems to be solved by parallel processors. This type of splitting is classified as the *Additive Splitting up Method.*

Another widely used method in the solution of the multi dimensional partial differential equations is the domain decomposition. The motivation for using domain decomposition method was to deal with complex geometries, equations that exhibits different behaviors in different regions of the domain and memory restriction for solving large scale problems.

Some of the studies in the domain decomposition area are motivated towards providing non iterative algorithms for solving parabolic problem. In 1990 Dawson et al. [3] presented the Explicit-Implicit non overlapping domain decomposition algorithm to solve the one or two dimensional heat equation by non iterative method. The simplicity and efficiency of the FS method for the solution of the parabolic problem, have attracted the researchers to consider this concept in the solution of the multi dimensional heat equation over multi subdomains. In [2] the authors utilized the interface boundary condition proposed by Dawson et al. [3] for the solution of the non overlapping subdomain problem using Strang's splitting. In 1991 Dryja [7] also considered the fractional splitting to solve the non overlapping domain decomposition problem and presented a non iterative algorithm of multiplicative Schwartz method in non overlapping domain decomposition for the elliptic equation obtained from the Crank-Nicolson's discretization.

In this work we considered the second order fractional splitting up algorithms for the solution of the multi dimensional parabolic problem over non overlapping subdomains. The interface boundary conditions for the non

*Department of Mathematics, Eastern Mediterranean University, Famagusta, North Cyprus, via Mersin 10 Turkey Email: daoud.daoud@emu.edu.tr, nil.gurbuz@emu.edu.tr

overlapping subdomains are defined by the Time Lagging (TL) method. The interface prediction are corrected by using the adjacent subdomains with updated solution.

In section 2, we will present the additive second order splitting method and in section 3 we will outline the considered TL interface prediction method. The non overlapping domain decomposition algorithm for the two dimensional parabolic problem, and also with the additional truncation error caused by the interface prediction are presented in section 4. Our numerical results from solving two model problems to demonstrate the difference in the performance and the accuracy will be given in section 5.

2 Parallel Splitting Up Method

In the additive splitting up method by Lu et al. [9] the multi dimensional parabolic problem (1) is split into a series of one dimensional operators using the continuous spatial operators definition. The general definition of the considered multi dimensional model problem is given by

$$\frac{\partial \phi}{\partial t} = \sum_{l}^{m} \frac{\partial}{\partial x_l}\left(a_l \frac{\partial \phi}{\partial x_l}\right) + f \quad \in \Omega \times [0,T]\,, \tag{1}$$

where $\phi(0) = \phi_0$, with Dirichlet boundary conditions over $\partial\Omega \times [0,T]$.

The domain Ω is a bounded domain in R^m $(m = 2,3)$, a_l are scalars and the assembled coefficients from the central difference discretization for $\frac{\partial}{\partial x_l}\left(a_l \frac{\partial \phi}{\partial x_l}\right)$, and for $\sum_l^m \frac{\partial}{\partial x_l}\left(a_l \frac{\partial \phi}{\partial x_l}\right)$ are denoted by A_l and A, respectively.

In this section we will briefly present the second order splitting up algorithm by Lu et al. [9] and the related theorems to show the order of accuracy in δt together with the necessary conditions for the stability. Details of the proof for the presented theorems are given by the first author in [1].

Algorithm 2.1 *The second order splitting algorithm:*

Step1: Let $\delta t > 0$ be any small time step. For $l = 1, \ldots, m$, compute $\phi^{n+\frac{l}{2m}}$ by:

$$\left(I - \frac{m}{2}\delta t A_l\right)\phi^{n+\frac{l}{2m}} = \left(I + \sum_{k,k\neq l}^{m} \frac{m}{2}\delta t A_k\right)\phi^n + \frac{m}{2}\delta t f^n, \tag{2}$$

where $f^n = f\left(\left(\frac{n+1}{2}\right)\delta t\right)$.

Step2: $\phi^{n+1} = \frac{2}{m^2}\left[\left(\frac{m^2}{2} - m\right)\phi^n + \sum_l \phi^{n+\frac{l}{2m}}\right]$.

Step3: If $T < (n+1)\,\delta t$ go to step 1, otherwise stop.

The above algorithm is a parallel type of algorithm with respect to each spatial variable because the split solution with respect to certain spatial variable is independent on the solution from any other spatial variables, not like the case of traditional Fractional Splitting algorithms (cf. e.g. [4, 6, 8, 11]). Also it should be noted that the solution by algorithm 1 possesses high flexibility for using non equal mesh spacings provided that the stability constraints by theorem 2.3 remains valid, for each spatial mesh spacings individually.

Theorem 2.2 *The splitting up algorithm 2.1 is of a second order accuracy in δt i.e. of $O\left(\delta t^2\right)$.*

Proof For the proof see [1].

The above theorem shows that the accuracy of algorithm (2.1) is equivalent to the accuracy of Crank-Nicolson scheme for the discrete form of the multi dimensional differential parabolic model problem (1).

The stability of the second order splitting up method is studied with respect to each spatial variable and also for the overall solution ϕ^{n+1}. The necessary condition for the stability is given by the following theorem.

Theorem 2.3 *The necessary condition for the stability of the solution $\phi^{n+\frac{l}{2m}}$ by step 1 and the solution ϕ^{n+1} by step 2 of algorithm 2.1 in l_2 norm is when $r_l = \frac{\delta t}{\delta x_l^2} < \frac{1}{m(m-1)}$, for $l = 1, \ldots m$.*

Proof For the proof see [1].

3 Interface Predictions-Time Lagging Method (TL- method)

For solving the time dependent problem over a set of non overlapping subdomains it requires numerical boundary conditions at the boundaries of the subdomains. The TL method use the solution values from the solution at the time level t^n to calculate(predict) the solutions at t^{n+1} and this is often called Time lagging method [10, 12].

The domain of definition Ω is decomposed into sets of K non overlapping sub domains defined along the x and y variables. The model problem (1) will then be splitted over the subdomains Ω_k where $k = 1, \ldots, K$ with the solution $\phi_k = \phi|_{\Omega_k}$ and defined as follows

$$\frac{\partial \phi_k}{\partial t} = \sum_{l}^{m} \frac{\partial}{\partial x_l}\left(a_l \frac{\partial \phi_k}{\partial x_l}\right) + f \quad over\ \Omega_k \times [0,T], \quad k = 1, ..., K. \tag{3}$$

For the solution of (3) over the time interval $[t_n, t_n + \delta t] = [t_n, t_{n+1}](T = \delta t N)$ the interface boundary condition at each point on the boundary of Ω_k are defined by $\phi(x_k, y, t^{n+1}) = g(x_k, y, t^n)$ and $\phi(x, y_k, t^{n+1}) = h(x, y_k, t^n)$ where (x_k, y) and (x, y_k), are the interface points with respect to x and y variables, respectively.

The subproblems in (3) are solved using the second order splitting up algorithm 2.1 for each time interval $[t_n, t_{n+1}]$ and for K non overlapping subdomains over $[0, T]$.

In this article we will consider time lagging interface prediction along one spatial variable (such as x), and also along the two spatial variables x and y for the solution of the two dimensional heat equation.

Let $\phi\left(x_k, y, t^{n+1}\right) = \phi(x_k, y, t^n) = g(x_k, y, t^n)$ be the predicted interface boundary value with respect to spatial variable x, defined along the lines (x_k, y), $k = 1, ..., K-1$, then by the TL method the interface boundary conditions between subdomains are generated by setting

$$\begin{aligned} \phi(x_{k-\alpha}, y, t^{n+1}) &= \phi(x_{k-\alpha}, y, t^n), \qquad \alpha = 1, 2 \text{ and}, \\ \phi(x_k, y, t^{n+1}) &= \phi(x_k, y, t^n), \qquad k = 1, ..., K-1. \end{aligned} \tag{4}$$

It should be noted that corresponding to $\alpha = 1, 2$ the left side of the non overlapping subdomains Ω_k, $k = 2, ..., K-1$ are shifted by δx and $2\delta x$ units to the left of (x_k, y), respectively. The estimation of the interface values along the line (x_k, y) by (4) are updated using the following relation

$$\phi(x_k, y, t^{n+1}) = \frac{1}{2}\left(\phi_k(x_{k-\alpha}, y, t^{n+1}) + \phi_{k+1}(x_{k-\alpha}, y, t^{n+1})\right)$$

where ϕ_k, ϕ_{k+1} are the most recent solution from the adjacent subdomains Ω_k and Ω_{k+1} with interface line (x_k, y).

For the interface prediction with respect to the y variable it is defined by $\phi\left(x, y_k, t^{n+1}\right) = \phi(x, y_k, t^n) = h(x, y_k, t^n)$ and will follow same aspects as in (4).

4 Non Overlapping Domain Decomposition

In this section we will present the generic non overlapping domain decomposition algorithm for the solution of (3) over each subdomain Ω_k. The interface predictions are given by the time lagging method. The interface boundary conditions are defined along the x-variable and in the other applications we defined the interfaces boundary conditions along the x and y variables as well.

We also proposed a correction for the interface prediction. This correction will enable us to get all the solution values including the interface boundaries to be of order $O\left(\delta t^2\right)$. The correction is performed by using algorithm 2.1 for the interfaces points using the most recent solution from the adjacent subdomains. The generic algorithm for the solution of (3) over each subdomain Ω_k is given by the following algorithm 4.1 using interface prediction by TL method given by (4).

Algorithm 4.1 *TL-Non overlapping Domain Decomposition Algorithm*

Step 1: Over the subdomain Ω_k let $\phi\left(x_k, y, t^{n+1}\right)$ and $\phi\left(x_{k-1}, y, t^{n+1}\right)$ be the interfaces prediction where (x_{k-1}, y) and (x_{k-1}, y) are left and right boundaries of Ω_k, respectively.

Step2: Solve the sub problems (3) over Ω_k partitioned by the interface lines x_{k-1}, x_k using algorithm (2.1) as follows:

$$(I - \delta t A_x)\, \phi_k^{n+\frac{1}{4}} = (I + \delta t A_y)\, \phi_k^n + \delta t f^j, \tag{5}$$

$$(I - \delta t A_y)\, \phi_k^{n+\frac{1}{2}} = (I + \delta t A_x)\, \phi_k^n + \delta t f^l, \tag{6}$$

$$\phi_k^{n+1} = \frac{1}{2}\left(\phi_k^{n+\frac{1}{4}} + \phi_k^{n+\frac{1}{2}}\right), \tag{7}$$

for $k = 1, \ldots, K$.

Step 3: If $T < (n+1)\delta t$ go to step 1, otherwise stop.

Where A_x and A_y corresponds to the central difference approximation for the differential operators defined by the spatial variables x and y.

Notee: The interface boundary conditions are predicted by (4) corresponding to the points (x_k, y) and corrected by using algorithm (2.1) with respect to the x variable only, using the solution values from the adjacent subdomains. The correction for the interfaces defined along the interface points (x, y_k) will follow similarly.

In algorithm 4.1, the error caused by the interface prediction will be estimated using (5) for the interface lines defined along the x-variable only.

To estimate the additional error caused by the TL method for $\alpha = 1$ with $\delta x \neq \delta y$ we consider the discrete form of (5) at the point (x_{k-1}, y_j) connected to the interface boundary point (x_k, y_j)

$$\begin{aligned} &\left\{\frac{u_{k-1,j}^{n+1} - u_{k-1,j}^n}{\delta t}\right\} - \left\{\frac{u_{k-2,j}^{n+1} - 2u_{k-1,j}^{n+1} + u_{k,j}^n}{\delta x^2}\right\} - \\ &\left\{\frac{u_{k-1,j-1}^n - 2u_{k-1,j}^n + u_{k-1,j+1}^n}{\delta y^2}\right\} = 0. \end{aligned} \tag{8}$$

To obtain the full discretization of the term u_{xx} at the point (x_k, y_j), at time t_{n+1} we add and subtract the term u_{x_k,y_j}^{n+1} to the equation (8), then (8) is given by;

$$\begin{aligned} &\left\{\frac{u_{k-1,j}^{n+1} - u_{k-1,j}^n}{\delta t}\right\} - \left\{\frac{u_{k-2,j}^{n+1} - 2u_{k-1,j}^{n+1} + u_{k,j}^{n+1} - u_{k,j}^{n+1} + u_{k,j}^n}{\delta x^2}\right\} - \\ &\left\{\frac{u_{k-1,j-1}^n - 2u_{k-1,j}^n + u_{k-1,j+1}^n}{\delta y^2}\right\} = 0. \end{aligned} \tag{9}$$

Therefore

$$\begin{aligned} &\left\{\frac{u_{k-1,j}^{n+1} - u_{k-1,j}^n}{\delta t}\right\} - \left\{\frac{u_{k-2,j}^{n+1} - 2u_{k-1,j}^{n+1} + u_{k,j}^{n+1}}{\delta x^2}\right\} - \\ &\left\{\frac{u_{k,j}^n - u_{k,j}^{n+1}}{\delta x^2}\right\} - \left\{\frac{u_{k-1,j-1}^n - 2u_{k-1,j}^n + u_{k-1,j+1}^n}{\delta y^2}\right\} = 0. \end{aligned} \tag{10}$$

By considering the Taylor's series with respect to the point (x_{k-1}, y_j) for (10) and since

$$\{u_t^{n+1} - u_{xx}^{n+1} - u_{yy}^n\}_{(}x_{k-1}, y_j) = 0,$$

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then the truncation error due to the interface prediction is given by;

$$\begin{aligned}
&\{-\tfrac{\delta t}{2}u_{tt}^{n+1}+\tfrac{(\delta)^2}{6}u_{ttt}^{n+1}+...\}_{(x_{k-1},y_j)}\\
&-\{\tfrac{(\delta x)^2}{12}u_{xxxx}^{n+1}+...\}_{(}x_{k-1},y_j)\\
&-\{-\tfrac{\delta t}{(\delta x)^2}u_t^{n+1}+\tfrac{(\delta t)^2}{2(\delta x)^2}u_{tt}^{n+1}-\tfrac{(\delta t)^3}{6(\delta x)^2}u_{ttt}^{n+1}+...\}_{(x_k,y_j)}\\
&-\{\tfrac{(\delta x)^2}{12}u_{yyyy}^{n+1}+...\}_{(x_{k-1},y_j)}=0.
\end{aligned} \tag{11}$$

The additional truncation error caused by time lagging interface prediction is given by the third pair of bracketed terms in (3). This shows that TL interface prediction method causes an additional truncation error is of order $O\left(\frac{\delta t}{\delta x^2}\right)$ at the boundaries between the subdomains. The error for the interface points (x, y_k) along $y-$ axis will be considered for the mesh points (x_i, y_{k-1}) using (6) and its also of order $O\left(\frac{\delta t}{\delta y^2}\right)$.

5 Numerical Results and Discussion

In this section we will present the numerical results obtained by solving the following two model problem using algorithm 4.1 with the two different definitions of the interface prediction.

Model problem 1

$$\phi_t = \phi_{xx} + \phi_{yy}, \tag{12}$$

defined over the domain $\Omega = [0,1]\times[0,1]$ and $T = [0,1]$ with initial and boundary conditions defined by the exact solution:

$$\phi(x,y,t) = e^{-1.25\pi^2 t}\sin(\pi x)\cos(0.5\pi y). \tag{13}$$

The model problem have been discretized using central difference discretization with respect to the space variables with equal and also non equal mesh spacing δx and δy. In order to visualize the stability constraints in the solution algorithm we considered different values of $r = \frac{\delta t}{\delta x^2}$ $r = 1,\ 0.7,\ 0.5,\ 0.1$ and the time stepping are selected, accordingly. The model problem are solved using algorithm 4.1 over different number of non overlapping subdomains (e.g. 2, 5, 10) with the interface prediction TL. The interfaces points are selected to be along the x variable and also along x and y variables to simulate the solution over the time interval $[0,1]$ for different time steps. The numerical solution is compared with the exact solution and the errors for model problem 1 are given in Table 1 and Table 2. From the numerical solution for different number of subdomains it is observed that the accuracy by the TL method is influenced by the value of α considered in the algorithms and that causes an improvement in accuracy when $\alpha = 2$ and, naturally, decay in the accuracy of the solution for large number of subdomains.

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Table 1: The error due to the solution of model problem 1 for different δx and δt over different number of subdomains

δx(no. of subdomain)	0.02(2)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	9.107e-5(6.691e-5)	8.981e-5(5.081e-5)	5.497e-5(1.049e-5)
δx(no. of subdomain)	0.02(5)			
$\delta t/\delta x^2$	1	0.7	0.4	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	2.291e-4(8.108e-5)	0.901e-4(6.486e-5)	8.671e-5(6.098e-5)
δx(no. of subdomain)	0.02(10)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	8.981e-4(7.761e-4)	7.051ee-4(4.071e-4)	1.936e-4(5.915e-4)
δx(no. of subdomain)	0.01(2)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	4.247e-6(2.012e-6)	3.217e-6(1.021e-6)	1.009e-6(7.802e-7)
δx(no. of subdomain)	0.01(5)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL$_{\alpha=1}$(TL$_{\alpha=2}$)	$\infty(\infty)$	9.985e-6(3.408e-6)	6.093e-6(8.983e-5)	6.101e-6(4.093e-6)
δx(no. of subdomain)	0.01(10)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL$_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	6.997e-5(1.031e-5)	4.597e-5(1.102e-5)	3.029e-5(1.009e-5)
δx(no. of subdomain)	0.005(2)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL$_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	4.149e-6(1.187e-6)	4.021e-6(8.91e-7)	9.927e-7(5.237e-7)
δx(no. of subdomain)	0.005(5)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	5.327e-6(3.178e-6)	4.981e-6(1.092e-6)	4.783e-6(1.007e-6)
δx(no. of subdomain)	0.005(10)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	5.103e-6(2.741e-6)	5.897e-6(2.897e-6)	5.801e-6(1.105e-6)

Table 2: The error due to the solution of model problem 1 for different δx, δy and δt over different number of subdomains

δx(no. of subdomain) δy(no. of subdomain)	0.02(2) 0.01(2)			
$\delta t/\delta x^2$	1	0.7	0.5	0.1
TL $_{\alpha=1}$(TL $_{\alpha=2}$)	$\infty(\infty)$	6.056e-5(4.019e-5)	4.983e-5(1.037e-5)	1.086e-5(8.431e-6)
δx(no. of subdomain) δy(no. of subdomain)	0.02(5) 0.01(5)			
$\delta t/\delta x^2$	1	0.7	0.4	0.1
TL $_{\alpha=1}$(TL$_{\alpha=2}$)	$\infty(\infty)$	3.481e-5(1.039e-5)	2.017e-5(1.813e-5)	8.671e-6(1.387e-6)
δx(no. of subdomain) δy(no. of subdomain)	0.02(10) 0.01(10)			
$\delta t/\delta x^2$	1	0.7	0.4	0.1
TL $_{\alpha=1}$(TL$_{\alpha=2}$)	$\infty(\infty)$	6.474e-5(3.934e-5)	5.891e-5(3.439e-5)	4.053e-5(4.941e-5)